Quantum Theory of Neutrino Oscillations for Pedestrians - Simple Answers to Confusing Questions

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Abstract

A simple rigorous calculation confirms the standard formula and clarifies some confusing difficulties arising in the standard textbook recipe converting the unobserved frequency of time oscillations between neutrino states with different energies to the observed oscillation wave length in space Including the quantum fluctuations in the position of the detector and in the transit time between source and detector enables the treatment of: 1. The difference in velocity and transit time between neutrinos with different energies. 2. The destruction of all phases between states with different masses by an ideal detector which measures the energy and momentum of the neutrino. 3. The destruction of all phases between states with different energies by a realistic detector in thermal equilirium with its macroscopic environment. 4. The difficulty for relativistic treatments and relativistic field theory to treat the crucial quantum mechanics of a macroscopic detector at rest in the laboratory.

I. INTRODUCTION

A. Problems causing confusion in understanding neutrino oscillations

Although an extensive review of neutrino oscillations does not mention time at all [1] and bypasses all confusion, the continuing argument about the roles of energy, momentum and time in neutrino oscillations [2] still causes confusion [3]. The "equal energy assumption" of one paper [4] was criticized by another [5] while shown to arise naturally from the interaction of the neutrino with its environment [6]. This interaction is still often ignored [7,8] and the controversy continues [9–11].

Why are different mass states coherent? What is the correct formula for the oscillation phase? How can textbook formulas for oscillations in time describe experiments which never measure time? How can we treat the different velocities and different transit times of different mass eigenstates and avoid incorrect factors of two? How can textbook forumulas which describe coherence between energy states be justified when Stodolsky's theorem states there is no coherence between different energies? Is covariant relativistic quantum field theory necessary to describe neutrino oscillations? How important is the detector, which is at rest in the laboratory and cannot be Lorentz tranformed to other frames?

These questions are answered by a simple rigorous calculation which includes the quantum fluctuations in the position of the detector and in the transit time between source and detector. The commonly used standard formula for neutrino oscillation phases is confirmed. An "ideal" detector which measures precisely the energy and momentum of the neutrino destroys all phases in the initial wave packet and cannot observe oscillations. A realistic detector preserves the phase differences between neutrinos having the same energy and different momenta and confirms the standard formula. Whether phase differences between neutrinos with different energies are observable or destroyed by the detector is irrelevant.

The confusion begins with the textbooks that claim that a neutrino in a mixture of mass eigenstates with different energies will oscillate in time. Nobody measures time. What is measured is distance. Confusion arises in converting the period of oscillation in time to a wave length in distance, combined with the observation that neutrinos with different energies travel with different velocities and arrive at the detector at different times. How to treat this is clear to anyone who understands first year quantum mechanics and wave-particle duality. Confusion arises because too many papers on neutrino oscillations are written by people who don't and double counting leads to the factor of two.

A second source of confusion is that neutrino oscillations are observable only if the neutrino detector is able to detect the relative phase of components in an incident wave packet. Leo Stodolsky [6] has pointed out that any stationary detector is described by a density matrix that is diagonal in energy and that phases between incident neutrino amplitudes with different energies are not observable. Only interference between components with the SAME ENERGY and different momenta are observable. But since this point is continuously ignored in the literature and in conferences it is spelled out clearly and unambiguously in the present paper. It may be only first year quantum mechanics, but unfortunately it seems to be needed.

These problems are illustrated in the section on neutrino mass, mixing and flavor change in the review of particle physics [12] by the Particle Data Group. The section contains this caveat "The quantum mechanics leading to the same result is somewhat subtle. To do justice to the physics requires a more refined treament than the one we have given. Sophisticated treatments continue to yield new insights". Unfortunately most readers ignore the caveat and a literal interpretation of the text contributes to the confusion.

The time-dependent neutrino wave function described by eq. (13.4) has a "Lorentz-invariant phase factor". The text does not note that this phase factor is not gauge invariant and is not observable; only phase differences are observable. Although first year quantum mechanics tells us that the overall phase of a wave function is not observable, many readers fall into this trap and take the phase seriously.

The text considers a neutrino produced with momentum p; all components have same momentum. The time-dependent phase is replaced by a distance-dependent phase by noting that the neutrino is extremely relativistic and sets $t \approx L$. However, this is not strictly true for finite mass neutrinos. Neutrinos with the same momentum and different masses have different velocities and travel the distance L in sufficiently different times to cause confusion.

This has recently been pointed out to the Particle Data Group by a reader who noted that the exact relation between t and L reads $L = t \cdot v$, with $v = p/E$. Substituting $L/v = L \cdot E/p$ into the "Lorentz-invariant phase factor" he gets $E \cdot t - p \cdot L = E^2 \cdot L/p - p^2 \cdot L/p = (E^2 - p^2) \cdot L/p = m^2 \cdot L/p$ which differs from the standard result by a factor of 2.

The text then examines the relative phases of the mass eigenstates at the distance L and converts the state to a linear combination of flavor eigenstates. This assumes that the states with different energies are coherent and that their relative phase is observable. But Stodolsky has shown [6] that neutrinos with different energies lose all coherence in the interaction with the detector and their relative phase is not observable.

Our purpose here is to present a coherent treatment that can be easily understood by physicists who are unaware of the "somewhat subtle quantum mechanics" needed to understand neutrino oscillations.

If the interaction with the environment is ignored, the problem reduces to the propagation of a single noninteracting Dirac fermion, easily solved without relativistic field theory. In a real experiment the neutrino passes the detector as a wave with a finite length in time and can interact with the detector any time during this finite interval. These quantum fluctuations in transit time affect the relative phase observed at the detector between two neutrino waves. Confusion and erroneous factors of two arise in incorrectly treating this transit time as a well-defined classical variable. We present here a rigorous treatment showing that the standard relation between phases of amplitudes having different masses always holds irrespective of whether states with equal energy, equal momentum or any other combination are used. Effects of transit time variations are shown to be negligible.

The observability of these phases depends on the detector. An ideal detector; e.g. a nucleon at rest, measures the neutrino mass and destroys all coherence and relative phases. This "missing mass" experiment has no oscillations. More realistic detectors preserve the relative phases of amplitudes having the same energy and different momenta; the detecting nucleon is confined to a region of space small compared to the wave length of the observed neutrino oscillations. The relative amplitudes for final states containing an e, μ or τ are thus determined for each energy. Since these relative magnitudes do not change appreciably with energy in realistic experiments, the flavor output of the detector is determined without needing Stodolsky's stationary environment assumption [6]. Even if a judicious time measurement preserves some coherence between different energies, the flavor output is unchanged.

B. The basic problem

The source of the confusion is easily seen by examining the state of a neutrino wave packet incident on a detector: a mixture of states having different masses, energies and momenta. The phase change in the wave function of a component of the wave packet in traveling a distance x from the source to the detector in a time t is

$$\phi(x,t) = px - Et \tag{1.1}$$

This phase by itself is unobservable, like any overall phase of a wave function. It is not gauge invariant since a gauge transformation can multiply the wave function by an arbitrary function of x.

What is gauge invariant and observable is the phase difference between any two components of the wave function having masses m_1 and m_2

$$\delta\phi(12) = \phi_1(x,t) - \phi_2(x,t) = (p_1 - p_2)x - (E_1 - E_2)t = \frac{p_1^2 - p_2^2}{p_1 + p_2} \cdot x - \frac{E_1^2 - E_2^2}{E_1 + E_2} \cdot t \tag{1.2}$$

where p_1, p_2, E_1 and E_2 are respectively the momenta and energies of these two components in the wave packet.

To relate this result (1.2) to the phase difference actually observed in a detector, we set x equal to the distance between the source and the detector. The problem arises in choosing a value for t. We parametrize the problem by setting t equal to the time required to traverse the distance x using an average group velocity $\bar{v}$ for the two mass eigenstates, with a correction δt to describe any discrepancies, including quantum fluctuations.

$$\bar{v} \equiv \frac{p_1 + p_2}{E_1 + E_2}; \quad t = \frac{x}{\bar{v}} + \delta t = \frac{E_1 + E_2}{p_1 + p_2} \cdot x + \delta t \tag{1.3}$$

Thus

$$\delta\phi(12) = \frac{p_1^2 - p_2^2}{p_1 + p_2} \cdot x - \frac{E_1^2 - E_2^2}{E_1 + E_2} \cdot t = \frac{m_2^2 - m_1^2}{p_1 + p_2} \cdot x - \frac{E_1^2 - E_2^2}{E_1 + E_2} \cdot \delta t \tag{1.4}$$

This gives the standard formula for the phase shift independent of the momentum and energy values of the two interfering waves with a correction term which vanishes when the two waves have the same energy. The physics here is evident. The relative phase at the same point in space between two amplitudes with different energies is proportional to the product of the energy difference and time and sensitive to the exact time when the neutrino is detected. The δt correction is appreciable for the understanding of time behavior. But a quantitative analysis given below shows the correction to the formula (1.4) to be negligible. This explains why treatments using waves with the same momentum and different energies give the same answer as those using the same energy and different momenta.

The time behavior has caused much confusion in the community and in the literature. We therefore we investigate in detail its physical meaning and quantum fluctuations.

II. THE ELEMENTARY QUANTUM MECHANICS OF SPACE AND TIME

Neutrinos arrive at a detector at distance x from the source as a coherent linear combination of states having different masses, energies and momenta. Since the source and absorber are quantum systems, quantum fluctuations arise in both the distance x and in the time interval t between the emission from the source and its observation at the detector. The distance fluctuations can be shown to be small in any experiment where the positions of the source and absorber are known to a precision much smaller than the wave length of the oscillation to be measured. The nature of the quantum fluctuations in the time interval t are not generally well understood and are the source of much confusion.

The neutrino wave function at the detector defines a probability amplitude for its detection at any time during the arrival of the wave packet. The value of the transit time t is thus not precisely defined and subject to the usual quantum fluctuations of any variable described by a quantum-mechanical probability amplitude.

The waves for different mass eigenstates travel with different group velocities, and their centers may arrive at the detector at different times. But the detector does not measure the arrival time of a wave packet center. Each event has a single value of the time t. The flavor output of the detector is determined by the relative phases of the different components of the wave packet at that time. The quantum fluctuations in time can be observed if there is a measurement of the time spectrum of events.

We now define the quantities necessary to evaluate the phase shifts in eq.(1.4).

The time variable t to use in eq. (1.2) is defined as the interval between the time when the neutrino is detected and a time at the source when the neutrino is in a flavor eigenstate. This is then subject to the uncertainty denoted by δt. The neutrino flavor at the source can be measured by placing a detector a short distance from the source. This was in fact the first experiment that showed that there were different flavors of neutrinos.

We now check the effect of δt for the case of equal momenta $p_1 = p_2 \equiv p$ in (eq.1.4).

$$\delta\phi(12) = \frac{m_2^2 - m_1^2}{p_1 + p_2} \cdot \left(x + \frac{E_1^2 - E_2^2}{m_1^2 - m_2^2} \cdot \bar{v} \cdot \delta t \right) = \frac{m_2^2 - m_1^2}{2p} \cdot x \cdot \left(1 + \frac{\bar{v} \cdot \delta t}{x} \right) \tag{2.1}$$

Since the spatial size of the wave packet, $\bar{v} \cdot \delta t$ is much smaller than the distance x between source and detector the neglect of δt in eqs. (1.4) and (2.1) is clearly justified. But δt is also seen to be much larger than the time separation δt_{wc} between the centers of the two mass eigenstate wave packets, which have moved through the distance x with group velocities $v_1 = p/E_1$ and $v_2 = p/E_2$ respectively.

$$\bar{v} \cdot \delta t_{wc} = \bar{v} \cdot \left(\frac{x}{v_1} - \frac{x}{v_2} \right) \approx \frac{m_2^2 - m_1^2}{2p^2} \cdot x << \bar{v} \cdot \delta t << x \tag{2.2}$$

The transit time fluctuations δt are seen to be much larger than δt_{wc} but sufficiently small so that the correction to eq. (1.4) is negligible. The time difference δt_{wc} is automatically included in the above calculation and needs no further correction. Unnecessary additional corrections motivated by the erroneous classical picure of a particle traveling on a classical path with a well defined velocity has led to the spurious factors of 2.

III. WHAT IS OBSERVABLE

Whether the phase (1.4) is experimentally observable depends upon the detector.

A. An "ideal" detector

Consider an "ideal" detector which is a nucleon at rest. It absorbs the incident neutrino and emits a lepton. Since energy and momentum are conserved, the energy and momentum transfers from the neutrino to the detector denoted by δE and δp are equal respectively to the energy and momentum of the incident neutrino, denoted by E_ν and p_ν

$$\delta E = T_N + E_{lepton} = E_\nu; \quad \delta p = p_N + p_{lepton} = p_\nu \tag{3.1}$$

where T_N and p_N are the kinetic energy and momentum of the final nucleon, and E_{lepton} and p_{lepton} are the total energy and momentum of the emitted lepton.

This "missing mass experiment" determines the neutrino mass

$$m_\nu = \sqrt{E_\nu^2 - p_\nu^2} \tag{3.2}$$

Since the detector determines the neutrino mass, waves arriving at the detector with different masses are not coherent. All phase information is destroyed in this "ideal" detector.

B. A realistic localized detector

Realistic detectors are not ideal. The localization of the detector nucleon in a small region of space makes its momentum uncertain. There is no comparable localization in time. This asymmetry between energy and momentum in the initial detector state destroys the apparent symmetry between energy and momentum noted in all covariant descriptions of neutrino oscillations. A fully covariant description of any experiment which can be used also to consider detectors moving with relativistic velocities is not feasible at present. A covariant description which neglects the quantum mechanics of the neutrino-detector interaction is neglecting essential physics of all realistic oscillation experiments.

The following section gives a rigorous treatment of the handwaving argument using the uncertainty principle to relate the momentum spread of the wave function to the size of the detector. The phase between components of the neutrino wave with the same energy and different masses and momenta is shown to be observable if quantum fluctuations in the position of the detector nucleon are very small in comparison with the wave length of the neutrino oscillations in space produced by the mass difference between neutrino mass eigenstates.

The simple hand-waving argument for this physics states that the uncertainty principle and the localization in space of the detector nucleon that absorbs the neutrino prevents the detector from knowing the difference between components of the incident neutrino wave packet with slightly different momenta and the same energy.

IV. A DETAILED RIGOROUS CALCULATION OF THE DETECTION PROCESS

The rigorous quantum-mechanical argument notes that the product $\delta p \cdot \delta x$ of the quantum fluctuations in the position of the detector nucleon δx and the range of momenta δp in relevant neutrino states having the same energy is a small quantity. Taking the leading terms in the expansion of the transition matrix elements in powers of $\delta p \cdot \delta x$ gives the result that the flavor spectrum of the charged leptons emitted from the detector at a given energy is determined by the relative phase of the components of the incident neutrino wave packet having the same energy and different momenta. Whether coherence between amplitudes with different energies is destroyed, as required by Stodolsky's theorem, or can be preserved by judicious time measurements is irrelevant to this result.

The detection of the neutrino is a weak interaction described in first order perturbation theory by transition matrix elements between the initial state of the neutrino-detector system before the interaction and all possible final states.

Consider the transition matrix element between an initial state $|i(E)\rangle$ with energy E of the entire neutrino - detector system and a final state $|f(E)\rangle$ of the system of a charged muon and the detector with the same energy E, where a neutrino ν_k with energy, mass and momentum E_ν, m_k and $\vec{P}_o + \delta\vec{P}_k$ is detected via the transition

$$\nu_k + p \rightarrow \mu^+ + n \tag{4.1}$$

occurring on a proton in the detector. We express the neutrino momentum as the sum of the mean momentum $\vec{P}_o$ of all the neutrinos with energy E_ν and the difference $\delta\vec{P}_k$ between the momentum of each mass eigenstate and the mean momentum,

The transition matrix element depends upon the individual mass eigenstates k only in the momentum difference $\delta\vec{P}_k$ and a factor $c_{k\mu}$ for each mass eigenstate which is a function of neutrino mixing angles. $c_{k\mu}$ describes the transition amplitude for this mass eigenstate to produce a muon when it reaches the detector, multiplied by a phase given by the generalization to the case of three neutrinos of eq. (1.4) with $E_1 = E_2$ The transition matrix element can thus be written in a factorized form with one factor T_o independent of the mass m_k of the neutrino eigenstate and a factor depending on m_k.

$$\langle f(E)|\, T\, |i(E)\rangle = \sum_k \langle f(E)|\, T_o \cdot c_{k\mu} e^{i\delta\vec{P}_k \cdot \vec{X}}\, |i(E)\rangle \tag{4.2}$$

where $\vec{X}$ denotes the co-ordinate of the nucleon that absorbs the neutrino. Then if the product $\delta\vec{P}_k \cdot \vec{X}$ of the momentum spread in the neutrino wave packet and the fluctuations in the position of the detector nucleon is small, the exponential can be expanded and approximated by the leading term

$$\langle f(E)|\, T\, |i(E)\rangle = \sum_k \langle f(E)|\, T_o \cdot c_{k\mu} e^{i\delta\vec{P}_k \cdot \vec{X}}\, |i(E)\rangle \approx \sum_k \langle f(E)|\, T_o \cdot c_{k\mu}\, |i(E)\rangle \tag{4.3}$$

The transition matrix element for the probability that a muon is observed at the detector is thus proportional to the coherent sum of the amplitudes $c_{k\mu}$ for neutrino components with the same energy and different masses and momenta to produce a muon at the detector. A similar result is obtained for the probability of observing each other flavor. The final result is obtained by summing the contributions over all the energies in the incident neutrino wave packet. But as long as the flavor output for each energy is essentially unchanged over the energy region in the wave packet, the flavor ouput is already determined for each energy, and is independent of any coherence or incoherence between components with different energies.

For the case of two neutrinos with energy E and mass eigenstates m_1 and m_2 the relative phase of the two neutrino waves at a distance x is given by eq. (1.4) with $E_1 = E_2$

$$\phi_m^E(x) = (p_1 - p_2) \cdot x = \frac{(p_1^2 - p_2^2)}{(p_1 + p_2)} \cdot x = \frac{m_2^2 - m_1^2}{2\bar{p}} \cdot x \tag{4.4}$$

The flavor output of the detector is thus seen to be determined by the interference between components in the neutrino wave paclet with the same energy and different masses and momenta. All the relevant physics is in the initial state of the nucleon in the detector that detects the neutrino and emits a charged lepton, together with the relative phases of the components of the incident neutrino wave packet with the same energy.

This result (4.3-4.4) is completely independent of the neutrino source and in particular completely independent of whether the source satisfies Stodolsky's stationarity condition [6]. No subsequent time measurements or additional final state interactions that mix energies can change this flavor output result.

V. CONCLUSIONS

A. Problems in calculating phase differences

1. A reliable calculation of neutrino oscillations that avoids confusion requires an understanding of the effects of variations in the transit time of a neutrino between source and detector and its quantum fluctuations.

2. When the small quantum fluctuations in time are neglected, the phase difference between two components of a narrow wave packet with different masses depends only on their squared mass difference and the mean momentum in the wave packet and is otherwise independent of their momenta and energies.

3. The value of this phase difference is given by the standard formula used in all analyses of neutrino oscillations.

B. Problems in observing phase differences

1. The observability of this phase difference depends upon the quantum mechanics of the detector.

2. All coherence is destroyed in an "ideal" detector, which precisely measures the momentum and energy of the neutrino.

3. A realistic detector preserves the relative phase between states having the same energy and different momenta. This is sufficient to uniquely determine the flavor output of the detector and show that it satisfies the standard formula for neutrino oscillations.

4. Whether the coherence between states having different energies is destroyed, as required by Stodolsky's stationarity theorem [6], or can somehow be preserved by judicious time measurements is irrelevant to the flavor output of the detector.

5. The standard formula used in all analyses of neutrino oscillations can be rigorously justified. All sources of confusion are resolved by using states with the same energy and different momenta to calculate neutrino oscillations.

C. Some concluding remarks

The initial uncertainty in the momentum of the detector nucleon in a localized detector destroys all memory of the initial neutrino momentum and of the initial neutrino mass after the neutrino has been absorbed. The hand-waving justification of the result (4.3) uses the uncertainty principle to say that if we know where the detector is we don't know its momentum and can't use momentum conservation to determine the mass of the incident neutrino. The above rigorous justification shows full interference between the contributions from different neutrino momentum states with the same energy as long as the product of the momentum difference and the quantum fluctuations in the initial position of the detector nucleon is negligibly small in the initial detector state.

This treatment of the neutrino detector is sufficient to determine the flavor output of any experiment in which the incident neutrino wave packet is the same well defined linear combination of mass eigenstates throughout the whole wave packet.

Time measurements and all possible coherence between amplitudes from components of the incident neutrino wave packet with different energies are not considered and seen to be unnecessary. There may be fancy time measurements which can introduce such coherence. But the coherence between incident neutrino states with the same energy and different momentum already determines the flavor output of the detector for each incident neutrino energy and cannot be destroyed by time measurements.

The question remains of possible variation of detector flavor output as a function of energy. As long as the wave packet is sufficiently narrow in momentum this flavor output does not change appreciably over the relevant energy range in the wave packet and the standard neutrino oscillation formulas are valid. When the flavor output varies widely as a function of energy, oscillations are no longer observed. This can be seen in the case of neutrinos traveling large distances with many oscillation wave lengths, as in neutrinos arriving from the sun or a supernova. Here the neutrino wave packet can separate into components with different mass eigenstates, traveling with different velocities and reaching the detector at measurably different times. All this time variation appears simply [6] in the energy spectrum, which is the fourier transform of the time behavior.

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