

Solving the leading order evolution equation for GPDs

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An analytic method for the solution of the evolution equation for GPDs is presented. The small x, ξ asymptotics of GPDs are calculated.

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One of the outstanding problems of quantum-chromodynamics is to understand how hadrons are built from quarks and gluons and how hadron properties result from their microscopic quark-gluon structure. In recent years the formalism of generalized parton distributions (GPDs) was developed [1, 2], which offers the most economic and comprehensive possibility to describe the internal structure of hadrons. GPDs provide a general framework covering all relevant processes, from inclusive ones (deep inelastic scattering) to exclusive ones, and, therefore, allow us to combine the information contained in different measurements in an optimal manner. In addition, GPDs link this information to fundamental elements of internal hadron structure, which cannot be directly determined by a measurement, like information on the total angular momentum of quarks or the spatial quark distribution in a fast moving hadron.

For instance, knowledge of the transverse spatial structure of quark and gluon distributions in a very high-energy proton is needed to describe proton-proton collisions at the LHC (CERN) which will start operating next year. The reason is that at LHC-energies several simultaneous hard interactions will occur in a single proton-proton collision. The rates for such events depend crucially on how quarks and gluons of different momentum fractions x are distributed in the transverse direction. Without a quantitative understanding of this QCD-background e.g. the experimental search for physics beyond the standard model will be less sensitive.

In practice, the determination of GPDs is highly non-trivial, because they typically enter only in convolutions. Probably only global fits to all relevant experimental [3] and lattice data [4] will be selective enough to really determine more than the most dominant GPDs. Such fits naturally rely heavily on Q^2 evolution. The evolution equations for GPDs are well known [5]; however, analytic methods to solve them were lacking. As a result, one has to adhere to numerical methods which are usually cumbersome and often unstable. Obviously, this whole field would profit enormously, if a faster and numerically more stable method to calculate the Q^2 evolution of GPDs could be found. In this contribution we present a simple algorithm, defined by Eqs. (2) - (4) which fulfills these demands for LO evolution. This algorithm is based on

an analytic method for the determination of the GPD scale dependence, which relies heavily on the symmetries of the evolution equations.

The analytic structure of GPD evolution is intrinsically related to that of Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution [6] for inclusive processes and Efremov-Radyushkin-Brodsky-Lepage (ERBL) evolution [7] for exclusive processes, because all three sets of evolution equations are only different facets of one and the same equation – the renormalization group equation for the corresponding composite operators. In this Letter we consider as concrete examples the gluon GPDs of the nucleon ($H^g(x, \xi, t)$, $E^g(x, \xi, t)$) related to the matrix element of the twist-two gluon operator

$$\mathcal{O}(z_1, z_2) = \text{tr} \{ G^{+\mu}(z_1 n) G_{\mu}^{+}(z_2 n) \}, \quad (1)$$

but the method is general. We also neglect mixing with quark-antiquark operators. (For the exact definitions of the GPDs and kinematical variables, x, ξ, P^+ see Ref. [2].) Since all of these GPDs obey the same evolution equation we introduce the common notation, $\varphi_{\xi}(x)$.

Because the derivation of our solution is rather technical let us state already here the basic result. We find it crucial to treat the two kinematic regions $|\xi| < |x|$ (the DGLAP region) and $|x| < |\xi|$ (the ERBL region) differently. Given some GPD parametrization at some input scale μ_1 , one calculates the coefficients $c_{\xi}(j)$, $c_{\xi}^{\pm}(j)$ from

$$c_{\xi}(j) = \frac{2j-1}{2} \int_{-1}^1 dx \partial_x^2 p_j(x) \partial_x^2 [\varphi_{\xi}^{\mu_1}(x|\xi)], \quad (2)$$

$$c_{\xi}^{\pm}(j) = \frac{2j-1}{2} \int_1^{1/|\xi|} dx \partial_x^2 p_j(x) \partial_x^2 [\varphi_{\xi}^{\mu_1}(\pm x|\xi)] \quad (3)$$

and plugs these into [15]

$$\begin{aligned} \varphi_{\xi}^{\mu_2}(x) &= \Theta(|\xi| - |x|) \sum_{j=3}^{\infty} c_{\xi}^{\mu_1}(j) L^{-\gamma(j)} p_j \left(\frac{x}{|\xi|} \right) \\ &+ \sum_{a=\pm} \frac{1}{\pi i} \int_C dj c_{\xi}^{\mu_1, a}(j) L^{-\gamma(j)} q_j \left(\frac{ax}{|\xi|} \right). \quad (4) \end{aligned}$$

Here $L = \frac{\alpha_s(\mu_1)}{\alpha_s(\mu_2)}$, and the anomalous dimension $\gamma(j)$ is specified later. The integration goes along the line parallel to the imaginary axis such that $2 < \text{Re} j < 3$. The

functions $p_j(x)$, $q_j(x)$ are expressed in terms of the Legendre functions of the first and second kinds [8]

$$p_j(x) = (1-x^2)P_{j-1}^{-2}(x), \quad (5)$$

$$q_j(x) = (1-x^2)\frac{e^{i\pi j}Q_{j-1}^{-2}(x_+) - e^{-i\pi j}Q_{j-1}^{-2}(x_-)}{2i \sin \pi j}, \quad (6)$$

where $x_{\pm} = x \pm i0$ [16]. The sums and integrals in (4) converge absolutely. As it is clear from (4) the first term contributes only to the ERBL region, while the integral gives contributions both to the DGLAP and ERBL regions. We notice also that function $q_j(x) = 0$ for $x < -1$, while for $x > 1$ and $|x| < 1$ it can be simplified to

$$q_j(x) = (1-x^2)Q_{j-1}^{-2}(x), \quad x > 1 \quad (7)$$

$$q_j(x) = -\frac{\pi}{2 \sin \pi j}(1-x^2)P_{j-1}^{-2}(-x), \quad |x| < 1. \quad (8)$$

It vanishes fast with $j \rightarrow \pm i\infty$. The formula (4) can be used both for numerical and analytic studies of GPD evolution. The main point is that if one chooses the GPD as a simple analytic form at the starting value μ_1 , the integrals in (2) and (3) can be done analytically. Then to restore GPD at the scale μ_2 one has to evaluate the sum and the one-dimensional integral. We also stress that the form of Eq. (4) is completely determined by the symmetry properties of the evolution equation.

Below we explain the main steps in the derivation of Eq. (4), the details will be given elsewhere. The evolution equation for the GPD $\varphi_{\xi}(x)$ follows from that for the operator (1). The corresponding kernel (see refs. [5, 9]) is usually given in momentum space (i.e. as a function of x). For our purposes it is more convenient to use the coordinate-space formulation [10]

$$\Phi_{\xi}(z) = (P^+)^{5/2} \int_{-\infty}^{\infty} dx e^{iP^+xz} \varphi_{\xi}(x), \quad (9)$$

$$\varphi(z_1, z_2) = \frac{(P^+)^{1/2}}{2\pi} \int_{-\infty}^{\infty} d\xi e^{-iP^+\xi(z_1+z_2)} \Phi_{\xi}(z_{12}),$$

where z_1 and z_2 are real variables, and $z_{12} = z_1 - z_2$. The LO evolution equation for $\varphi(z_1, z_2)$ reads

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right) \varphi(z_1, z_2) = -N_c \frac{\alpha_s}{\pi} [\mathbb{H} \varphi](z_1, z_2). \quad (10)$$

The integral operator \mathbb{H} [10] can be cast into the form

$$\begin{aligned} [\mathbb{H} \varphi](z_1, z_2) &= -4 \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta (\bar{\alpha}\bar{\beta} + 2\alpha\beta) \varphi(z_{12}^{\alpha}, z_{21}^{\beta}) \\ &+ \int_0^1 d\alpha \frac{\bar{\alpha}^2}{\alpha} [2\varphi(z_1, z_2) - \varphi(z_{12}^{\alpha}, z_2) \\ &- \varphi(z_1, z_{21}^{\alpha})] + \frac{7}{6} \varphi(z_1, z_2). \end{aligned} \quad (11)$$

Here we use shorthand notations $z_{ik}^{\alpha} = z_i \bar{\alpha} + z_k \alpha$, $\bar{\alpha} = 1 - \alpha$. The Hamiltonian (11) commutes with the

generators of collinear conformal transformations $S^a = (S_1^a + S_2^a)$, $a = \pm, 0$; $[\mathbb{H}, S^a] = 0$. The one particle operators, S_k^a , are the generators of the $SL(2, R)$ group

$$S_k^- = -\partial_k, \quad S_k^+ = z_k^2 \partial_k + 2s_k z_k, \quad S_k^0 = z_k \partial_k + s_k, \quad (12)$$

in the representation with spin $s_k = 3/2$. The corresponding finite transformations $T(g)$ maps the function $f(z)$ of the real variable z onto another one according to

$$[T(g^{-1})f](z) = \frac{1}{(cz+d)^3} f\left(\frac{az+b}{cz+d}\right), \quad (13)$$

where a, b, c, d are the entries of the matrix $g \in SL(2, R)$. To solve the LO evolution equation this symmetry plays a crucial role.

The GPD $\varphi(z_1, z_2)$ depends on two real variables z_1, z_2 and transforms under $SL(2, R)$ transformations according to the tensor product of two representations (13). Solving the evolution equation, one seeks for the complete system of functions that diagonalize the Hamiltonian. Usually such a basis is provided by the eigenfunctions of the Casimir operator of the symmetry group. In the case under consideration the Casimir operator is

$$\mathbb{J}^2 = S^+ S^- + S^0 (S^0 - 1) = -z_{12}^{-1} \partial_1 \partial_2 z_{12}^3. \quad (14)$$

To solve the eigenvalue problem for the Casimir operator (14) one should specify the scalar product. Our choice is restricted by the requirement of the $SL(2, R)$ invariance of the latter and by the condition for the ‘‘physical’’ GPD to be normalizable with respect to this scalar product. These requirements lead us to

$$\|\varphi\|^2 = \int dz_1 dz_2 z_{12}^4 |\varphi(z_1, z_2)|^2 = \frac{1}{2} \int d\xi dx |\partial_x^2 \varphi_{\xi}(x)|^2. \quad (15)$$

One can check that (15) is invariant with respect to the transformations (13) and that the total spin operators $S^{\pm, 0}$ are antihermitean. The Casimir operator (14) is self-adjoint and its eigenfunctions form a basis in the Hilbert space with the scalar product (15).

The eigenfunctions of the Casimir operator can be easily found and they are in accordance with the decomposition of the tensor product of two representations (13) into irreducible ones [11]. The operator \mathbb{J}^2 has both a discrete and continuous spectrum. The corresponding eigenfunctions can be expressed as

$$P_{j=n+1}^{\xi}(z_1, z_2) = \sqrt{P^+} e^{-iP^+\xi(z_1+z_2)} \Psi_j^{\xi}(z_{12}), \quad (16)$$

$$P_{j=\frac{1}{2}+i\rho}^{\xi, \pm}(z_1, z_2) = \sqrt{P^+} e^{-iP^+\xi(z_1+z_2)} \Psi_j^{\xi, \pm}(z_{12}), \quad (17)$$

where $n \in Z_+$ and $\rho \in R_+$. The functions $\Psi_j^{\xi, \pm}(z)$ and $\Psi_j^{\xi}(z)$ are defined for arbitrary complex j as follows

$$\Psi_j^{\xi, \pm}(z) = \frac{1}{2 \cos \pi j} \left[\Psi_j^{\xi}(z_{\pm}) - \Psi_{1-j}^{\xi}(z_{\pm}) \right], \quad (18)$$

$$\Psi_j^{\xi}(z) = e^{-i\frac{\pi}{2}(j-1/2)} z^{-5/2} J_{j-1/2}(|P^+\xi|z), \quad (19)$$

where $J_\nu(z)$ is the Bessel function and $z_\pm = \pm z + i0$. (Let us note that by construction $\Psi_j^{\xi,\pm}(z) = \Psi_{1-j}^{\xi,\pm}(z)$.) The eigenvalue of the Casimir operator in both cases is $E = j(j-1)$. Taking the Fourier transform of (16) one finds that the eigenfunctions of the discrete and continuous spectrum have support in the ERBL and DGLAP regions, respectively.

The decomposition of the arbitrary function $\varphi(z_1, z_2)$ into eigenfunctions of the Casimir operator thus reads

$$\varphi(z_1, z_2) = \int \frac{d\xi}{2\pi} \left\{ \sum_{j=1}^{\infty} \omega(j) a_\xi(j) P_j^\xi(z_1, z_2) - i \int_{1/2-i\infty}^{1/2+i\infty} dj \omega^c(j) a_\xi^\pm(j) P_j^{\xi,\pm}(z_1, z_2) \right\}, \quad (20)$$

where $\omega(j) = 2j-1$ and $\omega^c(j) = (j-1/2) \cot \pi j$. The expansion coefficients $a_\xi(j)$ and $a_\xi^\pm(j)$ are given by the scalar product of $\varphi(z_1, z_2)$ with the eigenfunctions (16) and (17). We get rid of the integral over ξ and write down the expansion directly for the function $\Phi_\xi(z)$ (9) as follows

$$\Phi_\xi(z) = \sum_{j=3}^{\infty} (2j-1) a_\xi(j) \Psi_j^\xi(z) - \frac{i}{2} \int_{5/2-i\infty}^{5/2+i\infty} \frac{dj}{\sin \pi j} (2j-1) a_\xi^\pm(j) \Psi_j^\xi(z_\pm). \quad (21)$$

To derive (21) we made use of the explicit form of the functions (18) and the symmetry, $a_\xi^\pm(j) = a_\xi^\pm(1-j)$, of the expansion coefficients. We notice also that the difference between the integrals in (20) and (21) can be calculated by residues and reproduces the missing terms ($j=1, 2$) in the sum in the Eq. (21).

Now we are ready to discuss the evolution of the function $\Phi_\xi(z)$ driven by the Hamiltonian (11). The eigenfunctions of the discrete spectrum of the operator \mathbb{J}^2 are the eigenfunctions of the Hamiltonian with energies

$$E(j) = 2 \left[\psi(j) - \psi(3) - \frac{1}{j(j+1)} - \frac{1}{(j-1)(j-2)} \right] + \frac{7}{6}.$$

Let us notice that $E(j)$ is defined only for integer $j \geq 3$. Further, one can check that the eigenfunctions of the continuous spectrum (17) do not diagonalize \mathbb{H} . This seems to contradict the commutativity of \mathbb{H} and \mathbb{J}^2 . The explanation is simple, the Hamiltonian \mathbb{H} is not self-adjoint with respect to the scalar product (15) and thus does not have the same eigenfunctions as the operator \mathbb{J}^2 [17].

Nevertheless, one can find functions which diagonalize the Hamiltonian (of course, they are not mutually orthogonal and do not form a basis of the Hilbert space). Indeed, the equation $\mathbb{J}^2 \Psi(z_1, z_2) = j(j-1) \Psi(z_1, z_2)$ (here j is an arbitrary complex number) after separation of the factor $e^{-iP^+ \xi(z_1+z_2)}$, turns into a second order differential equation with the two independent solutions $\Psi_j^\xi(z)$

and $\Psi_{1-j}^\xi(z)$. Because of the commutativity of the integral operator \mathbb{H} and the differential operator \mathbb{J}^2 , one concludes that

$$\mathbb{H} \Psi_j^\xi(z) = A(j) \Psi_j^\xi(z) + B(j) \Psi_{1-j}^\xi(z). \quad (22)$$

Substituting $\varphi(z_1, z_2) = e^{-iP^+ \xi(z_1+z_2)} \Psi_j^\xi(z)$ into (11) one finds that the integrals converges when $\text{Re} j > 2$. Next, let us notice that when the variables α and β run over the integration region the argument of the function Ψ_j^ξ varies from 0 to z . Thus to fix the coefficients $A(j)$ and $B(j)$ it is sufficient to study the $z \rightarrow 0$ asymptotics of the r.h.s. and l.h.s. of (22). In this case one can substitute $\Psi_j^\xi(z)$ by its leading term $\sim z^{j-3}$ to get $A(j) = E(j)$, $B(j) = 0$. Thus we have shown that the function $e^{-iP^+ \xi(z_1+z_2)} \Psi_j^\xi(z)$ diagonalizes the Hamiltonian \mathbb{H} .

Now we are in the position to solve the evolution equation for the GPD $\Phi_\xi(z)$. Indeed, the decomposition (21) involves only the functions $\Psi_j^\xi(z)$ which diagonalize the Hamiltonian, and the integration follows the line $\text{Re} j = 5/2$. So we can apply the Hamiltonian \mathbb{H} , Eq. (11), and change the order of integration between j and α, β . Thus the solution of the evolution equation can be written in the form (cf. [10, 12])

$$\Phi_\xi^{\mu_2}(z) = \sum_{j=3}^{\infty} (2j-1) a_\xi^{\mu_1}(j) L^{-\gamma(j)} \Psi_j^\xi(z) - \frac{i}{2} \int_C \frac{dj}{\sin \pi j} (2j-1) a_\xi^{\mu_1, \pm}(j) L^{-\gamma(j)} \Psi_j^\xi(z_\pm). \quad (23)$$

Here $\gamma(j) = 2N_c E(j)/b_0$, $b_0 = \frac{11}{3} N_c - \frac{2}{3} n_f$. After Fourier transformation, Eq. (23) can be cast into the form (4).

Let us discuss the properties of the solution (4). We represent GPD φ_ξ as $\varphi_\xi(x) = \varphi_\xi^I(x) + \varphi_\xi^{II}(x)$, where $\varphi_\xi^I(x)$ is given by the sum in Eq. (4) and $\varphi_\xi^{II}(x)$ by the integrals. The function $\varphi_\xi^I(x) = 0$ outside the ERBL region. The function $\varphi_\xi^{II}(x) = \varphi_\xi(x)$ in the DGLAP region and does not vanish for $|x| < |\xi|$. At the scale μ_1 it can be calculated for $|x| < |\xi|$ in closed form, $\varphi_\xi^{II}(x) = \varphi_\xi(\xi) + \varphi_\xi'(\xi)(x^2 - \xi^2)/2\xi$. This contribution is entirely due to the eigenfunctions of the discrete spectrum with $j=1, 2$ (see Eqs. (20), (21)).

As it was stated earlier the sum and the integrals in (4) are absolutely convergent. Thus having calculated the coefficients $c_\xi^\mu(j)$ and $c_\xi^{\mu, \pm}(j)$ at scale μ_1 one can easily restore the function at another scale μ_2 .

Let us also note that the evolution of the coefficients $c_\xi(j)$ and $c_\xi^\pm(j)$ is not autonomous. To find the coefficients at scale μ_2 one should insert the function $\varphi_\xi^{\mu_2}(x)$ into the Eqs. (2) and (3). Since the function $q_j(x)$ is not an eigenfunction of the Casimir operator one easily figures out, e.g. that the coefficient $c_\xi^{\mu_2}(j)$ cannot be expressed solely in terms of the coefficient $c_\xi^{\mu_1}(j)$. Rather,

after some algebra, one finds

$$c_{\xi}^{\mu_2}(j') = c_{\xi}^{\mu_1}(j')L^{-\gamma(j')} + \int_C \frac{dj}{2\pi i} \frac{1}{(j'+j-1)(j-j')} \\ \times L^{-\gamma(j)} \left[c_{\xi}^{\mu_1,+}(j) + (-1)^{j'-1} c_{\xi}^{\mu_1,-}(j) \right]. \quad (24)$$

The basic Eq. (4) can be analyzed analytically in different limits. As an example let us discuss the asymptotic expansion of the GPD at large μ_2 (see Refs. [5, 13]). To this end we shift the integration contour from the line $\text{Re } j = 5/2$ to the line $\text{Re } j = N - 1/2$, N being an integer. It follows from the Eqs. (7) and (8) that for $x > 1$ the function $q_j(x)$ is analytic for $\text{Re } j > 2$, while for $|x| < 1$ it has poles at integer j . Thus, shifting the contour one has to evaluate residues at these points. The integral over the new contour decreases faster than $L^{-\gamma(N-1/2)}$. Keeping the first $N - 3$ terms from the sum and adding the contribution from the residues one finds

$$\varphi_{\xi}^{\mu_2}(x) = \Theta(|\xi| - |x|) \sum_{j=3}^{N-1} D_{\xi}^{\mu_1}(j) L^{-\gamma(j)} \quad (25) \\ \times \left(1 - \frac{x^2}{\xi^2} \right)^2 C_{j-3}^{5/2} \left(\frac{x}{|\xi|} \right) + \mathcal{O}(L^{-\gamma(N)}).$$

Here $C_{j-3}^{5/2}$ is the Gegenbauer polynomial and the coefficient $D_{\xi}^{\mu_1}(j)$ reads

$$D_{\xi}^{\mu_1}(j) = r_j \int_{-1/|\xi|}^{1/|\xi|} dx C_{j-3}^{5/2}(x) \varphi_{\xi}^{\mu_1}(x|\xi|), \quad (26)$$

where $r_j = 9/2(2j-1)\Gamma(j-2)/\Gamma(j+2)$.

As another illustration let us study the small ξ behavior of GPDs. For small ξ (we take $\xi > 0$) the coefficient $c_{\xi}^+(j)$ can be represented in the form

$$c_{\xi}^{\mu_1,+}(j) = f(j) \bar{c}^{\mu_1,+}(j) (2/\xi)^{j-1} - (j \rightarrow 1-j), \quad (27)$$

where $f(j) = -\frac{\xi}{\sqrt{\pi}} \Gamma(j+1/2)/\Gamma(j-2)$ and the coefficient $\bar{c}^{\mu_1,+}(j) = \int_0^1 dx x^{j-3} \varphi_{\xi=0}^{\mu_1}(x) + \mathcal{O}(\xi^2)$. Denoting by $\varphi_{\xi}^{\mu_2,+}(x)$ the integral corresponding to the term with $a = +$ in the sum (4) and taking into account (27) one finds for $x \gg \xi$

$$\varphi_{\xi}^{\mu_2,+}(x) = \frac{1}{2\pi i} \int_C dj x^{-j+2} \bar{c}^{\mu_1,+}(j) L^{-\gamma(j)} + \mathcal{O}(\xi^2), \quad (28)$$

which corresponds to DGLAP evolution. It is interesting to compare the asymptotics in two regimes: $\xi \rightarrow 0$, x fixed and $\xi \rightarrow 0$, $x = a\xi$. Using (28) and (4) one derives for the ratio of the asymptotics

$$R^+(a) = \frac{\varphi_{x/a}^{\mu_2,+}(x)}{\varphi_0^{\mu_2}(x)} = \frac{e^{i\pi\nu} f_{\nu}(a_+) - e^{-i\pi\nu} f_{\nu}(a_-)}{2i \sin \pi\nu}, \quad (29)$$

where $a_{\pm} = a \pm i0$, $\nu = \left(\frac{4N_c}{b_0} \frac{\log L}{\log 1/x} \right)^{1/2}$ and

$$f_{\nu}(a) = e^{-i\pi\nu \arg(a)} {}_2F_1(\nu/2, \nu/2 + 1/2, \nu + 5/2 | a^{-2}).$$

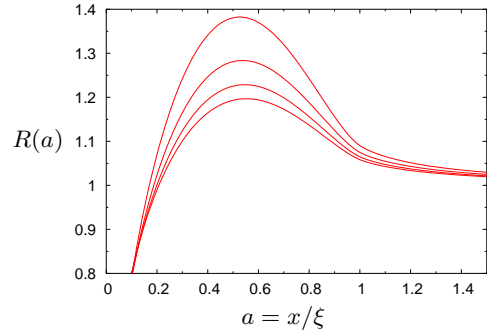


FIG. 1: The ratio $R(a) = R^+(a) + R^+(-a)$ for $L = 2.4$ and $x = 10^{-3}$ (upper curve), 10^{-4} , 10^{-5} and 10^{-6} (lowest curve).

Taking into account the similar contribution from the $a = -$ integral in (4) one obtains for the total ratio $R(a) = \varphi_{x/a}^{\mu_2}(x)/\varphi_0^{\mu_2}(x)$ the result shown in Fig. 1.

To summarize, we have developed an analytic method for solving the LO evolution equations for GPDs. This method is quite general and relies on the conformal symmetry of the evolution equation. Our approach can also be used to elucidate the analytic structure of three-particle parton distributions which possess additional hidden symmetry [14].

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- [15] Since the GPD $\varphi_\xi(x)$ is an even function of x and ξ the expansion coefficients $c_\xi(j)$ vanish for even j , and the functions $c_\xi^\pm(j)$ are equal to each other, $c_\xi^+(j) = c_\xi^-(j)$
- [16] Note that the definition of Legendre function $P_{j-1}^{-2}(x)$ for $|x| > 1$ and $|x| < 1$ is different [8].
- [17] It means that the Hamiltonian \mathbb{H} does not commute with finite $SL(2, R)$ transformations.