Covariant Model for Relativistic Three-Body Systems *

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Abstract

The system is described by three mass-shell constraints. When at least two masses are equal, this picture has a reasonable nonrelativistic limit. At first post-Galilean order and provided the interaction is not too much energy-dependent, the relativistic correction is tractable like a conventional perturbation problem. A covariant version of harmonic oscillator is given as a toy model.

A system of three particles can be covariantly described by three mass-shell constraints, involving an interaction term referred to as *potential*. These constraints must reduce to three independent Klein-Gordon (or Dirac) equations in the absence of potential. In any case, they determine the evolution of a wave function which depends on three four-dimensional arguments, say p_a with $a, b = 1, 2, 3$, if we chose the momentum representation of quantum mechanics.

Naturally, the potential depends on both configuration and momentum variables, q_a, p_b, and must allow for mutual compatibility of the constraints. Moreover it happens that, just like in the Bethe-Salpeter approach, manifest covariance is paid by the presence of redundant degrees of freedom of

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which the elimination is by no means straightforward (in contrast to the two-body case). These two important issues have been considered earlier by H. Sazdjian [1] who aimed at solving the general $n-$body case and proposed an approximate solution.
Specially dealing with the *three-boson* case, we have recently exhibited in closed analytic form a new set of variables q'_a, p'_b. In terms of these new variables, admissible expressions for the potential are explicitly available, and two superfluous degrees of freedom can be eliminated [2]. Setting $P = \sum p$ we linearly introduce relative variables

$$z_A = q_1 - q_A, \qquad y_A = \frac{P}{3} - p_A, \qquad A = 2, 3$$

and similar formulas for z'_A, y'_B in terms of q'_a, p'_b.
The mass-shell constraints can be equivalently replaced by their sum and differences; it is convenient to set $\nu_A = \frac{1}{2}(m_1^2 - m_A^2)$.

The difference equations, in their original form, yield no simplification. But we perform a *quadratic* change among the momenta, say $p_a \to p'_a$, or equivalently $P, y_A \to P', y'_A$. in order to ensure the elimination of two redundant degrees of freedom; this change is characterized by

$$(p_1 - p_A)(p_1 + p_A) = (p'_1 - p'_A) \cdot P$$

whereas $P' = P$ and the transverse parts of the momenta remain unaffected, say $\widetilde{y}' = \widetilde{y}$, where the *tilda* on any four-vector refers to its transverse part with respect to P.
Of course, this procedure generates a change of canonical variables [2], in particular we obtain new configuration variables, z'_A.

Three-dimensional Reduction
We impose a sharp value of the total linear momentum, it is a timelike vector k, and we define $k \cdot k = M^2$.
Notations: The *hat* on any vector refers to its transverse part with respect to k.
Underlining any dynamical variable indicates that, in its expression, we substitute k for P and take into account equation the difference equations

$$3y'_A \cdot k \ \Psi = (4\nu_A - 2\nu_B)c^2 \ \Psi \qquad (1)$$

We factorize out the relative energies; as a result the sum equation becomes

$$(3\sum m^2 - M^2)c^2\ \psi = 6(\widehat{y}_2^2 + \widehat{y}_3^2 + \widehat{y}_2 \cdot \widehat{y}_3)\psi + (6M^2c^2\underline{\Xi} + 18\underline{V})\ \psi \qquad (2)$$

for a *reduced* wave function ψ which depends only on the transverse relative momenta $\widehat{y}'_A = \widehat{y}_A$.

The meaning of Ξ is purely kinematic; this term depends only on the momenta and can be expressed in terms of their transverse part and P. Here V denotes the relativistic potential; it may be phenomenological or motivated by considerations of field theory. In particular it may be *formally* constructed as a sum of two-body terms, like in equation (5) below; so doing one uses *the shape* of two-body potentials but (for the sake of compatibility) with the *new* three-body variables as arguments. Not only the total momentum P but also the new configuration variables z'_A mix the two-body clusters, which amounts to automatically incorporate three-body forces. Admissible potentials entail that $\underline{V}$ is a function of the *new* variables $\widetilde{z}'_2, \widetilde{z}'_3$ and M^2c^2.

The reduced equation (2) is actually a nonconventional eigenvalue probem, where the operator to be diagonalized *explicitly depends* on its eigenvalue. This situation is by no means a special drawback of our model, in fact it is common in relativistic quantum mechanics [3], but it would make a general treatment rather involved.

On the other hand, it is natural to expand the formulas in powers of $1/c^2$ and to look for the nonrelativistic limit. For arbitrary masses, the term $M^2c^2\underline{\Xi}$ generally blows up, which leads to consider, instead of (2) an alternative combination of the mass-shell constraints.

Two equal masses.

Drastic simplifications arise when two masses are equal, say $m_2 = m_3 = m$, equivalently $\nu_2 = \nu_3 = \nu$. We find that the Galilean limit of our eigenvalue problem is a Schroedinger equation with effective (or *Galilean*) masses that are generally distinct from the constituent masses m_a. However they still coincide with the constituent masses, at first order in the "mass-dispersion index" ν/m^2.

Three Equal Masses.

When $m_a = m$ for all particles, equation (2) can be written as follows, using the rest frame

$$\lambda\psi = (\mathbf{y}_2^2 + \mathbf{y}_3^2 + \mathbf{y}_2 \cdot \mathbf{y}_3)\psi - 3\underline{V}\psi - M^2c^2\underline{\Xi}\psi \qquad (3)$$

with $6\lambda = (M^2 - 9m^2)c^2$. Now the last term in (3) remains finite in the nonrelativistic limit. Indeed we can write $Mc^2\underline{\Xi} = \frac{1}{M^2c^2}\Gamma_{(0)} + O(1/c^4)$ where

$$\Gamma_{(0)} = \frac{3}{4}\ \{(\widehat{y}_2^2)^2 + (\widehat{y}_3^2)^2 + 4(\widehat{y}_2 \cdot \widehat{y}_3)^2 + 2(\widehat{y}_2^2 + \widehat{y}_3^2)\ (\widehat{y}_2 \cdot \widehat{y}_3) - \widehat{y}_2^2\widehat{y}_3^2\} \tag{4}$$

At first order in $1/c^2$ we can, in $\underline{\Xi}$, replace M^2 by $9m^2$, which is independent from λ. Thus we replace $M^2c^2\underline{\Xi}$ by $\Gamma_{(0)}/9m^2c^2$. If the relativistic "potential" V doesnot depend on P^2, or if this dependence is of higher order, equation (3) becomes a conventional eigenvalue problem, tractable by perturbation theory. The last term in(3) brings a negative correction to the value $\lambda_{\rm NR}$ furnished by the nonrelativistic approximation, say

$$\lambda = \lambda_{NR} - < \Gamma_{(0)} > /9m^2c^2$$

if $\lambda_{\rm NR}$ corresponds to a nondegenerate level. One has to calculate $< \Gamma_{(0)} >$ in the eigenstate solution of the nonrelativistic problem.

Harmonic Oscillator
A covariant version of the harmonic potential is given by

$$V = 2\kappa\ \{(\tilde{z}_2')^2 + (\tilde{z}_3')^2 - \tilde{z}_2' \cdot \tilde{z}_3'\} \tag{5}$$

hence $\underline{V}$ in terms of $\tilde{z}_A' \cdot \tilde{z}_B' = -\mathbf{z}_A'^2 \cdot \mathbf{z}_B'^2$. In the nonrelativistic limit we recover the naive SU_6 invariant Schroedinger equation. At the first post-Galilean approximation, M^2 can be replaced by $9m^2$, neglecting the dependence on total energy in the reduced equation. At this stage, the eigenvalue problem amounts to diagonalize a nonrelativistic harmonic oscillator, with potential $V_{\rm NR} = -3\underline{V}/m$, submitted to a momentum-dependent perturbation. Expressed in terms of Jacobi-like coordinates, namely $\mathbf{R}_2 = -\mathbf{z}_2' + \mathbf{z}_3'$ $\quad \mathbf{R}_3 = (\mathbf{z}_2' + \mathbf{z}_3')/\sqrt{3}$ and their conjugate momenta, the unperturbed ground state is a Gaussian. If the unit of lenght is choosen such that $\kappa = \frac{2}{9}$, one finds $< \Gamma_{(0)} >= 11 + 1/4$.

This approach is intented for applications to confining interactions; future work should implement spin and investigate a possible contact with recents developments [4] of the BS approach.

References

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