

Speed-up of neutrino transformations in a supernova environment.

R. F. Sawyer¹

¹*Department of Physics, University of California at Santa Barbara, Santa Barbara, California 93106*

When the neutral current neutrino-neutrino interaction is treated completely, rather than as an interaction among angle-averaged distributions, or as a set of flavor-diagonal effective potentials, the result can be flavor mixing at a speed orders of magnitude faster than that one would anticipate from the measured neutrino oscillation parameters. It is possible that the energy spectra of the three active species of neutrinos emerging from a supernova are nearly identical.

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INTRODUCTION

Predictions regarding the neutrino flavor physics in the region of the supernova's neutrino-sphere have remained quite stable as treatments of other issues have evolved over time. These predictions, which involve significantly different energy spectra for the ν_e , $\bar{\nu}_e$, $\nu_{\mu,\tau}$, $\bar{\nu}_{\mu,\tau}$, are important to the theory of the region above the neutrino-sphere. They matter for the explosion dynamics, for heavy element nucleosynthesis, and for analyzing the neutrino burst from the galactic supernova scheduled for some time in the twenty-second century.

As the three known species of neutrinos move through layers of the supernova that are immediately under the effective neutrino-sphere, their physics can be much more complex than is assumed in current simulations of the whole supernova process. Before describing some of these complexities in detail, we look at parameters that describe the region: taking a density of 10^{11}gc^{-3} and temperature of 7 MeV. There is a characteristic energy that we define from the Fermi constant and the neutrino number density, $E_s \equiv 2\sqrt{2}G_F n_\nu$, where n_ν is the neutrino number density. which when translated into distance, under the above conditions, gives $E_s^{-1} = .04\text{cm}^{-1}$. Of course, this is many orders of magnitude less than the distance set by the density and cross-section (proportional to G_F^{-2}), a distance scale that is tens of km., the scale of the neutrino-sphere radius itself.

However, terms of order E_s do enter in forward (or "index of refraction") effects. In a medium containing electrons, the ν - e interaction energies create different potentials for ν_e 's, and for $\bar{\nu}_e$'s, than for ν_μ 's and ν_τ 's and their antiparticles. The induced terms of order E_s enter interesting physics in conjunction with the (δm^2) neutrino oscillation parameters, but these oscillation effects do not enter significantly out to a distance of many times the neutrino-sphere radius, if we use current values for the neutrino mass parameters and operate in the commonly accepted theoretical framework. The electron densities

have no effect on the oscillation between μ_ν and μ_τ , with oscillation length of the order of a kilometer for a 20 MeV neutrino, and uninteresting in any case by virtue of the near identities of their spectra and angular distributions. The ν_e oscillation length in vacuum is longer by a factor of 40 or so; furthermore, the large electron density ensures that oscillation amplitudes will be miniscule. Thus there is negligible "forward" neutrino physics of any consequence in the conventional picture of this region, barring a sterile neutrino that is attached to a much bigger mixing parameter.

All of this can change when we include the full neutral current interaction among the neutrinos, however. The effective forward interaction Hamiltonian for the system now contains terms that can exchange neutrino flavors. At first sight this seems fairly irrelevant in the region below the neutrino-sphere. The initial system coming from the supernova core can be taken as an incoherent sum of flavor eigenstates. Then the mixing of the ν_e remains small, according to the usual angle-averaged equations, which assume isotropic momentum distributions. A large, nearly flavor diagonal, surrounding neutrino density can serve to synchronize [1]-[8] these small oscillations, according to these equations, but not to enhance their amplitude when the amplitudes otherwise would be small.

In a region well above the neutrino-sphere the oscillation parameters can produce strong mixing and possible MSW transformations. Considerable attention has been given to the effects of the neutralcurrent coupling terms, including the forward flavor-exchange parts, in this region [9]-[12]. Here neutrino anisotropy was taken into account by using the isotropic form for the equations, but with an effective coupling constant that incorporates a single ("flux-averaged") angle of incidence for ν - ν encounters. However we now know that other non-linear phenomena can take over when anisotropy is incorporated into the evolution equations in a more correct way [13]. These phenomena, to some degree, can see the time scale $t_s = E_s^{-1}$ and overturn our intuitions about the problem.

In the present paper we look at the neutrino physics only over short distances, over which the surrounding

¹ We use units in which $\hbar = 1$, $c = 1$

density and temperature can be taken to be constant. Even this abstracted problem is extraordinarily complex. The simulations in [13], backed up by analytic considerations, were based on two-flavor models with simple anisotropies, essentially a few clusters of definite direction in momentum space. They show the possibility of one kind of instability, in the form of an exponentially increasing growth in flavor-mixing amplitudes seeded by the tiniest (coherent) initial mixing (which could come from earlier neutrino oscillation effects). In these examples the rise-time of the exponential is the shortest time scale in the problem, of order E_s^{-1} . This behavior is in contrast to the oscillatory behavior in the isotropic approximation. That said, in the domains considered in ref.[13] the conditions that can lead to this runaway evolution are delicately dependent on the initial flavor-momentum correlations, and are subject to change when new elements are added to the model.

Approaching an authoritative answer in a more realistic supernova environment requires the inclusion of all three active flavors and their antiparticles.² Let us think of the neutrino distributions in the transitional region just below the average depth of the last ν_μ and ν_τ scatterings, a region that is somewhat farther below the average depth of the last ν_e scattering. In this region the ν_e momentum distribution is fairly isotropic, whereas the $\nu_\mu, \nu_\tau, \bar{\nu}_\mu, \bar{\nu}_\tau$ distributions are already quite biased toward upward momenta. Both in angular distribution and in spectrum, the $\bar{\nu}_e$'s lie between the ν_e 's and the $\nu_{\mu,\tau}$'s. In our first idealization we picture this mixture as a core group of all six species in equal number with a momentum distribution peaked somewhat upwards, superimposed on a group in which we have $\nu_{\mu,\tau}$'s and their anti-particles in a distribution peaked strongly upward, and $\nu_e, \bar{\nu}_e$'s in a distribution peaked downward. We call the second group the valence group. The core group is irrelevant to the flavor evolution of the system, as may be obvious but in any case is supported by our later formalism. The problem that we now solve is one in which we replace the valence system by narrow angle beams of ν_μ and ν_τ neutrinos and antineutrinos directed outward, and beams of ν_e 's and $\bar{\nu}_e$'s directed inwards. In a second version we shall assign the $\bar{\nu}_e$'s to the outgoing group, corresponding to a less outwardly biased core group.

Over the time-scales that we consider below, the interactions that come into play, in addition to the neutral current couplings, are the neutrino mass-mixing terms and the difference between the forward $e-\nu_e$ and $e-\nu_{\mu,\tau}$ scattering amplitudes. These interactions would not even matter over our time scales, but for the instabilities in the non-linear equations that govern the evolution of the

system. The first initial state that we pick consists of down-moving ν_e 's and $\bar{\nu}_e$'s, each with a number density of n_ν , with the other four species of particles moving upward each with the same number density. The outcome, in this case and in two other cases with somewhat different initial conditions, is a scenario of extremely rapid mixing of flavors, leading to a rapid destruction of some of the initial correlations of flavor with angle, and in turn with energy spectrum.

EQUATIONS OF EVOLUTION

The equations are an extension of the equations of Sigl and Raffelt [14]. The main new element is the explicit inclusion of all three flavors of active neutrinos and antineutrinos, as well as perhaps a more incisive inclusion of angular factors, so that we can better treat anisotropy effects. The formalism is much simpler than that presented in ref.[14] because of our limitation to purely forward effects, justified by the short time scales to be considered.

All of the dynamics of the system can be described in terms of the annihilation and creation operators for the three neutrino flavors and the momentum states of the initial configuration; we define the annihilators of the mode p for the respective three flavors ν_e, ν_τ, ν_μ as a_p, b_p, c_p with the annihilators for the antiparticles denoted $\bar{a}_p, \bar{b}_p, \bar{c}_p$. The neutral-current effective interaction Hamiltonian, to be exhibited below, is independent of the neutrino energies and depends only on relative directions of motion. Therefore, for the purpose of discussing the evolution of the density matrix due to this interaction alone, we can bundle all energies together and define collective operators for each angle. We divide the solid angle of the momentum space into elements $d\Omega$ centered at angle Ω and define collective operators for each of these elements using a shorthand symbol S_Ω which operates according to the rule,

$$S_\Omega[a^\dagger b] = (d\Omega)^{-1} \sum_{p \subset d\Omega} a_p^\dagger b_p. \quad (1)$$

We choose,

$$\rho_1(\Omega) = S_\Omega[a^\dagger b] \quad , \quad \rho_2(\Omega) = S_\Omega[b^\dagger a] \quad ,$$

$$\rho_3(\Omega) = S_\Omega[(2c^\dagger c - a^\dagger a - b^\dagger b)/3] \quad ,$$

$$\rho_4(\Omega) = S_\Omega[b^\dagger c] \quad , \quad \rho_5(\Omega) = S_\Omega[c^\dagger b] \quad ,$$

$$\rho_6(\Omega) = S_\Omega[(2a^\dagger a - b^\dagger b - c^\dagger c)/3] \quad ,$$

$$\rho_7(\Omega) = S_\Omega[c^\dagger a] \quad , \quad \rho_8(\Omega) = S_\Omega[a^\dagger c] \quad ,$$

² Inclusion of a sterile species would complicate our considerations enormously.

$$\begin{aligned}\rho_9(\Omega) &= S_\Omega[(2b^\dagger b - a^\dagger a - c^\dagger c)/3], \\ \rho_0(\Omega) &= S_\Omega[a^\dagger a + b^\dagger b + c^\dagger c].\end{aligned}\quad (2)$$

Given the redundancy $\rho_3 + \rho_6 + \rho_9 = 0$, these embody the usual SU3 relation, $3^* \times 3 = 1 + 8$. We construct bilinears of the antiparticle operators $\bar{a}, \bar{b}, \bar{c}$ in the same fashion as above except that we make the exchanges in indices, relative to the neutrino case, of $1 \leftrightarrow 2$, $4 \leftrightarrow 5$, and $7 \leftrightarrow 8$. For example, we take,

$$\bar{\rho}_1(\Omega) = S_\Omega[\bar{b}^\dagger \bar{a}] \quad , \quad \bar{\rho}_2(\Omega) = S_\Omega[\bar{a}^\dagger \bar{b}] \quad , \quad \text{etc.} \quad (3)$$

The commutation rules are now given by,

$$\begin{aligned}[\rho_i(\Omega), \rho_j(\Omega')] &= \delta(\Omega - \Omega') \sum_{k=1}^9 f_{i,j,k} \rho_k(\Omega), \\ [\bar{\rho}_i(\Omega), \bar{\rho}_j(\Omega')] &= -\delta(\Omega - \Omega') \sum_{k=1}^9 f_{i,j,k} \bar{\rho}_k(\Omega), \\ [\bar{\rho}_i(\Omega), \rho_j(\Omega')] &= 0.\end{aligned}\quad (4)$$

The $f_{i,j,k}$ coefficients characterize the SU3 algebra of the operator arguments of the expressions (2) as they describe a single momentum state. They are listed in the appendix.

In terms of these operators the forward neutral current interaction Hamiltonian is,³

$$\begin{aligned}H_{\text{for}} &= \frac{G_F}{\sqrt{2}(\text{Vol.})} \int d\Omega d\Omega' (1 - \cos \theta_{\Omega, \Omega'}) \\ &\left\{ \frac{4}{3} [\rho_0(\Omega) - \bar{\rho}_0(\Omega)] \times [\rho_0(\Omega') - \bar{\rho}_0(\Omega')] + \right. \\ &\left. \sum_{i=1}^9 [\rho_i(\Omega) - \bar{\rho}_i(\Omega)] [\rho_i^\dagger(\Omega') - \bar{\rho}_i^\dagger(\Omega')] \right\}.\end{aligned}\quad (5)$$

The result (5) is obtained from the full neutral current four-Fermion interaction by retaining only those combinations of operators that give a forward amplitude, $\mathbf{p}, \mathbf{q} \rightarrow \mathbf{p}, \mathbf{q}$, and restricting these momenta to the set of states that are initially occupied. The terms in (5) for ν - ν (and $\bar{\nu}$ - $\bar{\nu}$) scattering come from two graphs. In one of these a zero-momentum Z is exchanged, giving rise to an interaction among the ρ_0 and $\bar{\rho}_0$ operators only, since for this term in the scattering $\mathbf{p} + \mathbf{q} \rightarrow \mathbf{p} + \mathbf{q}$, the operator that makes the final \mathbf{p} and the operator that annihilates the initial \mathbf{p} attach to the same end of the Z line. This singlet-singlet coupling (in the SU3 sense) gives 3/4 of the first term in (5) and contributes nothing to the time evolution of the density matrix. In the other (“crossed”) term for ν - ν scattering, a momentum $\mathbf{p} - \mathbf{q}$ is transferred through the Z . In this term we use a

Fierz identity, giving a multiplying minus sign, one anti-commutation, giving a second minus sign, and finally we use the appropriate crossing matrix in the internal SU3 space to produce the rest of the terms in (5). For the particle-anti-particle terms, the crossing is slightly different, since the “crossed” graph is a virtual annihilation into an intermediate Z , in which the Z carries momentum $\mathbf{p} + \mathbf{q}$. Our $1 \leftrightarrow 2$, etc., exchange in subscripts in defining the anti-particle densities reflects this change. Of course, since the neutrinos will be in the MeV energy range, we can neglect the neutrino momenta in the Z propagator.

To the above Hamiltonian we add a neutrino mass term, taking the ν and the $\bar{\nu}$'s all to have nearly enough a common energy, E , so that the oscillation term is the same for each. (We weaken this assumption later.) We choose,

$$\begin{aligned}H_{\text{mass}} &= \lambda_1 \int d\Omega [\rho_1(\Omega) + \rho_2(\Omega) + \bar{\rho}_1(\Omega) + \bar{\rho}_2(\Omega)] + \\ &\lambda_2 \int d\Omega [\rho_4(\Omega) + \rho_5(\Omega) + \bar{\rho}_4(\Omega) + \bar{\rho}_5(\Omega)],\end{aligned}\quad (6)$$

where λ_1 is given by $(2E)^{-1}$ times the δm^2 parameter for $\nu_{e,\tau}$ mixing and λ_2 is given by $(2E)^{-1}$ times a δm^2 parameter for $\nu_{\mu,\tau}$ mixing. Finally we include the (relative) energies of interaction that result from a net $n_e - \bar{n}_e$ electron-positron density difference,

$$H_e = \sqrt{2} G_F (n_e - \bar{n}_e) \int d\Omega [\rho_6(\Omega) - \bar{\rho}_6(\Omega)]. \quad (7)$$

From the above Hamiltonian and commutation rules we can write Heisenberg equations for our set of operators $\rho_i(\Omega)$, $\bar{\rho}_i(\Omega)$. The right hand sides will be quadratic functions of the densities themselves, with one angular integration. Thus if we are allowed to replace the expectation values of products of density operators, on the right hand sides, by products of expectation values, we obtain a closed set of non-linear, integro-differential equations for the expectations. More explicitly, we take an initial condition in which the expectation of any product of our density operators is the product of the expectations of the individual operators, e. g.,

$$\langle \rho_3(\Omega, t) \rho_6(\Omega, t) \rangle = \langle \rho_3(\Omega, t) \rangle \langle \rho_6(\Omega, t) \rangle, \quad (8)$$

at $t = 0$. The further assumption is that this factorization holds for all subsequent times. In the present work we assume that this is justified.⁴ To illustrate the

³ The kinetic term in the Hamiltonian takes the same value for every state in our set, and therefore can be ignored in all that follows.

⁴ Truncation schemes such as this should be regarded with some suspicion although we note that the whole literature pertaining to the effects of $\nu - \nu$ interactions is based on the factorization assumption. In previous work [13], [15], in cases of simpler algebras, we have found that the defect of the expectation-value assumption can be expressed as additional terms proportional to N_ν^{-1} in non-linear equations for the products of density op-

complexity of our system, we write the first of the ten independent equations of evolution, where now all density variables are understood as expectation values,

$$i\dot{\rho}_1(\Omega) = \frac{G_F}{\sqrt{2}\text{Vol.}} \int d\Omega' (1 - \cos\theta_{\Omega,\Omega'}) \left\{ \rho_7^*(\Omega) [\rho_4^*(\Omega') - \bar{\rho}_4^*(\Omega')] + \rho_1(\Omega) [\rho_9(\Omega') - \rho_6(\Omega') - \bar{\rho}_9(\Omega') + \bar{\rho}_6(\Omega')] - \rho_4(\Omega) [\rho_7^*(\Omega') - \bar{\rho}_7^*(\Omega')] + [\rho_6(\Omega) - \rho_9(\Omega)] [\rho_1(\Omega') - \bar{\rho}_1(\Omega')] \right\} + \lambda_1 [\rho_6(\Omega) - \rho_9(\Omega)] + \lambda_2 \rho_7^*(\Omega) - \sqrt{2}G_F(n_e - \bar{n}_e)\rho_1(\Omega). \quad (9)$$

The equations are unmanageable except in simplified situations. As mentioned above, there have been a number of treatments of the two-flavor case in which the angular distributions are isotropic. In work reported below we have found non-linear phenomena for the three-flavor case that are qualitatively different from the ‘‘synchronized’’ oscillations that can arise in the two-flavor system, even in the isotropic case. We return to this point, but we include anisotropy in all of the detailed solutions that we present later. Now we implement the two-beam idealization, as discussed in the introduction. Setting $1 + \cos\theta_{\Omega,\Omega'} = 2$, we define collective density coordinates for the two beams as,

$$x_i = (N_\nu)^{-1} \int_{\text{down}} d\Omega \rho_i(\Omega), \\ y_i = (N_\nu)^{-1} \int_{\text{up}} d\Omega \rho_i(\Omega), \quad (10)$$

where $N_\nu = n_\nu(\text{Vol.})$ is the initial number of each flavor of ν , in our ‘‘valence’’ group, and n_ν is the corresponding number density. We define \bar{x}_i, \bar{y}_i in parallel fashion. The commutation rules are now of the form,

$$[x_i, x_j] = (N_\nu)^{-1} \sum_k f_{i,j,k} x_k, \\ [\bar{x}_i, \bar{x}_j] = -(N_\nu)^{-1} \sum_k f_{i,j,k} \bar{x}_k, \quad (11)$$

The y_i, \bar{y}_i variables have the same commutation rules among themselves, and $[x_i, y_j] = 0$.

The operative neutral-current Hamiltonian for these variables is,

$$H_{\text{for}} = \frac{2\sqrt{2}N_\nu^2 G_F}{(\text{Vol.})} \left\{ \frac{4}{3} [x_0 - \bar{x}_0] [y_0 - \bar{y}_0] + \sum_i^9 [x_i - \bar{x}_i] [y_i^\dagger - \bar{y}_i^\dagger] \right\}. \quad (12)$$

The ν -mass and electron interaction terms are given by,

$$H_{\text{mass}} + H_e = N_\nu 2\sqrt{2}G_F n_\nu [\lambda'_1 (x_1 + x_2 + y_1 + y_2 + \bar{x}_1 + \bar{x}_2 + \bar{y}_1 + \bar{y}_2) + \lambda'_2 (x_4 + x_5 + y_4 + y_5 + \bar{x}_4 + \bar{x}_5 + \bar{y}_4 + \bar{y}_5) + \frac{(n_e - \bar{n}_e)}{2n_\nu} (x_6 + y_6 - \bar{x}_6 - \bar{y}_6)], \quad (13)$$

where dimensionless oscillation parameters have been introduced, $\lambda'_i = \lambda_i / (2\sqrt{2}G_F n_\nu)$. Writing the Heisenberg equations that come from the total Hamiltonian $H_{\text{for}} + H_{\text{mass}} + H_e$, and the commutation rules (11) and measuring time in units $(2\sqrt{2}G_F n_\nu)^{-1}$, we obtain a set of 24 coupled nonlinear equations for the 24 density matrix components. We show the first two here; more are provided in the appendix,

$$i\dot{x}_1 = x_1(y_9 - y_6 - \bar{y}_9 + \bar{y}_6) - x_4^*(y_7^* - \bar{y}_7^*) + x_7^*(y_4^* - \bar{y}_4^*) + (x_6 - x_9)(y_1 - \bar{y}_1) + \lambda'_1(x_6 - x_9) + \lambda'_2 x_7^* - \frac{n_e - \bar{n}_e}{2n_\nu} x_1, \\ i\dot{x}_3 = x_4^*(y_4 - \bar{y}_4) - x_4(y_4^* - \bar{y}_4^*) + x_7(y_7^* - \bar{y}_7^*) - x_7^*(y_7 - \bar{y}_7) + \lambda'_2(x_4^* - x_4). \quad (14)$$

We generate the non-linear parts (i. e., the parts coming from neutral current interactions) of the 22 remaining equations (four of them redundant) from the following successive steps: 1) simultaneous permutations of indices $123 \rightarrow 456$, and $456 \rightarrow 789$; 2) the same step repeated; 3) the interchange $x \leftrightarrow y$ (bringing us up to a total of 12 equations); 4) adding a bar to each unbarred symbol and removing the bar from each barred symbol, combined with an all-over change in sign of the right hand sides. The remaining linear terms on the right hand side, containing the ν -mass terms and the electron interaction term are better computed by hand from (13) using (11).

These equations are the generalization of those derived in ref.[14], and applied in [9]-[12], for a situation which has only two flavors and which is spherically symmetrical. The equations of ref [14], specialized to two flavors and to a single energy, can be easily recaptured from (9). We can improve a little on the set (15), as extended by the permutations listed above, by remembering that the up-moving particles have, on the average, higher energies than the down-moving ones. We incorporate this through reducing the oscillation terms for the y variables by a factor $\xi = \langle E_e \rangle / \langle E_{\mu,\tau} \rangle$, as calculated using the initial values of these energy averages, leading to equations of the form,

erators. However we also found that when the non-linear background equations had a certain instability, these terms could lead to rates of otherwise forbidden processes that are suppressed only by a factor $\log(N)$, and may need to be retained even when N is huge.

$$\begin{aligned}
i\dot{y}_1 &= y_1(x_9 - x_6 - \bar{x}_9 + \bar{x}_6) - y_4^*(x_7^* - \bar{x}_7^*) + \\
& y_7^*(x_4^* - \bar{x}_4^*) + (y_6 - y_9)(x_1 - \bar{x}_1) + \xi\lambda'_1(y_6 - y_9) + \\
& \xi\lambda'_2 y_7^* - \frac{n_e - \bar{n}_e}{2n_\nu} y_1, \\
i\dot{y}_3 &= y_4^*(x_4 - \bar{x}_4) - y_4(x_4^* - \bar{x}_4^*) + y_7(x_7^* - \bar{x}_7^*) \\
& - \xi y_7^*(x_7 - \bar{x}_7) + \xi\lambda'_2(y_4^* - y_4), \text{ etc.} \quad (15)
\end{aligned}$$

We take $\xi = .6$ in what follows. Turning to the solutions, we take a neutrino density for each species, $n_\nu = 18(\text{MeV})^3$, about 1/2 the thermal density for a temperature of 7 MeV. Thus our picture is that roughly half of the neutrinos fall into the “valence” group. Then the unit of energy in our scaled equations is $E_s = 2\sqrt{2}n_\nu G_F \approx 5 \times 10^{-10}\text{MeV}$ (corresponding to a distance parameter of $\approx .04$ cm.) We take an average ν energy of 17 MeV. The ν_e oscillation parameter, $\lambda'_1 = \delta m_{e-\tau}^2/2E_\nu$, in units of E_s , is now $\lambda'_1 \approx 6 \times 10^{-9}$, for $\delta m_{e-\tau}^2 = 10^{-4}$ eV, a number quite unfriendly to computations since there are oscillations in the key flavor-changing densities at periods of order E_s^{-1} . We take $\lambda'_2 = 50\lambda'_1$. For a mass density of $8 \times 10^{10}\text{gc}^{-3}$ and with $Y_e = .4$, we have $n_e/n_\nu \approx 8$.

In fig.1 we show a solution to the equations with artificially large values of the oscillation parameters, $\lambda'_1 = 10^{-5}$, and λ'_2 scaled to λ'_1 as described above. We take initial conditions that correspond to six groups of equal density, with $\nu_e, \bar{\nu}_e$ moving downward and $\nu_{\mu,\tau}, \nu_{\bar{\mu},\bar{\tau}}$ moving upward. This initial condition translates into initial values, $x_3 = \bar{x}_3 = x_9 = \bar{x}_9 = -1/3; x_6 = \bar{x}_6 = 2/3;$ and $y_3 = \bar{y}_3 = y_9 = \bar{y}_9 = 1/3; y_6 = \bar{y}_6 = -2/3,$ with all other initial values equal to zero.

From the plots of fig.1 we see that strong mixing sets in shortly after a time $t = (\lambda'_1)^{-1/2}$, with the down-moving states at some moments nearly totally occupied with $\nu_{\mu,\tau}$. Not exactly in synch with the down-moving mixing, the up-going states become up to 50% occupied with ν_e 's, although the percentage fluctuates wildly with time. We observe that the total number of ν_e 's and $\bar{\nu}_e$'s is not constant in the above, although their difference is constant. Owing to the perfectly symmetrical initial condition, the plots for anti-neutrinos are identical. These results are a prototype for all that follows; for smaller values of λ'_1 , the time required for big mixing of the down-moving states follows the $t = (\lambda'_1)^{-1/2}$ rule fairly closely as the oscillation parameters are reduced.

In fig 2. we show the case $\lambda'_1 = 10^{-7}$ (still 16 times our target value). The flipping in the up-moving states occurs after roughly $5000/(2\pi)$ periods of the rapid oscillation rate $5 \cdot 2\sqrt{2}n_\nu G_F$. Comparing fig.1 with fig.2, where

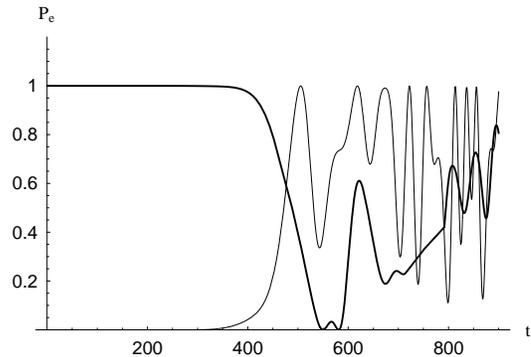


FIG. 1: Electron neutrino density in the up-moving beam (light curve) and in the down-moving beam (heavy curve) as a function of time, as expressed as a fraction P_e of the original density n_ν . The unit of time is the fast scale, as defined in text, $t_s = (2\sqrt{2}G_F n_\nu)^{-1}$. In this calculation the oscillation parameters have been taken to be 1.6×10^3 times as big as present oscillation parameters demand, with $\lambda'_1 = 10^{-5}$, $\lambda'_2 = 50\lambda'_1$, in order to better show features of the curve and to exhibit the scaling, discussed in text, which roughly relates these curves to those of fig. 2.

there is a factor of 100 difference in the oscillation parameters, and a factor of 10 difference in the time spans, we see good confirmation of the $t = (\lambda'_1)^{-1/2}$ condition for mixing. Therefore it appears that we can extrapolate reliably to the physical value of λ' . Doing so, we obtain a time of approximately 2.4×10^4 in our fast units, giving a mixing distance of around 1000 cm.

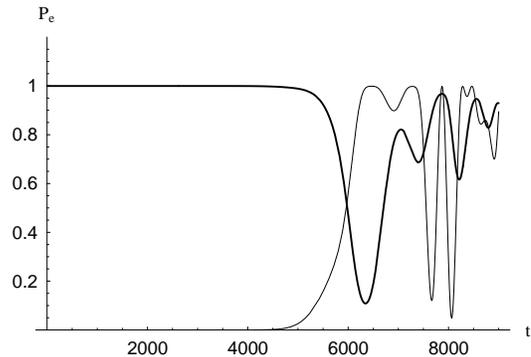


FIG. 2: The same as fig. 1, except that each oscillation parameter is reduced by a factor of 100, so that $\lambda'_1 = 10^{-7}$, etc. The unit of time is again the short time t_s , which is the oscillation period of the flavor-non-diagonal operators. Comparing with fig. 1 we see the $(\lambda')^{1/2}$ scaling discussed in text.

In the calculation leading to fig. 3 we take a five percent $\nu_e - \bar{\nu}_e$ surplus in the initial state, again for the case $\lambda'_1 = 10^{-7}$. We see from fig. 3 that in this case one-half of the up-going states become occupied with $\nu_e, \bar{\nu}_e$, as in

⁵ Luckily for our plots, this very rapid oscillation surfaces only in the expectations of the flavor-off-diagonal density operators.

the previous cases, but more steadily, whereas now the occupancy of the down-moving states is changed hardly at all. There is only a tiny difference in the behavior of the neutrino and antineutrino plots in this case, despite the fact that major features are driven by the small initial asymmetry.

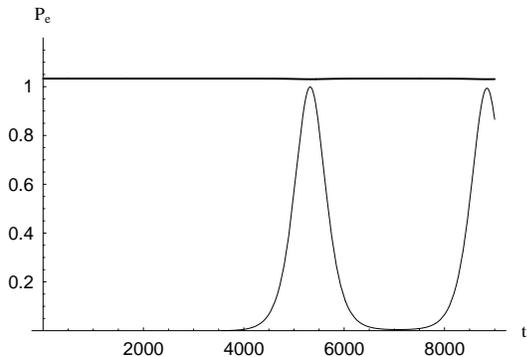


FIG. 3: The same conditions as fig. 2, except for the addition a 5% excess of ν_e 's over the other species in the initial up-moving beam.

In a third situation we change the rules so that the $\bar{\nu}_e$'s are placed in the upward beam. Since in typical supernova calculations the $\bar{\nu}_e$ are intermediate between the $\bar{\nu}_{\mu,\tau}$'s and the ν_e 's, both in energy spectrum and in angular distribution, this is a calculation complementary to the ones presented above. As mentioned earlier, it corresponds to a different choice for the “core” states. Again using values of λ'_1 and λ'_2 that are 16 times too large, because of computational constraints, we show in fig. 4 results for the electron neutrino distributions, in this case appearing to be a clean flip. In contrast to the earlier cases, shown in figs. 1-3, the anti-neutrino distributions are frozen at their initial values in this case.

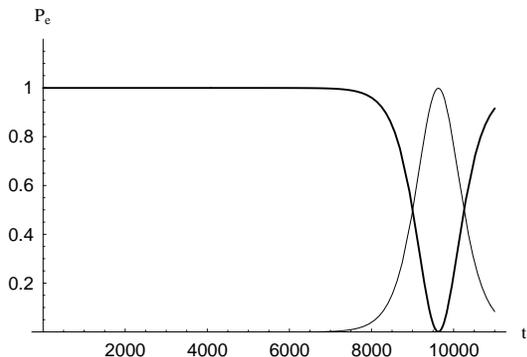


FIG. 4: The same parameters as for fig. 2, except that in the initial conditions the $\bar{\nu}_e$'s have been put in the up-moving group.

DISCUSSION

The purposes of this paper were:

1) To give a simple formulation of the complete 3-flavor equations that govern neutrino evolution, over time periods that are short compared to scattering times, when the density of neutrinos and antineutrinos is large. This is accomplished in the construction of the forward neutral-current Hamiltonian (5) and the commutation rules (4), used in conjunction with the usual oscillation terms. Of course, in a complete approach with radically non-monoenergetic distributions, the densities have still another index, E , and we would write equations for densities $\rho^{(E)}(\Omega)$. In this case the forward neutral current Hamiltonian has a double sum over energies but no explicit further energy dependence. The energy enters explicitly in the oscillation terms.

2) To find a way of treating these equations in the quasi-monoenergetic case, a case that at least can give some insight into possible behaviors. It appears to us that the approximation of using a few rays in the space of angles of the momentum distribution, each with its own complete set of 20 densities is both conceptually clear and practical to implement, the only caveat being that going beyond the two-ray approximation of the present paper will require large computing resources, owing to the extreme values of dimensionless parameters. A possible alternative would be to make a partial wave expansion of the density operators. Of course, because of the non-linearity, and the angular factor $1 - \cos\theta_{\Omega,\Omega'}$ in the kernel of (5), multipoles of all orders are coupled together. Thus, even though all reactions are forward, so that the manifold of occupied momentum states remains unchanged, and even though we might choose an initial configuration with, say, only $l = 0, 1, 2$ states present in the distributions, the flavor configuration could, over time, become arbitrarily complex in angle, developing modes of higher and higher multipolarity. In any case, because we cannot verify the stability of the truncation procedure we do not pursue the partial wave approach here, preferring instead the discrete rays approach. In the rays approach, we represent continuous distributions in the initial state less well, but we really solve the resulting dynamical evolution problem. On the basis of the previously investigated two-flavor models we expect that when we begin with more than two rays there will be instabilities that grow on a time scale of several times $(G_F n_\nu)^{-1}$ rather than $(G_F n_\nu \lambda_1)^{-1/2}$, under the conditions of this present paper, perhaps rendering all of our present results academic ⁶. Unfortunately, we are not equipped to deal computationally with the generaliza-

⁶ This is also true with a truncated partial wave expansion that includes $l = 0, 1$ and 2 states.

tion of the SU3 case, with antineutrinos included, to a case with more than the two rays of the present paper.

Turning to the specific results of the present paper: In three different idealizations, each capturing some of the features that could be important in the supernova neutrinosphere problem, we find an array of possible behaviors. In each case the time-scale is of the order $(E_s \lambda_1)^{-1/2}$, corresponding to a reaction distance of order 100 cm. In the first case we took an initial condition in which both ν_e and an equal number of $\bar{\nu}_e$ from the valence group were assigned to the down-going beam, while ν_τ , $\bar{\nu}_\tau$, ν_μ , $\bar{\nu}_\mu$ were assigned to the up-going beam in equal numbers. The result was a nearly complete flavor over-turn, in which at some moments in time one-half of the up-moving beam is composed of ν_e 's and the down-moving beam is entirely composed of $\nu_{\mu,\tau}$'s.

In the next example, we added a small extra number of ν_e 's in the initial down-moving group, uncompensated by anti-particles. In this case the system does a very amusing thing. The down-moving states remain occupied by ν_e , $\bar{\nu}_e$; but the up-moving states nevertheless make excursions into a configuration in which half of the total original number of ν_μ 's, ν_τ 's have transformed into ν_e 's, and similarly for the antiparticles.

In the final example we put the initial $\bar{\nu}_e$'s into the up-moving group. In this case there is perfect flavor neutrality in the anti-neutrino sector (i.e. a nonvanishing expectation only for the operator \bar{y}_0), and no antineutrino action. However, there is big mixing of the neutrinos in the (moderately) fast time scale, $(E_s \lambda_1)^{-1/2}$, as we saw in fig.4.

One thing in common in all of these simulations is that they require both parameters λ_1 and λ_2 to be non-zero, even though when $\lambda_1 \ll \lambda_2 \ll G_F n_\nu$ the effective time constant, $(E_s \lambda_1)^{-1/2}$, does not depend greatly on λ_2 . Thus all of the phenomena manifesting themselves here require having three flavors and two mixings.

We can try to separate the effects of the two innovations in this paper: 1) the two-group treatment of the angular effects; 2) the inclusion of all three flavors and their antiparticles. We perform all of the calculations described above but this time beginning from the completely isotropic equations. There are nearly twice as many terms on the right hand sides of the equations, as compared to (15), owing to the intra-group interactions ⁷ In the analogs of the calculations with completely symmetric conditions that led to figs 1,2, we find much diminished mixing effects at approximately ten times the time as in the case of our basic calculation. For the other

two examples, which led to the plots of fig. 3 and fig. 4., the isotropic equations gave no appreciable mixing.

It would be rash to extrapolate from any of the above results to draw firm conclusions for supernovas. Since our exploratory work [13] in the two-flavor case indicates the existence of instabilities that can lead to macroscopic changes on a shorter time scale than $(E_s \lambda_1)^{-1/2}$, once we introduce more complex angular dependence, someone with massive computing power should explore this case. The results could render much of the above obsolete, but not the basic equations or the method of attack. A reader could ask, "Why spend so much effort on a problem, if the author believes that it misses the main points of the ultimate solution?" The answer is that this appears to be quite a hard problem, and that systematic work around the edges may be likelier to bring us an understanding of what needs to be done than is plunging ahead, discarding this and that and approximating the other, in order to claim a conclusion for a physical system.

The main moral of our present results, as well as those presented in ref. [13], is that there are big nonlinear effects latent in the neutral-current couplings, effects which can lead to what we have called speed-ups. We speculate that there is a connection between the speed-ups of the present paper, which we would say are due to marginal instabilities, and those more rapidly growing instabilities described in ref. [13]. We think that the examples considered above sit exactly on the boundary of an unstable region in a parameter space. Indeed, the route to finding the growing modes in ref. [13] was to take seriously a demonstration that the two-beam case (as defined in the present paper) is on such a boundary. If the author were to guess at the ultimate outcome, he would guess that the end result of the non-linear effects will be total flavor mixing, and on a distance scale shorter than the 1000cm. that the calculations of this paper indicate. This would be perhaps a rather dull outcome, but it would be computationally convenient as far as incorporating the answers into big simulations of the complete supernova process are concerned.

We emphasize that there is no MSW transformation in any of our results. We deal in time spans that are so short that the density of the surrounding medium can be taken as constant, in the first place. In the second place, the effective time for the actual turn-over, after the waiting time $(E_s \lambda_1)^{-1/2}$, during which the system prepares itself, is many orders of magnitude faster than an MSW transition time can be for any time-changing (i.e. space-changing) medium, for the oscillation parameters that we choose. This distinguishes our considerations from those of refs. [9]-[12].

⁷ A group now consists of a set of neutrinos with momenta in all directions. In the neutral current interaction $\cos \theta$ is replaced by its average value of zero. We still use two groups of states, to take into account the different energy spectra.

APPENDIX

The coefficients $f_{i,j,k}$ which in our representation are antisymmetric only in the first two indices, have the following non-vanishing components,

$$\begin{aligned} f_{1,2,6} = f_{1,4,8} = f_{1,9,1} = f_{2,6,2} = f_{2,8,4} = f_{3,5,5} = \\ f_{3,7,7} = f_{4,5,9} = f_{4,7,2} = f_{5,9,5} = f_{6,8,8} = f_{7,8,3} = 1, \\ \\ f_{1,2,9} = f_{1,6,1} = f_{1,7,5} = f_{2,5,7} = f_{2,9,2} = f_{3,4,4} = \\ f_{3,8,8} = f_{4,5,3} = f_{4,9,4} = f_{5,8,1} = f_{6,7,7} = f_{7,8,6} = -1, \end{aligned} \quad (16)$$

as well as the components obtained by interchange of the first two indices. Although this set is slightly less economical than the canonical set of Gell-Mann (which are based on a hermitean set of ρ_i), the present representation serves us well for the purpose of writing down the Heisenberg equations. In this representation it suffices to write only the equations for the time derivatives of ρ_1 and ρ_3 , with the rest found by permuting indices, as we discussed earlier. The non-hermitean operators that we use also made it easy for us to express the quantities $\langle \rho_2 \rangle$, $\langle \rho_5 \rangle$, $\langle \rho_8 \rangle$ in terms of the respective complex conjugates of $\langle \rho_1 \rangle$, $\langle \rho_4 \rangle$, $\langle \rho_7 \rangle$, reducing the number of integro-differential equations from sixteen to ten, or the number of independent ordinary DE's from thirty two to twenty, in the case of our two-beam approach.

We show the first five of these equations, beginning with the two presented previously. The remaining 15, now including the mass terms, are obtained from the final two substitution rules given in text: a.) the interchange $x \leftrightarrow y$; b.) adding a bar to each unbarred symbol and removing the bar from each barred symbol, combined with an all-over change in sign of the right hand sides,

$$\begin{aligned} i\dot{x}_1 = x_1(y_9 - y_6 - \bar{y}_9 + \bar{y}_6) - x_4^*(y_7^* - \bar{y}_7^*) + \\ x_7^*(y_4^* - \bar{y}_4^*) + (x_6 - x_9)(y_1 - \bar{y}_1) + \lambda'_1(x_6 - x_9) + \\ \lambda'_2 x_7^* - \frac{n_e - \bar{n}_e}{2n_v} x_1, \end{aligned}$$

$$\begin{aligned} i\dot{x}_3 = x_4^*(y_4 - \bar{y}_4) - x_4(y_4^* - \bar{y}_4^*) + x_7(y_7^* - \bar{y}_7^*) \\ - x_7^*(y_7 - \bar{y}_7) + \lambda'_2(x_4^* - x_4), \end{aligned}$$

$$\begin{aligned} i\dot{x}_4 = x_4(y_3 - y_9 - \bar{y}_3 + \bar{y}_9) - x_7^*(y_1^* - \bar{y}_1^*) + \\ x_1^*(y_7^* - \bar{y}_7^*) + (x_9 - x_3)(y_4 - \bar{y}_4) + \\ -\lambda'_1 x_7^* + \lambda'_2(x_9 - x_3), \end{aligned}$$

$$\begin{aligned} i\dot{x}_6 = x_7^*(y_7 - \bar{y}_7) - x_7(y_7^* - \bar{y}_7^*) + x_1(y_1^* - \bar{y}_1^*) \\ - x_1^*(y_1 - \bar{y}_1) + \lambda'_1(x_1 - x_1^*), \end{aligned}$$

$$i\dot{x}_7 = x_7(y_6 - y_3 - \bar{y}_6 + \bar{y}_3) - x_1^*(y_4^* - \bar{y}_4^*) +$$

$$\begin{aligned} x_4^*(y_1^* - \bar{y}_1^*) + (x_3 - x_6)(y_7 - \bar{y}_7) \\ \lambda'_1 x_4^* - \lambda'_2 x_1^* + \frac{n_e - \bar{n}_e}{2n_v} x_7. \end{aligned} \quad (17)$$

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