

# Deep-inelastic scattering and the operator product expansion in lattice QCD

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(Dated: November 26, 2024)

We discuss the determination of deep-inelastic hadron structure in lattice QCD. By using a fictitious heavy quark, direct calculations of the Compton scattering tensor can be performed in Euclidean space that allow the extraction of the moments of structure functions. This overcomes issues of operator mixing and renormalisation that have so far prohibited lattice computations of higher moments. This approach is especially suitable for the study of the twist-two contributions to isovector quark distributions, which is practical with current computing resources. While we focus on the isovector unpolarised distribution, our method is equally applicable to other quark distributions and to generalised parton distributions. By looking at matrix elements such as  $\langle \pi^\pm | T[V^\mu(x)A^\nu(0)] | 0 \rangle$  (where  $V^\mu$  and  $A^\nu$  are vector and axial-vector heavy-light currents) within the same formalism, moments of meson distribution amplitudes can also be extracted.

## I. INTRODUCTION

Lattice QCD offers the prospect of exploring the structure functions probed in deeply inelastic scattering (DIS) and other high-energy experiments from first principles. By comparing to accurate experimental data, such calculations provide stringent tests of QCD. They also allow the extraction of information on hadron structure which is not currently available from experiment, *e.g.*, the transversity distribution  $\delta q(x)$ . The structure functions describe the hadronic part of the DIS process, *viz.*, the hadronic tensor

$$W_S^{\mu\nu}(p, q) = \int d^4x e^{iq \cdot x} \langle p, S | [J^\mu(x), J^\nu(0)] | p, S \rangle, \quad (1)$$

where  $p$  and  $S$  are the momentum and spin of the external state,  $q$  is the momentum transfer between the lepton and the hadron, and  $J^\mu$  is the electromagnetic current. Using the optical theorem,  $W_S^{\mu\nu}$  can be related to the imaginary part of the forward Compton scattering tensor

$$T_S^{\mu\nu}(p, q) = \int d^4x e^{iq \cdot x} \langle p, S | T[J^\mu(x)J^\nu(0)] | p, S \rangle. \quad (2)$$

Since lattice QCD is necessarily formulated in Euclidean space, direct calculation of the structure functions is challenging because of the analytical continuation to Minkowski space that is required [1, 2]. In addition, such a calculation would involve all-to-all light-quark propagators, and is therefore numerically demanding. For these reasons, beginning with pioneering works of Refs. [3, 4], lattice studies of the deep-inelastic structure of hadrons have focused on calculations of matrix elements of local operators that arise from the light-cone operator product expansion (OPE) [5, 6, 7, 8, 9] of the currents

$$T[J^\mu(x)J^\nu(0)] = \sum_{i,n} \mathcal{C}_i(x^2, \mu^2) x_{\mu_1} \dots x_{\mu_n} \mathcal{O}_i^{\mu\nu\mu_1 \dots \mu_n}(\mu), \quad (3)$$

where the  $\mathcal{C}_i$  are the perturbatively calculable Wilson coefficients that incorporate the short-distance physics, and the sum is over all local operators,  $\mathcal{O}_i^{\mu\nu\mu_1 \dots \mu_n}$  with the correct symmetries.  $\mu$  is the renormalisation scale. This expansion enables the investigation of  $T_S^{\mu\nu}$  via the knowledge of hadronic matrix elements of local operators. The analytical continuation of these matrix elements from Euclidean space to Minkowski space is straightforward. However, a number of difficulties arise in this approach because of the lattice regularisation. Firstly, the non-zero lattice spacing breaks the symmetry group of Euclidean space-time from  $O(4)$  to the discrete hyper-cubic subgroup  $H(4)$ , consequently modifying the transformation properties of the local operators in the OPE. In general, operators belonging to different irreducible representations of  $O(4)$ , which span the right-hand side of the OPE in Eq. (3), mix unavoidably in the lattice theory since  $H(4)$  has only a finite set of irreducible representations. For twist-two (twist = dimension - spin) contributions, this becomes particularly severe for operators of spin  $n > 4$  as they mix with lower dimensional operators and the mixing coefficients contain power divergences. Currently this restricts the available lattice calculations to operators of spin  $n = 1, 2, 3, 4$ . For higher-twist operators, such power divergences are generally unavoidable [10, 11, 12]. A second issue is that the matching of the lattice regularisation to continuum renormalisation schemes [13, 14, 15, 16], in which the Wilson coefficients are calculated, becomes more involved as  $n$  increases.

In this paper, we discuss an approach to determining matrix elements of higher-spin, twist-two operators in Eq. (3). This approach is based upon directly studying the OPE on the lattice, as was first investigated in kaon physics in Ref. [17]. A similar technique has also been applied to determine Wilson coefficients non-perturbatively [18] and extract the lowest moment of the isovector twist-two quark distribution [19] (our method is related to this latter work but improves on it in a number of ways). In our proposal, one simulates the Compton scattering tensor using lattice QCD, with currents coupling the physical light quarks,  $\psi(x)$ , present in the hadron to a non-dynamical (purely valence), unphysically heavy quark,  $\Psi(x)$ .<sup>1</sup> The introduction of this heavy quark significantly simplifies the calculation of isovector matrix elements because it removes the requirement of all-to-all propagators. After performing an extrapolation to the continuum limit, the lattice data for the Compton tensor are compared to the predictions of the OPE in Euclidean space to extract the matrix elements of local operators in Eq. (3), directly in the continuum renormalisation scheme in which the Wilson coefficients are calculated. This approach also removes the power divergences, thereby enabling extraction of matrix elements of higher spin ( $n > 4$ ) operators for twist-two operators with a simple renormalisation procedure. These matrix elements determine the Mellin moments of the structure functions which are identical in Euclidean space and Minkowski space and their analytical continuation is trivial. Finally, the chiral and infinite volume extrapolations can now be performed at the level of the local matrix elements using chiral perturbation theory [21, 22, 23, 24, 25, 26, 27, 28, 29].

The matrix elements obtained via the above procedure are completely independent of the mass of the unphysical, heavy quark and are indeed physical quantities. This is because such a quark can only propagate between the bilocal currents, and the OPE relegates its short-distance information to the Wilson coefficients. In addition to the numerical advantage, it also proves useful to introduce a fictitious heavy quark for other reasons. Firstly, the presence of the heavy scale suppresses long distance correlations between the currents in a similar way to a large Euclidean momentum. Combining both the heavy quark mass,  $m_\Psi$ , and momentum injection,  $q$ , at the current allows us to control the behaviour of the OPE precisely at moderate  $m_\Psi$  and  $q^2$ . The only constraint is

$$\Lambda_{\text{QCD}} \ll m_\Psi \sim \sqrt{q^2} \ll \frac{1}{\hat{a}}, \quad (4)$$

where  $\hat{a}$  is the coarsest lattice spacing used in the calculation. Secondly, the non-dynamical nature of the heavy quark automatically removes many contributions (for example, so-called ‘‘cat’s ears’’ diagrams – see Fig. 1(d) below) that are higher-twist contaminations in traditional DIS.

In Section II, we review the formalism of DIS with heavy quarks before discussing the extraction of the moments of twist-two parton distributions from lattice correlators in Section III. Finally in Section IV, we broaden the analysis to investigate moments of meson distribution amplitudes.

## II. FLAVOUR CHANGING CURRENTS AND HEAVY QUARKS IN LEPTON-HADRON DEEP-INELASTIC SCATTERING

The roles of quark and hadron masses in deep-inelastic scattering have been well studied. Target mass effects were first discussed by Nachtmann [30] and extensively investigated throughout the 1970s, following the observation of the precocious scaling of the structure functions [31, 32]. Away from the Bjorken limit, they result in significant contributions which arise from the OPE being an expansion in terms of operators belonging to definite irreducible representations of the Lorentz group. These contributions scale as powers of  $M^2/Q^2$ , where  $M$  is the target mass and  $Q^2 = -q^2$ , and can be summed exactly [30, 33, 34, 35]. The effects of the struck and produced quark masses were also comprehensively investigated [33, 34, 36]. These target and quark mass effects lead to  $\xi$  scaling [30, 33, 34, 35], and are particularly relevant at moderate values of  $Q^2$ . Since currently available lattice cut-offs are  $1/a \sim 3$  GeV, it is important to include these mass effects in the application of the OPE on the lattice, because of the condition in Eq. (4). In this section we present the OPE in Euclidean space relevant for computing higher moments of parton distributions on the lattice with these mass effects taken into account.

We consider fictitious currents that couple light up and down quarks to unphysical heavy quarks of mass  $m_\Psi$ . We focus on a purely vector coupling, leaving the discussion of other possible currents to the end of the section. We define

$$J_{\Psi,\psi}^\mu(x) = \bar{\Psi}(x)\gamma^\mu\psi(x) + \bar{\psi}(x)\gamma^\mu\Psi(x), \quad (5)$$

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<sup>1</sup> Such fictitious currents have been used to study quark-hadron duality in heavy quark effective theory [20]. However, in this context the heavy quark is not a valence quark.

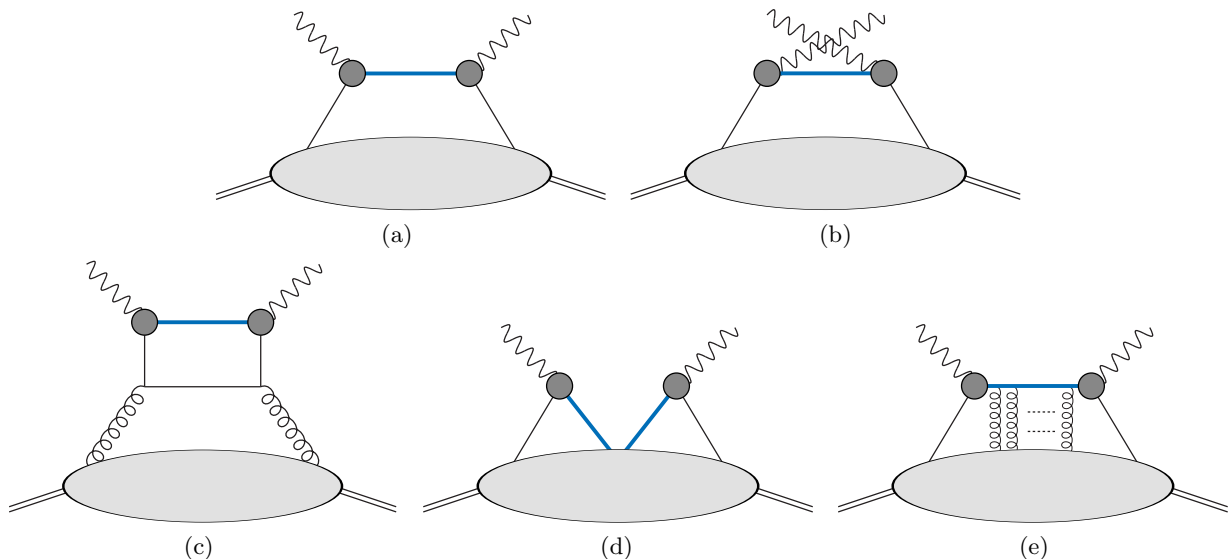


FIG. 1: Contributions to the Compton scattering tensor. Diagrams (a), (b) and (c) correspond to the leading twist contributions. Diagram (c) (the “box diagram”) involves gluonic operators and vanishes for the isovector combination, Eq. (7). Diagram (d) (the “cat’s ears diagram”) is higher-twist and absent in our analysis. Diagram (e) includes leading- and higher-twist terms and is discussed in the main text. The thick lines correspond to the heavy-quark propagators, the shaded circles to the heavy-light currents and the large shaded regions to the various parton distributions.

and construct the Euclidean Compton scattering tensor

$$T_{\Psi,\psi}^{\mu\nu}(p, q) \equiv \sum_S \langle p, S | t_{\Psi,\psi}^{\mu\nu}(q) | p, S \rangle = \sum_S \int d^4x e^{iq \cdot x} \langle p, S | T [ J_{\Psi,\psi}^\mu(x) J_{\Psi,\psi}^\nu(0) ] | p, S \rangle, \quad (6)$$

(henceforth all momenta are Euclidean).

In the limit  $q^2 \rightarrow \infty$  or  $m_\Psi \rightarrow \infty$ ,  $T_{\Psi,\psi}^{\mu\nu}$  is given by the leading-twist contribution, the “handbag diagrams” in Figs. 1 (a) and (b). The “box diagram”, Fig. 1 (c)<sup>2</sup>, which involves purely gluonic operators after the OPE, is strongly suppressed in our approach and is completely absent in the study of the OPE of the isovector Compton scattering tensor

$$T_{\Psi,v}^{\mu\nu} = T_{\Psi,u}^{\mu\nu} - T_{\Psi,d}^{\mu\nu}. \quad (7)$$

This makes the extraction of moments of the isovector quark distributions practical, and we focus on this case in this paper.

At moderate  $q^2$  and  $m_\Psi$ , higher-twist terms also contribute. However, the non-dynamical nature of the fictitious heavy quark entirely eliminates the higher-twist contributions involving more than one quark propagator between the currents, *e.g.*, the “cat’s ears diagram” in Fig. 1 (d). The diagrams in Fig. 1 (e) contain pieces that contribute to the twist-two operators in Eqs. (12) and (13), and also higher-twist terms that are discussed below.

The twist-two contributions to the OPE in  $T_{\Psi,v}^{\mu\nu}$  are from

$$t_{\Psi,\psi}^{\mu\nu} = \bar{\psi} \gamma^\mu \frac{-i (\overleftrightarrow{D} + q) + m_\Psi}{(i\overleftrightarrow{D} + q)^2 + m_\Psi^2} \gamma^\nu \psi, \quad (8)$$

and a similar term, Fig. 1 (b), in which  $\mu \leftrightarrow \nu$  and  $q \rightarrow -q$ . The derivatives  $[\overleftrightarrow{D}^\mu = \frac{1}{2} (\overrightarrow{D}^\mu - \overleftarrow{D}^\mu)]$  are included to account for the soft transverse momentum of the struck quark; they are covariant in order to maintain gauge-invariance.

<sup>2</sup> In Fig. 1 (c), we specify that the large momentum,  $q^2$ , flows through the three light-quark lines; the contributions in which these quarks have soft momenta are already included in Figs. 1 (a) and (b). In principle, these gluonic contributions can be disentangled from their different  $q^2$  behaviour.

The OPE of the bilocal currents in Eq. (6) is now given by a Taylor expansion of the propagator in the above expression,

$$\frac{-i \left( i \vec{D} + q \right) + m_\Psi}{(i \vec{D} + q)^2 + m_\Psi^2} = -\frac{-i \left( i \vec{D} + q \right) + m_\Psi}{Q^2 + \vec{D}^2 - m_\Psi^2} \sum_{n=0}^{\infty} \left( \frac{-2i q \cdot \vec{D}}{Q^2 + \vec{D}^2 - m_\Psi^2} \right)^n, \quad (9)$$

where  $Q^2 = -q^2$ . The pole mass of the heavy quark is not a measurable quantity and we replace it with the mass of a meson composed of the unphysical heavy quark and a light quark,  $M_\Psi = m_\Psi + \frac{1}{2}\alpha$ , where  $\alpha$  is the binding energy and is of  $\mathcal{O}(\Lambda_{\text{QCD}})$ . This meson mass can be directly computed on the lattice. The parameter  $\alpha$  can be extracted from experiment or lattice calculations [37], however we view it as an unknown to be determined in the procedure of studying the OPE on the lattice. The  $\vec{D}^2$  term in the denominators in Eq. (9) gives rise to higher-twist contributions, such as those arising from Fig. 1 (e), if we instead Taylor expand with respect to  $\left( \frac{-2iq \cdot \vec{D} + \vec{D}^2}{Q^2 - m_\Psi^2} \right)$ . These higher-twist contributions scale as powers of  $\Lambda_{\text{QCD}}^2/(q^2 + m_\Psi^2)$  and can be neglected for  $q^2$  or  $m_\Psi^2$  sufficiently large. For more moderate scales, they may become important (particularly for large  $n$ ) and can in principle be studied in our approach. However, this is beyond the scope of this work. Instead, we replace the denominator in Eq. (9) by

$$\tilde{Q}^2 = Q^2 - M_\Psi^2 + \alpha M_\Psi + \beta, \quad (10)$$

where the unknown parameter  $\beta$  represents terms of  $\mathcal{O}(\Lambda_{\text{QCD}}^2)$ , including these higher-twist effects and sub-leading heavy-quark mass effects. In doing so we have neglected the  $n$  dependence of  $\beta$  since it is suppressed by powers of the strong coupling. To remove these higher-twist and mass uncertainties, one might consider using a fictitious heavy particle that does not interact strongly. However, issues such as the gauge dependence of the resulting Compton tensor would need to be investigated thoroughly and are not discussed here.

After making this expansion and considering only the symmetric combination ( $m$  is the light quark mass)  $t_{\Psi, \psi}^{\{\mu\nu\}} = \frac{1}{2} \left( t_{\Psi, \psi}^{\mu\nu} + t_{\Psi, \psi}^{\nu\mu} \right)$ , combining the two leading contributions then leads to

$$t_{\Psi, \psi}^{\{\mu\nu\}}(q) = \frac{i}{\tilde{Q}^2} \left( 4 \sum_{\substack{n=0 \\ \text{even}}}^{\infty} \mathcal{C}_{n+2} \frac{(2q_{\mu_1}) \cdots (2q_{\mu_n})}{\tilde{Q}^{2n}} \mathcal{O}_\psi^{\mu\nu\mu_1 \cdots \mu_n} + \tilde{Q}^2 \delta^{\mu\nu} \sum_{\substack{n=2 \\ \text{even}}}^{\infty} \mathcal{C}'_n \frac{(2q_{\mu_1}) \cdots (2q_{\mu_n})}{\tilde{Q}^{2n}} \mathcal{O}_\psi^{\mu_1 \cdots \mu_n} \right. \\ \left. + 2i \delta^{\mu\nu} (m_\Psi - m) \sum_{\substack{n=0 \\ \text{even}}}^{\infty} \hat{\mathcal{C}}_n \frac{(2q_{\mu_1}) \cdots (2q_{\mu_n})}{\tilde{Q}^{2n}} \hat{\mathcal{O}}_\psi^{\mu_1 \cdots \mu_n} - 4q^\mu \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \mathcal{C}''_{n+1} \frac{(2q_{\mu_1}) \cdots (2q_{\mu_n})}{\tilde{Q}^{2n}} \mathcal{O}_\psi^{\nu\{\mu_1 \cdots \mu_n\}} \right), \quad (11)$$

where we have introduced the operators (the braces indicate symmetrisation of the enclosed indices)

$$\mathcal{O}_\psi^{\mu_1 \cdots \mu_n} = \bar{\psi} \gamma^{\{\mu_1} \left( i \vec{D}^{\mu_2} \right) \cdots \left( i \vec{D}^{\mu_n} \right) \psi - \text{traces}, \quad (12)$$

and

$$\hat{\mathcal{O}}_\psi^{\mu_1 \cdots \mu_n} = \bar{\psi} \left( i \vec{D}^{\{\mu_1} \right) \cdots \left( i \vec{D}^{\mu_n} \right) \psi - \text{traces}, \quad (13)$$

and the various perturbatively calculable Wilson coefficients  $\mathcal{C}_n, \mathcal{C}'_n, \mathcal{C}''_n$  and  $\hat{\mathcal{C}}_n$  depend on  $q^2, \mu^2$  (the renormalisation scale) and the heavy quark mass [38, 39, 40, 41]. The first set of operators are the usual twist-two vector operators that enter into textbook DIS analyses. The second set are chiral-odd, twist-three operators whose matrix elements correspond to the parton distribution  $e_\psi(x)$  [42, 43].

Taking spin-averaged hadron matrix elements of this expression then leads to

$$T_{\Psi, \psi}^{\{\mu\nu\}}(p, q) = i \sum_{\substack{n=2 \\ \text{even}}}^{\infty} A_\psi^n(\mu^2) \zeta^n \left\{ \delta^{\mu\nu} \left[ \mathcal{C}_n \frac{\tilde{Q}^2 n C_n^{(1)}(\eta) - 2\eta C_{n-1}^{(2)}(\eta)}{q^2 n(n-1)} + \mathcal{C}'_n C_n^{(1)}(\eta) \right] + \frac{p^\mu p^\nu \tilde{Q}^2}{(p \cdot q)^2} \mathcal{C}_n \left[ \frac{8\eta^2 C_{n-2}^{(3)}(\eta)}{n(n-1)} \right] \right. \\ \left. + 4 \frac{p^\mu q^\nu}{p \cdot q} \left[ \mathcal{C}_n \frac{\tilde{Q}^2 (n-1) \eta C_{n-1}^{(2)}(\eta) - 4\eta^2 C_{n-2}^{(3)}(\eta)}{q^2 n(n-1)} - \mathcal{C}''_n \frac{\eta}{n} C_{n-1}^{(2)}(\eta) \right] \right. \\ \left. + \frac{q^\mu q^\nu}{q^2} \left[ \mathcal{C}_n \frac{\tilde{Q}^2 n(n-2) C_n^{(1)}(\eta) - 2\eta(2n-3) C_{n-1}^{(2)}(\eta) + 8\eta^2 C_{n-2}^{(3)}(\eta)}{q^2 n(n-1)} - 2\mathcal{C}''_n \left( C_n^{(1)}(\eta) - 2 \frac{\eta}{n} C_{n-1}^{(2)}(\eta) \right) \right] \right\} \\ - 2i \frac{M(m_\Psi - m)}{\tilde{Q}^2} \delta^{\mu\nu} \sum_{\substack{n=0 \\ \text{even}}}^{\infty} \hat{\mathcal{C}}_n \hat{A}_\psi^n(\mu^2) \zeta^n C_n^{(1)}(\eta), \quad (14)$$

where we have defined

$$\zeta = \frac{\sqrt{p^2 q^2}}{\tilde{Q}^2}, \quad \eta = \frac{p \cdot q}{\sqrt{p^2 q^2}}, \quad (15)$$

( $M$  is the proton mass,  $p^2 = -M^2$ ) and the hadronic matrix elements of the local operators in Eqs. (12) and (13) as

$$\sum_S \langle p, S | \mathcal{O}_\psi^{\mu_1 \dots \mu_n} | p, S \rangle = A_\psi^n(\mu^2) [p^{\mu_1} \dots p^{\mu_n} - \text{traces}], \quad (16)$$

$$\sum_S \langle p, S | \hat{\mathcal{O}}_\psi^{\mu_1 \dots \mu_n} | p, S \rangle = i M \hat{A}_\psi^n(\mu^2) [p^{\mu_1} \dots p^{\mu_n} - \text{traces}], \quad (17)$$

(the  $A_\psi^n$  and  $\hat{A}_\psi^n$  are dimensionless and real). In Eq. (14) the  $C_n^{(\lambda)}(\eta)$  are Gegenbauer polynomials that arise from the trace subtractions in Eqs. (16) and (17) and account for the target mass effects [30, 33, 34, 35]. If we now choose the rest frame of the proton,  $p = (0, 0, 0, i M)$  and parameterise  $q = (0, 0, \sqrt{q_0^2 - Q^2}, i q_0)$ , then the symmetric combination of  $\{\mu, \nu\} = \{3, 4\}$  is

$$T_{\Psi, \psi}^{\{34\}}(p, q) = \sum_{\substack{n=2 \\ \text{even}}}^{\infty} A_\psi^n(\mu^2) f(n), \quad (18)$$

where

$$f(n) = -\sqrt{q_0^2 - Q^2} \zeta^n \left\{ \frac{2}{q_0} \left[ C_n \frac{\tilde{Q}^2 (n-1) \eta C_{n-1}^{(2)}(\eta) - 4\eta^2 C_{n-2}^{(3)}(\eta)}{n(n-1)} + C_n'' \frac{\eta}{n} C_{n-1}^{(2)}(\eta) \right] \right. \\ \left. + \frac{q_0}{Q^2} \left[ C_n \frac{\tilde{Q}^2 n(n-2) C_n^{(1)}(\eta) - 2\eta(2n-3) C_{n-1}^{(2)}(\eta) + 8\eta^2 C_{n-2}^{(3)}(\eta)}{n(n-1)} + 2C_n'' \left( C_n^{(1)}(\eta) - 2\frac{\eta}{n} C_{n-1}^{(2)}(\eta) \right) \right] \right\}. \quad (19)$$

Equation (18) is the central object studied in the following section where we will see that lattice calculations of it would allow extraction of the even moments  $A_\psi^n(\mu^2)$ , since the Wilson coefficients and kinematic factors that determine  $f(n)$  are known. However we discuss some more general aspects of  $T_{\Psi, \psi}^{\mu\nu}$  first.

Once we have determined the  $A_\psi^n(\mu^2)$ , we can also use the diagonal elements of  $T_{\Psi, \psi}^{\mu\nu}$  to extract information on the moments of the twist-three quark distributions  $e_\psi(x)$  from the same vector-vector current correlator. Experimentally, this distribution is difficult to extract, being measurable only in charged current DIS, semi-inclusive DIS or Drell-Yan processes [42, 43]. The only currently available determination uses data from the CLAS collaboration [44] and requires assumptions about the fragmentation functions that enter semi-inclusive DIS [45]. Any information on the moments of  $e_\psi(x)$  from lattice QCD (whether in our approach or by direct calculation) would be useful. Additionally, the odd- $n$  moments  $A_\psi^n(\mu^2)$  and  $\hat{A}_\psi^n(\mu^2)$  (which determine the valence combinations of quark distributions) can be extracted from a correlator of vector and axial-vector currents just as for  $F_3(x)$  in neutrino scattering.

Similar methods can be used to calculate moments of the helicity and transversity parton distributions. The moments of both the twist-two and twist-three helicity distributions and the twist-two transversity distribution may be determined from suitable antisymmetric pieces of  $T_{\Psi, \psi}^{\mu\nu}$ . Since we are not restricted by physical scattering processes, we can also consider correlators of unphysical currents: *e.g.* scalar, pseudo-scalar and tensor. The analysis for such currents is again very similar to the case that we have discussed and by using such currents we can investigate distributions that are not experimentally accessible in DIS; for example the transversity distribution can be accessed independent of the quark masses by looking at the scalar-vector correlator. Since these currents are not conserved, they are no longer scale independent and the appropriate anomalous dimensions must be used [46, 47]. Finally, the off-forward matrix elements of the various twist-two and twist-three operators that determine the moments of generalised parton distributions can also be studied through analysis of the off-forward Compton scattering amplitude.

### III. EXTRACTION OF MOMENTS FROM LATTICE CALCULATIONS

With numerical investigation of  $T_{\Psi, \psi}^{\{34\}}(p, q)$  for varying  $Q^2$ ,  $q_0$  and  $M_\Psi$ , Equation (18) provides a means to extract the moments  $A_\psi^n(\mu)$  for  $n > 4$  without requiring power subtractions and complicated renormalisations. In order to

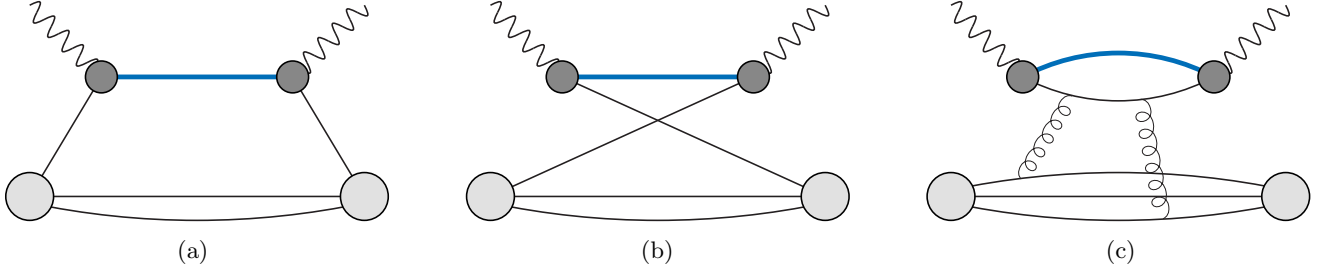


FIG. 2: Quark contractions in the four point correlator. Light shaded circles are the proton source and sink whilst the dark shaded circles are the heavy-light currents. The thick line indicates the heavy quark propagator. Diagram (c) is quark-line disconnected (as indicated by the representative gluons).

calculate  $T_{\Psi,\psi}^{\{\mu\nu\}}(p, q)$  we consider the following four-point Euclidean correlator [2, 48] (with  $\tau > 0$ )

$$\begin{aligned}
G_{(4)}^{\mu\nu}(\mathbf{p}, \mathbf{q}, t, \tau; \Gamma) &= \sum_{\mathbf{x}, \mathbf{z}} \sum_{\mathbf{y}} e^{i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{q}\cdot\mathbf{y}} \Gamma_{\beta\alpha} \langle 0 | \chi_{\alpha}(\mathbf{x}, t) \bar{J}_{\Psi,\psi}^{\mu}(\mathbf{y} + \mathbf{z}, \tau + \frac{t}{2}) \bar{J}_{\Psi,\psi}^{\nu}(\mathbf{z}, \frac{t}{2}) \bar{\chi}_{\beta}(\mathbf{0}, 0) | 0 \rangle \\
&= \sum_{\mathbf{x}, \mathbf{z}} \sum_{\mathbf{y}} \sum_{N, N'} \sum_{s, s'} e^{i(\mathbf{p}-\mathbf{p}_N)\cdot\mathbf{x}} e^{i(\mathbf{p}_N-\mathbf{p}_{N'})\cdot\mathbf{z}} e^{i\mathbf{q}\cdot\mathbf{y}} e^{-(E_N+E_{N'})\frac{t}{2}} \Gamma_{\beta\alpha} \\
&\quad \times \langle 0 | \chi_{\alpha}(0) | E_N, \mathbf{p}_N, s \rangle \langle E_N, \mathbf{p}_N, s | \bar{J}_{\Psi,\psi}^{\mu}(\mathbf{y}, \tau) \bar{J}_{\Psi,\psi}^{\nu}(\mathbf{0}) | E_{N'}, \mathbf{p}_{N'}, s' \rangle \langle E_{N'}, \mathbf{p}_{N'}, s' | \bar{\chi}_{\beta}(0) | 0 \rangle \\
&\xrightarrow{t \rightarrow \infty} e^{-E_0 t} \sum_{\mathbf{y}} e^{i\mathbf{q}\cdot\mathbf{y}} \sum_{s, s'} \Gamma_{\beta\alpha} \langle 0 | \chi_{\alpha}(0) | E_0, \mathbf{p}, s \rangle \langle E_0, \mathbf{p}, s | \bar{J}_{\Psi,\psi}^{\mu}(\mathbf{y}, \tau) \bar{J}_{\Psi,\psi}^{\nu}(\mathbf{0}) | E_0, \mathbf{p}, s' \rangle \langle E_0, \mathbf{p}, s' | \bar{\chi}_{\beta}(0) | 0 \rangle,
\end{aligned} \tag{20}$$

where  $\chi$  is a dimensionless interpolating field for the proton and  $\bar{J}_{\Psi,\psi}^{\mu}$  is the lattice version of Eq. (5). Here  $\Gamma$  is a Dirac matrix which can be chosen as  $\Gamma_4 = \frac{1}{2}(1 + \gamma_4)$  for the components of the amplitude we are considering and  $E_0(\mathbf{p})$  is the ground state energy. The sums on  $N$  and  $N'$  are over all possible eigenstates of the Hamiltonian. Then, defining the two-point Euclidean correlator as

$$G_{(2)}(\mathbf{p}, t; \Gamma) = \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} \Gamma_{\beta\alpha} \langle 0 | \chi_{\alpha}(0) \bar{\chi}_{\beta}(\mathbf{x}, t) | 0 \rangle, \tag{21}$$

we can determine the Compton amplitude from the Fourier transform of the ratio of these correlators:

$$T_{\Psi,\psi}^{\{\mu\nu\}}(p, q) = 4M a \sum_{\tau} e^{iq_4\tau} \left[ \lim_{t \rightarrow \infty} \frac{G_{(4)}^{\{\mu\nu\}}(\mathbf{p}, \mathbf{q}, t, \tau; \Gamma_4)}{G_{(2)}(\mathbf{p}, t; \Gamma_4)} \right], \tag{22}$$

where  $a$  is the lattice spacing. Since  $G_{(4)}^{\mu\nu}(\mathbf{p}, \mathbf{q}, t, \tau; \Gamma)$  falls off with  $\tau$  over a characteristic time  $(M_{\Psi}a)^{-1}$ , the Fourier transform in Eq. (22) will be well approximated in practice.

The above equations also hold if the heavy-light current is replaced by the usual light-light current. However, using a quenched heavy quark greatly simplifies the numerical work in extracting  $T_{\Psi,\psi}^{\{34\}}(p, q)$  without altering the non-perturbative physics, the Mellin moments, we are interested in. After performing the Wick contractions of the quark fields,  $G_{(4)}^{\mu\nu}(\mathbf{p}, \mathbf{q}, t, \tau; \Gamma)$  is given by the three different arrangements of quark propagators shown in Fig. 2 where the thick and thin lines represent the heavy and light (physical) quarks respectively. For the isovector channel obtained via Eq. (7), diagram 2 (c) does not contribute. In the remaining two diagrams, the Wick contractions can be computed with the technique of extended propagators. Many values of the heavy quark mass can be studied with only a very small increase in computational cost. For a light-light current, we would need the additional Wick contractions shown in Fig. 3 and would require light all-to-all propagators even in the isovector channel.

In order to avoid the problems of operator mixing discussed in the introduction, a continuum extrapolation of the Compton amplitude must be performed before it can be used to extract the moments  $A_{\psi}^n(\mu)$  in Eq. (18). This requires calculations at a number of different lattice spacings. On the other hand, the chiral and infinite volume extrapolations can only be performed after we have extracted the local matrix elements since the current-current correlator is not a low-energy observable and cannot be treated reliably in effective field theory. Also, to use Eq. (18), the heavy-light meson and proton masses must be extracted from appropriate two-point correlators.

Assuming these tasks have been performed, we may now consider what is required to obtain useful information from these calculations. Ideally we would extract all integer moments from the lattice and be able to uniquely determine

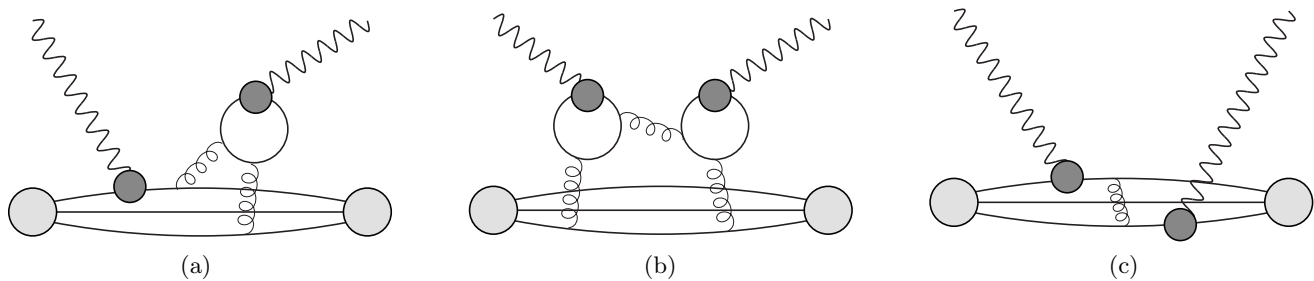


FIG. 3: Additional quark contractions present in the four point correlator for light-light currents but absent in the case we consider. These diagrams correspond to higher-twist contributions (diagram (d) in Fig. 1). Light shaded circles are the proton source and sink whilst the dark shaded circles are the light-light currents.

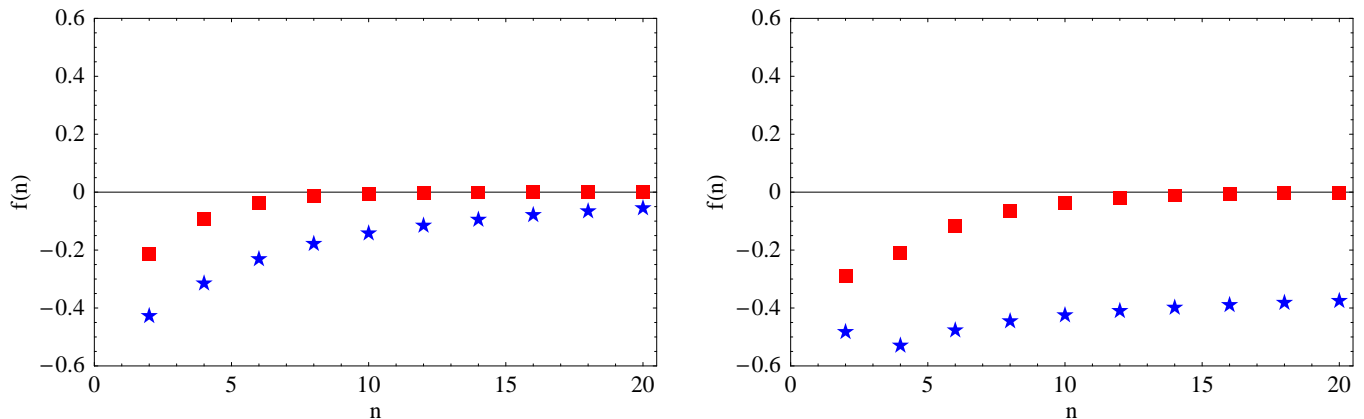


FIG. 4: Behaviour of the function  $f(n)$  for two different parameter sets. On the left,  $M_\Psi = 3.54$  GeV,  $Q^2 = 1.5$  GeV<sup>2</sup> and  $q_0 = 2.76$  GeV while on the right we have chosen  $M_\Psi = 2.1$  GeV,  $Q^2 = -3.85$  GeV<sup>2</sup> and  $q_0 = 1.98$  GeV. The stars and boxes correspond to  $\alpha = 0.4, 1.2$  GeV respectively and we have chosen  $M = 1.2$  GeV and  $\beta = 0$  GeV<sup>2</sup>.

the Minkowski space parton distributions through an inverse Mellin transform.<sup>3</sup> More realistically, we might envisage extracting 6–10 even moments (and the parameters  $\alpha$  and  $\beta$  in Eq. (10)) by fits using Eq. (18). Previous analysis [50] suggests that this is enough to reliably constrain standard parameterisations of parton distributions. Additionally, the low moments that have previously been computed directly from local operators can be used as input into our approach. In order to extract these higher moments, we must be careful of higher-twist contributions, as discussed in Sec. II. Since by using a heavy quark we have removed many higher-twist terms, we might expect that those that remain will be small. This would be indicated by the fitted value of  $\beta$  being small.

For simplicity, we work to zeroth order in the QCD coupling, taking the Wilson coefficients to be unity. In a complete analysis, the perturbative Wilson coefficients appropriate to the desired scheme and scale [38, 39, 40, 41] should be used. With this assumption, the only unknowns in Eq. (18) are the moments  $A_\psi^n(\mu)$  and the parameters  $\alpha$  and  $\beta$ . Then if we wish to determine six moments ( $n = 2, 4, \dots, 12$ ) for example, we need the Compton amplitude evaluated at eight or more different combinations of momenta and heavy quark masses. *A priori*, we would also want to choose the masses and momenta such that the convergence of the expansion on the RHS of Eq. (18) allows us to neglect terms beyond a certain  $n_{max}$  and extract the moments for  $n < n_{max}$ . For example, choosing  $M_\Psi = 3.54$  GeV,  $q_0 = 2.76$  GeV and  $Q^2 = 1.5$  GeV<sup>2</sup>, the expansion on the RHS of Eq. (18) falls off as shown in the left panel of Fig. 4. However, from experimental measurements and perturbative counting rules, we know that the moments of the parton distributions in fact fall off rapidly as  $n$  increases and it may be more useful to choose the masses and momenta so that  $f(n)$  decreases slowly, allowing the natural suppression of the moments to control how many can be extracted.

<sup>3</sup> This is guaranteed to be a unique reconstruction by Carlson's theorem [49]. Essentially we would be expanding the OPE in the Euclidean region, analytically continuing the Wilson coefficients and then re-summing the expansion in the Minkowski region.

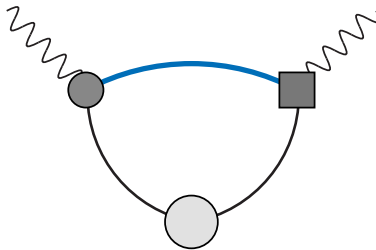


FIG. 5: Extraction of moments of meson distribution amplitudes. Here, the light-shaded circle denotes the pion interpolating operator and the dark circle and dark square indicate the vector and axial-vector currents, respectively.

In this case, choosing  $M_\Psi = 2.1$  GeV,  $q_0 = 1.98$  GeV and  $Q^2 = -3.85$  GeV<sup>2</sup> which gives the flatter behaviour shown in the RHS of Fig. 4 may be more appropriate. Without performing the large scale simulations that are required to determine the Compton amplitude, it is hard to be definite on the choices of parameters. However, it seems that this approach has significant potential to determine higher moments of isovector parton distributions than are currently available from QCD.

#### IV. DISTRIBUTION AMPLITUDES FROM CURRENT-CURRENT MATRIX ELEMENTS

A further application of the approach we have outlined is in computing moments of meson distribution amplitudes,  $\phi_M$ . In the lattice approach, we can extract moments of distribution amplitudes in the same way as DIS determines moments of parton distributions; for example, we may study the matrix element  $\langle \pi^\pm | T[V_{\Psi,\psi}^\mu(x) A_{\Psi,\psi}^\nu(0)] | 0 \rangle$ , where  $V_{\Psi,\psi}^\mu$  and  $A_{\Psi,\psi}^\nu$  are fictitious vector and axial vector heavy-light currents. This process is described by the tensor

$$S_{\Psi,\psi}^{\mu\nu}(p, q) = \int d^4x e^{i q \cdot x} \langle \pi^+(p) | T[V_{\Psi,\psi}^\mu(x) A_{\Psi,\psi}^\nu(0)] | 0 \rangle. \quad (23)$$

Following from Eq.(9), the OPE of the two currents leads to the same matrix elements of twist-two operators that determine the moments of the pion distribution amplitude:

$$\langle \pi^+(p) | \bar{\psi} \gamma^{\{\mu_1} \gamma_5 (i D)^{\mu_2} \dots (i D)^{\mu_n\} \psi | 0 \rangle = f_\pi \langle \xi^{n-1} \rangle_\pi [p^{\mu_1} \dots p^{\mu_n} - \text{traces}], \quad (24)$$

where

$$\langle \xi^n \rangle_\pi \equiv \int_0^1 d\xi \xi^n \phi_\pi(\xi). \quad (25)$$

These matrix elements can be determined by studying the various components of  $S_{\Psi,\psi}^{\mu\nu}$  for varying  $m_\Psi$  and  $q^\mu$  analogously to Eq. (14). As in the DIS case, many higher-twist contributions are absent because of the valence nature heavy quark and the problems that plague direct evaluation of higher moments due to the lattice cutoff are eliminated. Since only the zeroth (decay constant) and second moments of the pion distribution amplitude have been investigated in the direct approach [51, 52, 53, 54, 55, 56], any information on higher moments will be useful in constraining the distribution amplitude from QCD. For flavour non-diagonal mesons (*e.g.*  $\pi^\pm$ ,  $K^{\pm,0}$ ), extraction of the tensor  $S_{\Psi,\psi}^{\mu\nu}$  on the lattice only requires the computation of the Wick contraction shown in Fig. 5.

#### V. SUMMARY

To summarise, the direct study of Compton scattering tensor on the lattice using the operator product expansion can provide useful information on the moments of quark distributions. Using currents that couple an unphysical, quenched, heavy quark field to the physical light quarks renders the approach feasible without modifying the non-perturbative physics that can be extracted. This has the potential that a large enough number of moments can be extracted that the parton distributions can be reliably reconstructed from lattice calculations. Our analysis has focused on the unpolarised isovector quark distribution, but it can also be used to study the other twist-two and twist-three parton distributions and generalised parton distributions. Additionally, this method will allow computations of the moments of meson distribution amplitudes where even the lowest non-trivial moment is not known reliably from the lattice.



### Acknowledgments

We are grateful to W. Melnitchouk, G. A. Miller, D. B. Renner, S. R. Sharpe, J. I. Skullerud and R. D. Young for useful discussions. CJDL acknowledges the support of the National Centre for Theoretical Sciences, Taipei. This work was supported by DOE grants DE-FG03-97ER41014, DE-FG03-00ER41132 and DE-FG03-96ER40956.

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