

Spectrality, coupling constant analyticity and the renormalization group

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Abstract

Analytic structure in the strong coupling constant that emerges for some observables in QCD after duality averaging of renormalization group improved amplitudes is discussed. It is shown that perturbation theory calculations are justified for the proper observables related to the two-point correlators of hadronic currents the analytic properties of which are well-established. A particular case of gluonic current correlators is discussed in detail.

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General properties of quantum field theory impose strong constraints on model building for elementary particle phenomenology. Symmetry properties of interactions lie in the foundation of the standard model [1], causality leads to restrictions on the analytic structure of scattering amplitudes as functions of energy, the freedom of redefinition of ultraviolet subtraction procedures in renormalizable theories leads to the renormalization group invariance which is a basic property of theoretical quantities corresponding to physical observables [2]. While such general properties are supposed to be valid in a full theory there is little known about the very existence of realistic nontrivial quantum field models – only some simplified examples (mainly in two-dimensional space-time) have been considered (e.g. [3]). The realistic four-dimensional models suitable for particle phenomenology are analyzed within perturbation theory in the coupling and only first few terms of perturbative expansions are usually available. Some general results on asymptotic behavior at large orders of perturbation theory are obtained by the steepest descent method for the functional integral determining the generating functional for Green's functions. The results are known in some models of quantum field theory where classical solutions of equations of motion were found [4]. The classical solutions of field equations are also known in nonabelian gauge theories [5] that provides the appropriate saddle-point configurations for the steepest descent method of evaluating the functional integrals [6] and allows for deeper understanding the ground state structure in these models [7]. Besides the steepest descent methods for evaluating functional integrals the all-order perturbation theory results are also discussed using a particular way of resumming some special subsets of perturbation theory diagrams [8, 9].

At present the problem of evaluating the high-order perturbation theory contributions becomes a practical issue for high-precision tests of the standard model and new physics search as the accuracy of experimental data improves [10]. It is most important in perturbative QCD because the strong coupling constant α_s is numerically large. Since the perturbation theory expansion in α_s is, in general, asymptotic a resummation of all-order terms gives a possible way to improve the accuracy of theoretical predictions. An example of the infinite resummation of perturbation theory diagrams is an account for the Coulomb interaction for the processes of heavy quark production near the threshold [11] that allowed for an essential improvement in the description of top-antitop production [12]. Note that this resummation does not really include the strong coupling regime of QCD. For light quarks and massless gluons with a genuine strong interaction in the infrared domain there is no successful recipe of resumming the subsets of perturbation theory diagrams that could lead to the description of observables in terms of physical hadrons [13]. To deal with the region of strong coupling in the low-energy hadron phenomenology one exploits an idea of averaging over some energy range. It is assumed that the theoretical predictions for averaged quantities obtained with perturbation theory in the strong coupling constant in terms of quark-gluon degrees of freedom can be well confronted with experimental data measured in terms of observed hadrons. This assumption is known as duality concept (e.g. [14]). While the duality assumption is a real base for using perturbation theory

in the low-energy hadron phenomenology it is, however, difficult to quantitatively control the accuracy of this assumption in concrete applications. The most advanced quantitative study of the validity of duality concept is for two-point correlators of hadronic currents because of their simple analytic properties in momentum. The quality of the perturbation theory series for two-point correlators can be essentially improved by the renormalization group resummation that is an efficient tool of calculating various asymptotics of the Green's functions and related to the freedom of performing ultraviolet subtractions that leads to a possibility of redefinition of the coupling. The technical way to implement the renormalization group improvement of perturbation theory series is to use a running coupling normalized in the vicinity of a physical scale of the process in question. Such a choice of normalization for the coupling allows one to resum big logarithms related to the difference of scales in all orders of perturbation theory. Because of the final average over the energy interval as duality requires one has a choice whether the renormalization group improvement should be done before or after averaging. In general, these two operations – duality averaging and renormalization group improvement – do not commute. Performing renormalization group improvement before the final averaging allows one to resum a lot of regular corrections relevant to running only; one can consider this procedure as a determination of a proper scale for the averaged observables. The technique of renormalization group improvement for two-point correlators before the final averaging necessary for physical observables is known as the contour-improved perturbation theory and is especially important at low energies where the QCD coupling constant is large and higher order perturbation theory terms can be numerically important: they can change the results of finite-order perturbation theory by an amount comparable with experimental precision [15]. The precision of present experimental data on τ lepton decays, for instance, suffices for distinguishing the results of contour-improved and finite order perturbation theory [16].

An account for running in perturbation theory by using the renormalization group improvement under integration sign is close in spirit to the formulation of calculational scheme for the Green's functions within Schwinger-Dyson equations (skeleton expansion). Within Schwinger-Dyson formulation of perturbation theory one can use for the irreducible vertices which constitute the building blocks of the integral equations either finite-order perturbation theory or renormalization group improved one. The Schwinger-Dyson technique was intensively used for determination of the fermion propagator beyond the QCD perturbation theory approximation in relation to the problem of mass generation in massless theories and spontaneous symmetry breaking [17]. It is known that reiteration of running into loops can be infrared dangerous (just to have an idea what happens one can think of perturbation theory expansions in terms of a bare coupling in dimensional regularization and compare the results to the situation in superrenormalizable theories with a dimensional coupling constant). The reason is that the renormalization group summation is applicable to Green's functions at some values of momenta while the asymptotic behavior is determined by performing an analytic continuation which is sometimes implicit. Therefore, analytic properties of amplitudes in the whole complex

plane of momenta in finite-order perturbation theory can differ from those after renormalization group improvement. This difference of analytic properties can lead to some singularities when perturbative renormalization group running is extended to area where perturbation theory is not valid [18]. For asymptotically free QCD in the leading order of running the situation was discussed in [19].

In the present paper we discuss the resummation of effects of running on the example of two-point correlators of hadronic currents. The two-point correlators are about the simplest Green's functions and have well-established analytic properties in momenta. Two-point correlators are important for phenomenology, they are relevant for describing the processes of e^+e^- -annihilation into hadrons and/or τ -lepton hadron decays [20]. Note that the correlators of gauge invariant currents built from gluonic operators describe a spectrum of glueballs, the experimental observation of which would give a strong additional support for QCD as theory of hadrons. Gluonic current correlators is an actual choice for the analysis in the present paper.

We first discuss some generalities. The correlator of a hadronic current $j(x)$ has the form

$$i \int \langle T j(x) j^\dagger(0) \rangle e^{iqx} dx = \Pi(q^2) \quad (1)$$

where $\Pi(q^2)$ is an invariant scalar function. Analytic properties of the function $\Pi(q^2)$ in the variable q^2 are fixed by a dispersion relation (Källén - Lehmann, or spectral, representation)

$$\Pi(q^2) = \int \frac{\rho(s) ds}{s - q^2} + \text{subtractions} \quad (2)$$

where the spectral density $\rho(s)$ is determined by a sum over the states of the theory (e.g. [21]) and ultraviolet subtractions is a polynomial in q^2 . The spectrum of the correlator in eq. (1), or the support of the function $\rho(s)$ from eq. (2), is determined by singularities of the function $\Pi(q^2)$ in the complex q^2 plane. The spectral density $\rho(s)$ is then given by the discontinuity of the function $\Pi(q^2)$ across the spectrum

$$\rho(s) = \frac{1}{2\pi i} (\Pi(s + i0) - \Pi(s - i0)), \quad s \in [\text{spectrum}]. \quad (3)$$

In QCD with massless quarks and gluons a general assumption about the spectrum (spectrality condition) is $s \geq 0$ or $[\text{spectrum}] = [0, \infty]$. This assumption is based on the Fock representation for the states in terms of massless quarks and gluons (e.g. [21]). Note that this is an assumption and, in fact, analytic properties of $\Pi(q^2)$ and, therefore, the support of the spectral density $\rho(s)$ depend on interaction. The dependence of the spectrum on interaction can readily be seen in the example of heavy charged particles with Coulomb interaction. For a pair of heavy particles with masses m_1 and m_2 one would expect the spectrum start at the threshold $s_{\text{thr}} = (m_1 + m_2)^2$. However, if the Coulomb interaction is present it is true only for the repulsive interaction while the attractive interaction leads to the appearance of Coulombic poles below the threshold. In QCD the shape of the spectrum near the heavy quark threshold depends also on definition of the masses used to describe heavy quarks and other details of the interaction (e.g. theoretical

spectrum can be different at different orders of perturbation theory [22]). Such a situation is well known also from the analysis of simplified models [23]. Thus, the theoretical spectrum of a hadronic correlator is a dynamical quantity and constraints on the support of the spectral density coming from kinematical considerations based on values of masses of asymptotic states are not always valid in the full theory.

In asymptotically free QCD the function $\Pi(q^2)$ is computable theoretically in Euclidean domain (sufficiently far from the positive semiaxis $q^2 > 0$) that allows one to find theoretical predictions for observables. Still, to extract a theoretical prediction for the spectral density $\rho(s)$ from the function $\Pi(q^2)$ is not straightforward. The point is that $\Pi(q^2)$ is only known as a perturbation theory expansion at large Euclidean q^2 while $\rho(s)$ is given by a discontinuity across singularities of $\Pi(q^2)$ in the complex q^2 plane. However, the perturbation theory calculation of the function $\Pi(q^2)$ is not justified near its singularities. Therefore, the analytic continuation in the complex q^2 plane to the vicinity of positive semiaxis and into infrared region is necessary. The analytic continuation is an incorrectly set operation, i.e. small errors of the initial function $\Pi(q^2)$ at Euclidean points can produce large errors in $\rho(s)$. This instability is especially important for a theoretical evaluation of $\rho(s)$ at low energy. The problem of performing an analytic continuation can be also reformulated in the language of integral equations since the dispersion relation in eq. (2) gives the correlation function $\Pi(q^2)$ as an integral transformation of the spectral density $\rho(s)$. The relation (2) is a Fredholm integral equation of the second kind which is known to lead to an incorrectly set problem. Thus, the errors of $\rho(s)$ (as a solution of equation (2)) are not continuously related to the errors of $\Pi(q^2)$ (as initial data of equation (2)) and can be large even if errors of $\Pi(q^2)$ in Euclidean points are sufficiently small. The general procedure of constructing the approximate solutions to incorrectly set problems was suggested by Tikhonov and is known as regularization. Averaging the spectral density over a finite energy interval (sum rules) can be considered as a particular realization of Tikhonov's regularization procedure. One wants to theoretically study the function $\rho(s)$ at low energy because its experimental counterpart – the hadronic spectral density $\rho^{\text{had}}(s)$ – can be directly measured at low energy with high precision. Thus, while a pointwise description of the spectral density $\rho(s)$ at low energy is beyond perturbation theory, the appropriate quantities to analyze theoretically in perturbative QCD are the moments or integrals of $\rho(s)$ with a set of weight functions. This is a manifestation of the fact that the theoretical spectral density is, in general, a distribution rather than a continuous function of energy.

The moments of the spectral density $\rho(s)$ over a finite energy interval are defined by the relation

$$M_n = (n + 1) \int^{s_0} \rho(s) \frac{s^n ds}{(s_0)^{n+1}}. \quad (4)$$

The factor $(n + 1)$ in the definition of the moments is chosen to have all contributions of the leading order of perturbation theory uniformly normalized to unity. Equivalently one can say

that all measures defined on the interval $[0, s_0]$

$$(n+1) \frac{s^n}{s_0^{n+1}} ds = d \left(\frac{s}{s_0} \right)^{n+1} \quad (5)$$

are normalized to unity. Note that the accuracy of perturbation theory evaluation of a given moment depends on a particular weight function.

With the dispersion relation given in eq. (2) one can rewrite moments in eq. (4) as integrals over a contour in the complex q^2 plane [24]. For practical calculations of moments a convenient contour is a circle with the radius s_0 though the results are independent of the shape of the contour when it is deformed in the analyticity domain of the correlator. The contour representation of the moments reads

$$\begin{aligned} M_n &= (n+1) \frac{(-1)^n}{2\pi i} \oint_{|z|=s_0} \Pi(z) (z/s_0)^n dz/s_0 \\ &= (n+1) \frac{(-1)^n}{2\pi i} \oint_{|z|=1} \Pi(s_0 z) z^n dz \\ &= (n+1) \frac{(-1)^n}{2\pi} \int_{-\pi}^{\pi} \Pi(s_0 e^{i\varphi}) e^{i(n+1)\varphi} d\varphi. \end{aligned} \quad (6)$$

Note that the moments on the circle as given in eq. (6) are just Fourier coefficients of correlation functions that allows one to use a well-developed mathematical technique of Fourier analysis to study them.

Theoretical calculations of the moments are usually performed within operator product expansion (OPE) for the correlation function $\Pi(q^2)$ [25, 26, 27]. The OPE expression for the correlator contains a perturbation theory part and power corrections. The perturbation theory part can be further improved using renormalization group summation. In this paper we consider only the perturbation theory part of the theoretical correlator, or $\Pi(q^2)$ -function, for analyzing the moments. If the renormalization group improved $\Pi(q^2)$ is used under integration sign for the moments this means a resummation of the effects of running [15]. This technique was used for tau decay analysis [15, 28]. Power corrections within OPE – nonperturbative terms – appear by prescribing the nonvanishing vacuum expectation values to the local operators of higher dimensionality [27]. The contributions of these terms into the moments can be found with Cauchy theorem (e.g. [29]). At present the qualitative change in the phenomenology of sum rules is that high-order perturbation theory terms for hadronic correlators are known in various hadronic channels and the experimental value for the strong coupling constant is larger than that of the original papers therefore perturbation theory corrections are important numerically. It was already noticed that in some channels the perturbation theory corrections can numerically dominate over the power corrections that makes the study of perturbation theory corrections important for the present phenomenology [30].

As a concrete example we take a correlator of the gluonic current $G^2 = G_{\mu\nu}^a G_{\mu\nu}^a$ where $G_{\mu\nu}^a$ is a strength tensor of the gluon field. To the leading order of perturbation theory the

renormalization group invariant expression for the current can be chosen in the form

$$j_G = \alpha_s G^2 \quad (7)$$

where α_s is the strong coupling constant of QCD. This current is related to the trace of the QCD energy-momentum tensor θ_μ^μ and can serve as interpolating operators for glueballs. The full renormalization group invariant expression for θ_μ^μ in QCD with massless quarks is $(\beta(\alpha_s)/2\alpha_s)G^2$ where $\beta(\alpha_s)$ is the QCD β -function; this is not important for us in the following. The correlator reads

$$\frac{\pi^2}{2} i \int \langle T j_G(x) j_G^\dagger(0) \rangle e^{iqx} dx = q^4 \Pi_G(q^2). \quad (8)$$

Note that a kinematical factor q^4 is removed from the definition of the function $\Pi_G(q^2)$ which is justified within perturbation theory. The Adler's function

$$D_G(Q^2) = -Q^2 \frac{d}{dQ^2} \Pi_G(Q^2), \quad Q^2 = -q^2 \quad (9)$$

has a simple form at the leading order of perturbation theory

$$D_G(Q^2) = \alpha_s(Q^2)^2 (1 + O(\alpha_s)). \quad (10)$$

A theoretical prediction for the function $D_G(Q^2)$ has been calculated up to the third order of perturbation theory [31, 32]. Our main aim is to take into account the effects of running of the coupling for evaluating the moments of the spectral density, therefore, the introduction of an effective charge is convenient [33]. Indeed, high-order corrections can be accounted for by introducing the effective charge $\alpha_G(Q^2)$ in all orders of perturbation theory by the relation [31]

$$D_G(Q^2) = -Q^2 \frac{d}{dQ^2} \Pi_G(Q^2) = \alpha_G(Q^2)^2. \quad (11)$$

The effective strong coupling $\alpha_G(Q^2)$ obeys the renormalization group equation

$$Q^2 \frac{d}{dQ^2} \frac{\alpha_G(Q^2)}{\pi} = \beta \left(\frac{\alpha_G(Q^2)}{\pi} \right) \quad (12)$$

with

$$\beta(a) = -a^2 \left(\beta_0 + \beta_1 a + \beta_2^G a^2 + \beta_3^G a^3 \right) + O(a^6). \quad (13)$$

First two coefficients of the β -function are scheme independent, the higher order terms β_2^G, β_3^G depend on the effective charge definition in eq. (11). In QCD with n_f light quark flavors one has

$$\beta_0 = \frac{1}{4} \left(11 - \frac{2}{3} n_f \right). \quad (14)$$

For our purpose it suffices to use only the leading order running that contains all essential features of the whole phenomenon. Effects due to higher order corrections of the β -function are small and do not change the basic picture, slightly affecting the values of the moments

numerically [34]. Thus, we consider the leading order renormalization group equation for the effective charge

$$Q^2 \frac{d}{dQ^2} \frac{\alpha_G(Q^2)}{\pi} = -\beta_0 \left(\frac{\alpha_G(Q^2)}{\pi} \right)^2. \quad (15)$$

The renormalization group resummed correlation function reads

$$\Pi_G(Q^2) = \frac{\pi}{\beta_0} \alpha_G(Q^2) + \text{subtractions} \quad (16)$$

where

$$\alpha_G(Q^2) = \frac{\alpha_0}{1 + (\beta_0 \alpha_0 / \pi) \ln(Q^2/s_0)} \quad (17)$$

with $\alpha_0 = \alpha_G(s_0)$. Note that for the process of e^+e^- annihilation into hadrons the corresponding renormalization group resummed correlation function reads

$$\Pi_{e^+e^-}(Q^2) = \ln\left(\frac{\mu^2}{Q^2}\right) + \frac{1}{\beta_0} \ln\left(\frac{\alpha_{e^+e^-}(Q^2)}{\pi}\right) + \text{subtractions} \quad (18)$$

with the first term being a parton contribution independent of α_s . Setting $Q^2 = s_0 e^{i\varphi}$ on the contour one obtains an explicit expression for the correlator as a function of the angle φ

$$\Pi_G(s_0 e^{i\varphi}) = \frac{\pi}{\beta_0} \frac{\alpha_0}{1 + i\beta_0 \alpha_0 \varphi / \pi} + \text{subtractions} \quad (19)$$

With an explicit expression for the function $\Pi_G(z)$ from eqs. (16,19) the analysis of the moments M_n is straightforward. The explicit expression for the moments written through the contour representation reads

$$M_n = (n+1) \frac{(-1)^n}{2\pi} \int_{-\pi}^{\pi} \frac{\pi}{\beta_0} \frac{\alpha_0}{1 + i\beta_0 \alpha_0 \varphi / \pi} e^{i(n+1)\varphi} d\varphi. \quad (20)$$

Eq. (20) is a basic relation for further study. Note that the form of the representation in eq. (20) is rather general and gives a basis for other applications: higher powers of the running coupling α_s can be easily generated.

Let us discuss the above expressions in some detail. The main feature of the contour representation for the moments is that everything is explicit as it is in finite-order perturbation theory. After formulating the particular way of resummation for the moments, i.e. by defining them on the contour, there is no ambiguity in these quantities (they are not given by series in α_s but by close formulae). Therefore, the moments are explicit functions of α_0 that can be rigorously studied. One should, however, remember that a particular definition of the moments on the contour has been used.

Expanding eq. (20) in α_0 one reproduces all results of finite-order perturbation theory (e.g. [35]). Indeed, expanding the function $\Pi_G(Q^2)$ from eq. (16) one finds

$$\Pi_G(Q^2) = \frac{\pi}{\beta_0} \alpha_0 \left\{ 1 + \beta_0 \frac{\alpha_0}{\pi} \ln\left(\frac{s_0}{Q^2}\right) + \beta_0^2 \left(\frac{\alpha_0}{\pi}\right)^2 \ln^2\left(\frac{s_0}{Q^2}\right) + O(\alpha_0^3) \right\} + \text{subtractions}. \quad (21)$$

The first term (Q^2 independent) can be added to subtractions. Then, finally one has

$$\Pi_G(Q^2) = \alpha_0^2 \ln\left(\frac{s_0}{Q^2}\right) + \beta_0 \frac{\alpha_0^3}{\pi} \ln^2\left(\frac{s_0}{Q^2}\right) + O(\alpha_0^4) + \text{subtractions}. \quad (22)$$

While the expansion of the integrand in eq. (20) in α_0 with further integration gives nothing new in comparison with the finite-order perturbation theory analysis, new features appear if one retains a resummed expression under integration sign.

The moments in eq. (20) are expandable in a convergent series in α_0 for $\beta_0\alpha_0 < 1$. The existence of a finite radius of convergence in the complex α_0 plane within the contour technique of resummation for the moments is a general feature that persists for the running with the high-order perturbative β -function. However, in QCD the convergence radius in α_0 decreases when higher orders of the β -function are included [34]. Thus, the explicit result eq. (20) allows for an analytic continuation in the complex α_0 plane leading to the functions $M_n(\alpha_0)$ which are analytic in α_0 at the origin, i.e. near the point $\alpha_0 = 0$. This sounds a bit unusual as one implicitly assumes that perturbation theory objects should have an essential singularity in α_0 at the origin usually a cut along the negative semiaxis (e.g. [36]). Note that the moments of the heavy quark production with infinite resummation of Coulomb interaction are also given by convergent series in α_s (the explicit result at the leading order of perturbation theory is presented in [37]). The exact expression given in eq. (20) without expansion in α_0 provides one with an analytic continuation beyond the convergence radius even when α_0 lies outside the convergence circle.

Looking at eqs. (2,3,16,17,19) one notices that analytic properties in the variable q^2 declared for a general function $\Pi(q^2)$ built from massless fields in eqs. (2,3) differ from that of the explicit result given in eqs. (16,17,19): the explicit renormalization group improved expression $\Pi_G(q^2)$ has a pole in the Euclidean region of q^2 which is supposed to be the analyticity region from general assumptions about the spectrum. This is an important feature to notice: a concrete approximation $\Pi_G(q^2)$ in eq. (16) has different analytic properties in the whole complex q^2 plane than it is declared by general requirements. Contrary to the resummed expression given in eq. (16), at any finite order of perturbation theory given in eq. (21) one has only powers of logarithms that have correct analytic properties in the variable q^2 – a cut along the positive semiaxis. It is just a consistency feature – finite-order perturbation theory is a (trivial) example of the model of quantum field theory where all general requirements are valid. Thus, the renormalization group resummation for the hadronic correlator in asymptotically free QCD can change its analytic structure in the infrared region as compared to the finite-order perturbation theory approximation. In the leading order of the running in QCD a (Landau) pole is usually generated. This pole is included into the definition of the moments in eqs. (6,20) because one encircles the origin with a large contour. There is no other possibility to work consistently in perturbation theory because the infrared region is completely nonperturbative and one is not allowed to move the integration contour to that region. The requirement of integrating only along the positive axis is an external constraint on the theory rather than its attribute.

It cannot be realized in perturbation theory – the integration contour should go sufficiently far from the infrared region which is a requirement of the applicability of perturbation theory approximations. Note that if s_0 is not large enough in order the circular contour includes all infrared singularities the contour should be deformed to do so. To give the results for the moments that are justified in perturbation theory (at least formally) the integration contour should be chosen such that no singularity incompatible with general requirements lies outside it in the complex q^2 plane.

After defining the moments properly (written as eq. (20), for instance) the practical calculation of explicit functions $M_n(\alpha_0)$ can be done in different ways. Technically, one can shrink the integration contour back to singularities of $\Pi_G(q^2)$ which is a uniquely defined mathematical operation for the explicit function $\Pi_G(q^2)$ in the complex q^2 plane. Then one discovers a pole which is a pure computational fact without any meaning for the structure of the perturbation theory at high orders. The perturbation theory moments are constructed at high energies and cannot decipher the point-by-point structure of the spectrum in the infrared region (or singularities of $\Pi_G(q^2)$ at small q^2) – they just give a contribution from this region as it is seen from large energies (on the contour). If a high-order β -function is used for the renormalization group improvement of the correlator then the structure of singularities in the infrared region can drastically change [38]. However, this has little effect on the moments – they develop some small perturbation theory corrections independent of a particular structure of the correlation function in the infrared region obtained as a perturbation theory approximation. Of course, the parameter s_0 should be sufficiently large in order the perturbation theory expansion in the coupling α_0 would be justified. A discussion of the pointwise behavior of $\Pi(q^2)$ in the infrared region is beyond the scope of perturbation theory. Note that the possibility to restore moments as exact functions of the coupling from their (asymptotic) perturbation theory series depends on the behavior of $\Pi(q^2)$ in the infrared region.

Still some convenient representations of eq. (20) are worth studying for practical calculations. Let us first consider the leading order moment $M_0(\alpha_0)$ that reads

$$M_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\pi}{\beta_0} \frac{\alpha_0}{1 + i\beta_0\alpha_0\varphi/\pi} e^{i\varphi} d\varphi = \frac{\alpha_0}{2\beta_0} \int_{-\pi}^{\pi} \frac{e^{i\varphi} d\varphi}{1 + i\beta_0\alpha_0\varphi/\pi}. \quad (23)$$

With the expression (23) given one can work out an efficient computation technique for it. In applications the moments are usually computed numerically [15, 28, 34]. For a general analysis one can consider also analytical computations of the moments in various limits. Integrals of the type (23) are related to the exponential integral function and have been well studied [39]. Formally one can use a convergent series in α_0 but if an experimental value of α_0 is larger than the convergence radius then the expansion in α_0 is of no use and an analytic continuation of the function given by the series in α_0 beyond the convergence radius is necessary. Let us look at this issue closer. The convergence radius of the function $M_0(\alpha_0)$ in the complex α_0 plane for the leading order β -function is given by $|\alpha_0| < 1/\beta_0$. For a full perturbative β -function up to the fourth order in the $\overline{\text{MS}}$ scheme it is smaller [34]. In a realistic case of τ decays, for instance,

$s_0 = M_\tau^2 = (1.777 \text{ GeV})^2$ and $\beta_0 = 9/4$ that leads to

$$\alpha_0 \equiv \alpha_0(s_0 = M_\tau^2 = (1.777 \text{ GeV})^2) < \frac{4}{9} = 0.44\dots \quad (24)$$

i.e. the experimental value of the coupling $\alpha_0 \approx 0.3$ [40] lies rather close to the boundary of the convergence circle. Taking the scale s_0 for the moments smaller than the squared τ lepton mass M_τ^2 one can get the value of α_0 lying outside the convergence circle. The convergent power series may not be the best way of computing the moments for such numerical values of the coupling constant. The more efficient approximation can be obtained by constructing an asymptotic expansion for the zero moment. Integrating by parts one finds

$$M_0 = \frac{\alpha_0^2}{1 + \beta_0^2 \alpha_0^2} + \frac{\alpha_0^2}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\varphi} d\varphi}{(1 + i\beta_0 \alpha_0 \varphi/\pi)^2}. \quad (25)$$

Here the first term gives a perturbation theory expression for the spectral density at s_0 with all corrections due to analytic continuation resummed (so called π^2 corrections) [41]. This contribution can be obtained from the leading order running by retaining the highest power of π at every order of perturbation theory. It also corresponds to the calculation of the moments on the cut through the boundary value of the perturbation theory spectrum [35]. Further integration by parts gives

$$\begin{aligned} M_0 &= \frac{\alpha_0^2}{1 + \beta_0^2 \alpha_0^2} + \frac{\alpha_0^2}{\pi} \sum_{j=2}^n (j-1)! \left(\frac{\beta_0 \alpha_0}{\pi} \right)^{j-2} \frac{\sin\{j \arctan(\beta_0 \alpha_0)\}}{(1 + \beta_0^2 \alpha_0^2)^{j/2}} \\ &+ n! \frac{\alpha_0^2}{2\pi} \left(\frac{\beta_0 \alpha_0}{\pi} \right)^{n-1} \int_{-\pi}^{\pi} \frac{e^{i\varphi} d\varphi}{(1 + i\beta_0 \alpha_0 \varphi/\pi)^{n+1}}. \end{aligned} \quad (26)$$

This result can be obtained using the recurrence relation

$$\frac{1}{2} \int_{-\pi}^{\pi} \frac{e^{i\varphi} d\varphi}{(1 + i\beta_0 \alpha_0 \varphi/\pi)^k} = \frac{\sin(k\chi)}{r^k} + k \left(\frac{\beta_0 \alpha_0}{\pi} \right) \frac{1}{2} \int_{-\pi}^{\pi} \frac{e^{i\varphi} d\varphi}{(1 + i\beta_0 \alpha_0 \varphi/\pi)^{k+1}} \quad (27)$$

with quantities r and χ defined by

$$1 + i\beta_0 \alpha_0 = r e^{i\chi}, \quad r = \sqrt{1 + \beta_0^2 \alpha_0^2}, \quad \chi = \arctan(\beta_0 \alpha_0). \quad (28)$$

Retaining only leading powers of α_0 at every order of the expansion (26) one recovers an asymptotic series often discussed in the literature. Indeed, taking only the leading asymptotics of every term in eq. (26) one finds

$$M_0^{\text{leading asym}} = \alpha_0^2 \left(1 + 2\beta_0 \frac{\alpha_0}{\pi} + \dots + (n+1)! \beta_0^n \left(\frac{\alpha_0}{\pi} \right)^n \right) + O(\alpha_0^{n+3}). \quad (29)$$

The expansion (29) shows a nonalternating factorial growth of the coefficients that leads to a Borel nonsummable asymptotic series [42]. The approximation (29) for the expansion (26) is not accurate. Note that Borel summation (with some recipe for treating nonsummable

singularities) of the leading asymptotics (29) cannot restore the exact function (23) from some general principles.

The representation (26) gives an efficient way to numerically compute the result (23), it represents an asymptotic expansion of the function $M_0(\alpha_0)$ which is analytic at the origin $\alpha_0 = 0$. It is known that an asymptotic expansion of a function can be more efficient for its numerical evaluation than a convergent series even inside the convergence circle, it also gives an efficient way for the calculation outside the convergence circle (not too far though). One can see that the result (26) is an efficient asymptotic expansion which can give a better accuracy than a direct power series expansion in α_0 for some α_0 and n . Note that when the analytic structure of the function is known, or a concise expression for the function is given as in eq. (23), the asymptotic expansions which converge fast for the first few terms are more useful for practical calculations than formal convergent series that require many terms for getting a reasonable numerical accuracy [43]. Still one is left with the residual term which is represented by the integral in eq. (26). In practical calculations one can simply neglect it. However, in some cases one can do better than that. By extending the integration range in the variable φ from $-\infty$ to $+\infty$ the integral over φ can be readily computed

$$n! \left(\frac{\beta_0 \alpha_0}{\pi} \right)^{n-1} \int_{-\infty}^{\infty} \frac{e^{i\varphi} d\varphi}{(1 + i\beta_0 \alpha_0 \varphi / \pi)^{n+1}} = 2\pi \left(\frac{\pi}{\beta_0 \alpha_0} \right)^2 e^{-\frac{\pi}{\beta_0 \alpha_0}}. \quad (30)$$

Using the decomposition

$$\int_{-\pi}^{\pi} d\varphi = \int_{-\infty}^{\infty} d\varphi - \left(\int_{-\infty}^{-\pi} d\varphi + \int_{\pi}^{\infty} d\varphi \right) \quad (31)$$

one can write

$$\begin{aligned} & n! \left(\frac{\beta_0 \alpha_0}{\pi} \right)^{n-1} \int_{-\pi}^{\pi} \frac{e^{i\varphi} d\varphi}{(1 + i\beta_0 \alpha_0 \varphi / \pi)^{n+1}} \\ &= 2\pi \left(\frac{\pi}{\beta_0 \alpha_0} \right)^2 e^{-\frac{\pi}{\beta_0 \alpha_0}} - n! \left(\frac{\beta_0 \alpha_0}{\pi} \right)^{n-1} \left(\int_{-\infty}^{-\pi} + \int_{\pi}^{\infty} \right) \frac{e^{i\varphi} d\varphi}{(1 + i\beta_0 \alpha_0 \varphi / \pi)^{n+1}} \end{aligned} \quad (32)$$

for any n . Therefore, the residual term in eq. (26) is transformed into a sum of an explicit nonperturbative term proportional to $e^{-\pi/\beta_0 \alpha_0}$ and the term which can be smaller than the original residual term for some values of α_0 and n . One has

$$\begin{aligned} M_0 &= \left(\frac{\pi}{\beta_0} \right)^2 e^{-\frac{\pi}{\beta_0 \alpha_0}} + \frac{\alpha_0^2}{1 + \beta_0^2 \alpha_0^2} + \frac{\alpha_0^2}{\pi} \sum_{j=2}^n (j-1)! \left(\frac{\beta_0 \alpha_0}{\pi} \right)^{j-2} \frac{\sin\{j \arctan(\beta_0 \alpha_0)\}}{(1 + \beta_0^2 \alpha_0^2)^{j/2}} \\ &\quad - n! \frac{\alpha_0^2}{2\pi} \left(\frac{\beta_0 \alpha_0}{\pi} \right)^{n-1} \left(\int_{-\infty}^{-\pi} + \int_{\pi}^{\infty} \right) \frac{e^{i\varphi} d\varphi}{(1 + i\beta_0 \alpha_0 \varphi / \pi)^{n+1}}. \end{aligned} \quad (33)$$

The explicit nonperturbative term $e^{-\pi/\beta_0 \alpha_0}$ has appeared in the asymptotic expansion of the moment (23) written in the form of eq. (33). The difference between the expansions in eq. (26) and eq. (33) is not very noticeable, in fact, they are almost identical up to the residual terms.

What happened is the change of the residual term. Therefore, the choice of the representation for the moment (23) (eq. (26) or eq. (33)), i.e. with or without the explicit nonperturbative term $e^{-\pi/\beta_0\alpha_0}$, is a question of the choice of a particular form of the residual term. It can happen that after dropping the residual term (which is a common practice in asymptotic series calculations) the representation in the form of eq. (33) is more accurate numerically than that in the form of eq. (26) for some particular values of α_0 and n . However, a quantitative conclusion about the accuracy of the asymptotic series representation for a function can only be drawn if one has a concise expression for the function as eq. (23) in our case when the explicit form of the residual term is also known (see also ref. [44] where a simplified model in quantum mechanics was considered). Any conclusions based on the terms of the series itself (for instance, based on the representation (29)) can be not accurate numerically; they can also be unjustified in a general sense of analytic behavior as one can see from eq. (32).

The above results are valid for any moment M_l . Namely, the recurrence relation can be generalized to read

$$(l+1)\frac{1}{2}\int_{-\pi}^{\pi}\frac{e^{i(l+1)\varphi}d\varphi}{(1+i\beta_0\alpha_0\varphi/\pi)^k}=\frac{\sin\{k(l+1)\chi\}}{r^k}+k\left(\frac{\beta_0\alpha_0}{\pi}\right)\frac{1}{2}\int_{-\pi}^{\pi}\frac{e^{i(l+1)\varphi}d\varphi}{(1+i\beta_0\alpha_0\varphi/\pi)^{k+1}}. \quad (34)$$

The representation with integration by parts analogous to one given in eq. (26) shows an improvement in the convergence for large l moments equivalent to the replacement $\alpha_0 \rightarrow \alpha_0/l$. This agrees with conclusions drawn from the analysis of finite-order perturbation theory [35]. In general, one can also modify the residual term for any moment M_n . In the literature there are also some moments defined on the final energy interval with different weight functions [45]; our conclusion can be generalized to that moments as well.

Now we discuss the spectrum of the explicit resummed function $\Pi_G(q^2)$. The structure of the spectrum in the infrared domain is most interesting. Note that this part of the spectrum is obtained by the analytic continuation from the Euclidean region where $\Pi_G(q^2)$ is calculated as a perturbation theory expansion to a region where perturbation theory is not valid that means that the structure of the spectrum has no general physical meaning at small s pointwise. The spectrum of the explicit function $\Pi_G(q^2)$ given in eq. (16) is a well-defined mathematical quantity. It is straightforward to calculate it. Using the expression for the leading order coupling constant in the form

$$\alpha_G(Q^2)=\frac{\alpha_0}{1+(\beta_0\alpha_0/\pi)\ln(Q^2/s_0)}=\frac{\pi}{\beta_0\ln(Q^2/\Lambda_G^2)} \quad (35)$$

where

$$\Lambda_G^2=s_0\exp\left(-\frac{\pi}{\beta_0\alpha_0}\right) \quad (36)$$

one finds

$$\Pi_G(Q^2)=\frac{\pi}{\beta_0}\alpha_G(Q^2)+\text{subtractions}=\frac{\pi^2}{\beta_0^2\ln(Q^2/\Lambda_G^2)}+\text{subtractions}. \quad (37)$$

Therefore, the spectrum (a discontinuity across singularities) reads

$$\rho_G(s) = \frac{1}{2\pi i}(\Pi_G(s+i0) - \Pi_G(s-i0)) = \frac{\pi^2}{\beta_0^2} \left(\Lambda_G^2 \delta(\Lambda_G^2 + s) + \theta(s) \frac{1}{\pi^2 + \ln^2(s/\Lambda_G^2)} \right) \quad (38)$$

where $\delta(s)$ is a Dirac δ -distribution and $\theta(s)$ is a step-distribution. Explicit functions given in eqs. (37,38) satisfy integral equation (2). Note that the explicit spectrum in eq. (38) contains a contribution $\delta(\Lambda_G^2 + s)$ corresponding to a pole $1/(q^2 + \Lambda_G^2)$ of the function $\Pi_G(q^2)$ in the region $q^2 < 0$ which is supposed to be the analyticity domain of the two-point correlators from general requirements. The position of the pole Λ_G^2 is specific for a given channel if an effective charge is used. The expression for the theoretical spectrum given in eq. (38) can be used in a mathematical sense for calculating integrals (moments) in eq. (20) (an analogous approach may be used for the general case in eq. (4)) but a physical interpretation of the spectrum at small s is rather meaningless because perturbation theory is not applicable at small momenta.

The part of the spectrum on the positive real axis is a discontinuity of the function $\Pi_G(q^2)$ across the cut [41]. It can be conveniently written in the form

$$\rho_G^{cont}(s) = \frac{\pi^2}{\beta_0^2} \frac{1}{\pi^2 + \ln^2(s/\Lambda^2)} = \frac{\alpha(s)^2}{1 + \beta_0^2 \alpha(s)^2} \quad (39)$$

with a function

$$\alpha(s) = \frac{\pi}{\beta_0 \ln(s/\Lambda_G^2)}. \quad (40)$$

Note that the function $\alpha(s)$ has a pole on the physical cut at $s = \Lambda_G^2$. This is this pole that leads to problems of Borel nonsummability in the resummation of the effects of running directly on the cut when one integrates through the infrared region (cf. eq. (29)). However, the pole of the auxiliary function $\alpha(s)$ from eq. (40) has no physical meaning within perturbation theory. For instance, the spectral density (39) is a smooth function at this point. While the spectral density explicitly given in eq. (38) allows one to compute the moments by the direct integration in a pure mathematical sense it is not productive to ask whether this spectrum is physical or not because there is no possibility to answer this question within perturbation theory. Interpretations of this spectrum at low energies like specific recipes of resummation for Borel non-summable series (as in eq. (29), for instance, are additional assumptions beyond perturbation theory.

The continuous part of the spectral density in eq. (39) can be obtained uniquely from the finite-order perturbation theory expansion. However, the pole remains hidden and cannot be restored from the summation on the cut if only the discontinuity across the cut along the positive semiaxis is considered. Note that this is also a situation in heavy quark physics – no Coulombic poles can be restored from the summation on the cut (see discussion in [46] in relation to the precision determination of heavy quark masses).

It is worth stressing again that the moments in eq. (20) are analytic functions of the coupling α_0 at the origin. It means that the nonanalytic piece in eq. (32) cancels the corresponding part

in the residual term. Depending on a particular form of the residual term the formal analytic structure of the expansion for the moments in the coupling constant α_0 drastically changes. This demonstrates a danger of making conclusions about infrared power corrections emerging from the extrapolation of the running to the infrared region. Because the infrared region is not the perturbation theory domain the formal perturbative expansions originating from the integration over the infrared region can strongly be modified by making small changes in perturbation theory quantities like effective β -functions [38]. In practice, or from a phenomenological point of view, the use of power corrections stemming from the infrared modification of perturbation theory is difficult to appreciate if high order terms in the perturbative α_0 expansion are taken into account. For such observables as the moments of the spectral density one cannot distinguish numerically high-order perturbation theory corrections from power corrections (nonperturbative part of the expansion): the power corrections are numerically hidden by the high-order perturbation theory corrections.

Thus, for the observables related to two-point correlators the problem of resumming the running effects in perturbation theory is solved by the contour integration. We stress that the pole (or any singularity that may occur upon the formal analytic continuation of the perturbation theory expressions into the infrared region) is inside the integration circle (cf. the discussion in ref. [47]). One is not allowed to use integration contours that go close to the origin because this region is completely nonperturbative and should be avoided: perturbation theory cannot decipher the structure of amplitudes in this region pointwise, only contributions to the integrals are perturbative and can be computed. This situation is to some extent analogous to the situation with Coulombic poles especially for not very heavy quarks. For perturbation theory applications any type of infrared singularity should be avoided by moving the integration contour far from the origin and keeping infrared nonperturbative region inside, thereby including also the contribution of this region into the integral. The possibility to accurately apply perturbation theory for averaged quantities is a specific feature of two-point correlators with simple analytic properties in the momentum variable. In the cases when observables are obtained by the averaging of more complicated Green's functions where the analytic structure is not transparent the effects of running are accounted for by considering a model field theory with a one-loop gluon propagator reiterated in all orders of perturbation theory. To respect gauge invariance in QCD in such a model the technique of naive nonabelianization is used [48]. Note that in pure gluodynamics which is a proper theoretical model for studying glueballs this trick is not straightforward. If analytic properties of the amplitude are unknown one has no clear way to avoid going through infrared singularities of the running coupling and one is trying to perform the integration across the infrared region directly (as in applications of infrared renormalons [49]). In this case an infrared structure of the running is important for the analysis, however, it is completely nonperturbative. Therefore, the obtained results depend on additional assumptions about the infrared behavior of the running coupling.

As a last remark we give an expression for the resummed function $\Pi_G(q^2)$ in the second

order of the β -function. Taking the approximation for the β -function in the form

$$\beta(a) \equiv \beta_2(a) = -a^2 (\beta_0 + \beta_1 a) \quad (41)$$

one finds the expression for the resummed function $\Pi_G(q^2)$:

$$\Pi_G(Q^2) = \frac{\pi^2}{\beta_1} \ln \left(\beta_0 + \beta_1 \frac{\alpha_G^{(2)}(Q^2)}{\pi} \right) + \text{subtractions} \quad (42)$$

where the function $\alpha_G^{(2)}(Q^2)$ is a solution of the renormalization group equation with the second order β -function

$$Q^2 \frac{d}{dQ^2} \left(\frac{\alpha_G^{(2)}(Q^2)}{\pi} \right) = -\beta_0 \left(\frac{\alpha_G^{(2)}(Q^2)}{\pi} \right)^2 - \beta_1 \left(\frac{\alpha_G^{(2)}(Q^2)}{\pi} \right)^3. \quad (43)$$

The generalization of our analysis to this case is straightforward.

To conclude, it has been shown that for the observables related to two-point current correlators the summation of the effects of running can be done in perturbation theory. In more complicated cases without simple analytic structure of the respective Green's functions the interpretation of running in the infrared region is not unique and is outside the scope of perturbation theory. The asymptotic structure of the perturbation theory series depends on the actual treatment of the observables (there is no true asymptotic structure unless explicit assumptions are formulated). The series can be analytic at the origin for some approximations as it is for the widely used approximation with resummation on the contour. Possible power corrections stemming from such resummation are of rather computational origin and simply reflect a particular way of approximating the relevant integrals; no general conclusions on the analytic structure of the exact theory can be drawn. Theoretically, there is no invariant meaning in splitting the results into nonperturbative infrared power corrections and perturbation theory part (as opposite to OPE where power corrections are related to high-dimension operators and determined by the projections onto other perturbation theory states than the vacuum). Phenomenologically, the high-order perturbation theory terms (with high powers of inverse logarithms) can numerically mimic the renormalon-type power corrections well. In this situation, the way to go beyond perturbation theory for improving the accuracy of theoretical formulae would be just a convention to use for the observables an effective scheme where all perturbative corrections are explicitly resummed into the redefinition of the coupling.

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