

# Properties of the vector meson nonet at large $N_c$ beyond the chiral limit

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Masses and especially coupling constants of the vector meson nonet are determined in the large- $N_c$  limit, but beyond the chiral limit taking into account terms up to quadratic order in the Goldstone boson masses. With two input parameters five coupling constants for hadronic and dilepton decays are determined which agree very well with the experimental results. The obtained parameters are also used to calculate the pion and kaon decay constant in the large- $N_c$  limit. A consistent picture is only obtained, if the correct assignment of the  $N_c$ -dependence of the electromagnetic charges of the quarks is taken into account.

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## I. INTRODUCTION

It is nowadays common wisdom that the spontaneous breaking of chiral symmetry leads to the appearance of a Goldstone boson octet. This causes a large gap in the excitation spectrum of the observed hadrons. The lowest pseudoscalar octet appears to be light<sup>1</sup> while all other hadrons are heavy. Therefore at low energies QCD reduces to an effective theory where only the pseudoscalar mesons appear which interact with each other and with external sources. Spontaneous chiral symmetry breaking also demands that the meson self-interaction vanishes with vanishing energy. Therefore a systematic expansion in terms of the derivatives of the meson fields is possible. These considerations lead to the effective Lagrangian of chiral perturbation theory ( $\chi$ PT) [1, 2] presented in the following for three light flavors:

$$\mathcal{L}_{\chi\text{PT}} = \mathcal{L}_1 + \mathcal{L}_2 + \text{higher order derivatives} \quad (1)$$

with the term which contains two derivatives of the Goldstone boson fields,

$$\mathcal{L}_1 = \frac{1}{4} F_0^2 \text{tr}(\nabla_\mu U^\dagger \nabla^\mu U) + \dots, \quad (2)$$

and all possible terms which contain four derivatives,

$$\begin{aligned} \mathcal{L}_2 = & L_1 [\text{tr}(\nabla_\mu U^\dagger \nabla^\mu U)]^2 + L_2 \text{tr}(\nabla_\mu U^\dagger \nabla_\nu U) \text{tr}(\nabla^\mu U^\dagger \nabla^\nu U) \\ & + L_3 \text{tr}(\nabla_\mu U^\dagger \nabla^\mu U \nabla_\nu U^\dagger \nabla^\nu U) + \dots \end{aligned} \quad (3)$$

Note that we have only displayed explicitly the terms which are relevant for later use, i.e. the ones which remain present once all external fields and explicit chiral symmetry breaking terms are put to zero. All the other terms are indicated by the dots in (2) and (3). In  $U$  the pseudoscalar meson fields are encoded.  $F_0$  denotes the pion decay constant in the chiral limit. We refer to [2] for further details. In principle, the low-energy constants  $F_0$  and  $L_1$ - $L_3$  and the ones which we have not explicitly displayed ( $B_0$  and  $L_4$ - $L_{10}$ ) can be obtained from QCD by integrating out all degrees of freedom besides the Goldstone bosons. Such a task, however, would be more or less equivalent to solving QCD in the low-energy regime. Lattice QCD [3] has started to determine some of these low-energy constants (see e.g. [4] and references therein). In practice, one determines these coupling constants from experiment [1, 2, 5] and/or from hadronic [6, 7, 8, 9] or quark models [10, 11, 12, 13, 14, 15, 16]. Indeed, already in the seminal work [2] it was pointed out that it is difficult to pin down in particular all ten constants  $L_1$ - $L_{10}$  solely from experimental inputs. Large- $N_c$  considerations were involved in addition yielding

$$L_1 - \frac{1}{2} L_2, L_4, L_6, B_0 = o(1), \quad (4)$$

while

$$L_1, L_2, L_3, L_5, L_8, L_9, L_{10}, F_0^2 = O(N_c). \quad (5)$$

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<sup>1</sup> The meson masses deviate from zero on account of the (small) current quark masses which explicitly break the chiral symmetry.

Here  $N_c$  denotes the number of colors. We note in passing that  $L_7$  is special since it involves the chiral anomaly. See [17, 18] for details.

At low energies  $\chi$ PT yields an excellent description of the hadron phenomenology (see e.g. the reviews [14, 19, 20]). However, as an expansion in powers of energies and momenta of the involved states the approach is limited to the low-energy regime. One limitation comes from the resonances which appear in the meson-meson scattering amplitudes. A derivative expansion like the one present in the  $\chi$ PT Lagrangian (1) can only give polynomials in the kinematic variables while a resonance appears as a pole. Therefore to make contact between the low-energy region governed by  $\chi$ PT and the region of mesonic resonances non-perturbative methods are needed — which, of course, introduce some model dependence.

In the following, we will make use of the resonance saturation approach developed in [6, 7] (cf. also [8]). The purpose of the present work is to extend this approach beyond the chiral limit and study the properties of the lowest-lying vector meson nonet ( $\rho$ ,  $\omega$ ,  $K^*$  and  $\phi$ ). We will concentrate on the masses of the vector meson nonet and their hadronic and dilepton decays. Concerning the masses we basically repeat the calculations of [21]. The new aspect of the present work are the decay properties. The basic ideas are the following:

- A Lagrangian of vector mesons coupled to the Goldstone bosons is proposed. Contact between this Lagrangian and  $\chi$ PT is made by integrating out the vector mesons in the low-energy regime. This task can be performed systematically in the framework of a large- $N_c$  expansion. Following [6] we will work in leading order of  $1/N_c$  throughout the present work. (In [22] the resonance saturation approach is extended beyond the large- $N_c$  limit, but is restricted to the chiral limit and to two flavors.)
- We perform all calculations beyond the chiral limit by including systematically terms which are of quadratic order in the Goldstone boson masses (linear order in the current quark masses). Concerning the decay properties of the vector mesons this constitutes the main new aspect of the present work.
- Additional high-energy constraints are used to relate various coupling constants [7]. As we will see, a consistent picture (beyond the chiral limit) emerges only, if the correct  $N_c$ -dependence of the electromagnetic charges of the quarks [23, 24, 25, 26] is taken into account. This new finding (which apparently involves aspects of the electroweak model) does not show up in the chiral limit.

The paper is organized in the following way: In the next section we will introduce the basic ideas of resonance saturation by presenting the whole approach in the chiral limit. This section can be regarded as a brief summary of the works [6, 7] as far it concerns our purposes. In section III we extend the approach beyond the chiral limit. Numerical results, especially for coupling constants, are presented in section IV. Finally we summarize in section V.

## II. RESONANCE SATURATION IN THE CHIRAL LIMIT

One way to generalize  $\chi$ PT to higher energies is to introduce additional degrees of freedom, i.e. mesonic resonances with various quantum numbers. Of course, they have to be included in accordance with chiral symmetry. Contact with  $\chi$ PT is made — at least in principle — by integrating out these resonance fields. In that way, the low-energy coefficients of  $\chi$ PT are expressed in terms of the resonance parameters [6, 7, 8]. In practice, interacting fields cannot be integrated out exactly. Here the large- $N_c$  expansion comes into play: Meson interaction vertices are more suppressed [27] the higher the number of external legs is. This leads to the fact that resonance loops are subleading as compared to tree diagrams. In addition, we recall that  $\chi$ PT is an expansion around the chiral limit in derivatives and masses of the Goldstone bosons. Both aspects together yield a systematic way to integrate out resonances in the low-energy regime [6].

From a phenomenological point of view [28] one could introduce arbitrary many of such resonances. On the other hand, it was already observed in [1] for  $SU(2)$ - $\chi$ PT that the  $\rho$ -meson basically saturates all low-energy coefficients to which it contributes. This issue has been extended to  $SU(3)$  and studied more systematically in [6, 7]. The lowest-lying scalar and vector mesons were included in that analysis. The starting point is the following Lagrangian:

$$\mathcal{L}_{\text{res.sat.}} = \mathcal{L}_1 + \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{int}} . \quad (6)$$

Here  $\mathcal{L}_1$  is again the lowest-order Lagrangian of  $\chi$ PT as given in (2) and  $\mathcal{L}_{\text{kin}}$  denotes the kinetic part of the resonance

Lagrangian (see [6] for details).<sup>2</sup> The last term is given by

$$\mathcal{L}_{\text{int}} = \frac{1}{2\sqrt{2}} (F_V \text{tr} (V_{\mu\nu} f_+^{\mu\nu}) + iG_V \text{tr} (V_{\mu\nu} [u^\mu, u^\nu])) + \dots \quad (7)$$

where we have only displayed the contributions from vector mesons explicitly. In the large- $N_c$  limit the lowest-lying vector mesons can be collected in a nonet. In the chiral limit this nonet is completely degenerate. Here this nonet (in the tensor representation) is encoded in  $V_{\mu\nu}$ , the Goldstone bosons in  $u_\mu$  and the external currents in  $f_+^{\mu\nu}$  (see [6] for details).

As shown in [6] the lowest-lying resonances practically saturate the low-energy coefficients. E.g. for  $L_1$ - $L_3$  one obtains (in leading order of  $1/N_c$ )

$$L_2 = 2L_1 = \frac{G_V^2}{4M_V^2} \quad (8)$$

and

$$L_3 = -\frac{3G_V^2}{4M_V^2} + \text{contributions from scalar mesons}. \quad (9)$$

Here  $M_V$  denotes the mass of the vector meson nonet in the chiral limit. Note that there are no contributions from scalar resonances to  $L_2$ . In [7] the question was addressed whether the results (8) and (9) depend on the details how the resonances are introduced, e.g. in which representation. It turned out that at least for vector mesons the results are model independent, if constraints from high-energy QCD are involved in addition. Even more, further relations between  $F_V$ ,  $G_V$  and  $F_0$  can be obtained by studying the electromagnetic and the axial form factor of the pion [7]: In the framework of (6) the former is given by

$$F(t) = 1 + \frac{F_V G_V}{F_0^2} \frac{t}{M_V^2 - t}. \quad (10)$$

Assuming that  $F(t)$  vanishes at infinity (a quite natural assumption for a form factor) one gets

$$F_V G_V = F_0^2. \quad (11)$$

The axial form factor describes the coupling of a Goldstone boson to a photon and an axial-vector current and controls e.g. the decay  $\pi^+ \rightarrow e^+ \nu \gamma$ . For a pion in the chiral limit it is given by

$$G_A(t) = \frac{F_V (2G_V - F_V)}{M_V^2} + \text{contributions from axial vectors}. \quad (12)$$

The vector meson contribution comes from the process where the photon transforms to a vector meson (coupling constant  $F_V$ ) which in turn couples to the pion and the axial-vector current with strength  $2G_V - F_V$ . Again, we demand that  $G_A$  vanishes at infinity. The additional contributions not displayed explicitly fulfill this requirement. Thus we obtain [7]

$$F_V = 2G_V \quad (13)$$

which finally leads to

$$F_V = 2G_V = \sqrt{2} F_0 \quad (14)$$

and therefore

$$L_2 = 2L_1 = \frac{F_0^2}{8M_V^2}. \quad (15)$$

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<sup>2</sup> Note that already here interactions between the resonances and the Goldstone bosons are included via a chirally covariant derivative. It is, however, subleading in  $1/N_c$  and not relevant for our purposes.

Since the tensor representation of vector mesons is rather unusual it is illuminating to translate the results into the language of a standard  $\rho\pi\pi$ - and  $\rho\gamma$ -Lagrangian [29]

$$\mathcal{L}_{\text{int}} = ig\rho^\mu (\pi^+\partial_\mu\pi^- - \pi^-\partial_\mu\pi^+) + g^2\rho^\mu\rho_\mu\pi^+\pi^- - \frac{e}{2g_\gamma}F_\rho^{\mu\nu}F_{\mu\nu}^A \quad (16)$$

where  $\rho^\mu$  denotes the neutral  $\rho$ -meson,  $\pi^\pm$  the charged pions,  $A_\mu$  the photon field,  $e$  the electromagnetic coupling and  $F_B^{\mu\nu}$  the field strength corresponding to the field  $B_\mu = \rho_\mu$  or  $A_\mu$ , respectively. The connections of  $G_V$  and  $F_V$  to the usual  $\rho\pi\pi$  and  $\rho\gamma$  couplings are provided by

$$g = \frac{G_V M_V}{F_0^2} \quad (17)$$

and

$$g_\gamma = \frac{M_V}{F_V}. \quad (18)$$

Relations (17) and (18) can be obtained by calculating the decay widths  $\Gamma(\rho \rightarrow \pi\pi)$  and  $\Gamma(\rho \rightarrow e^+e^-)$  in both approaches (6) and (16). The equations (14) and (15) translate to

$$g^2 = g_\gamma^2 = \frac{M_V^2}{2F_0^2} = \frac{1}{16L_2}, \quad (19)$$

i.e. we have obtained the universality of the  $\rho$ -meson coupling and the KSFR relation (in the chiral limit) [30, 31]. The fact that the  $\rho$ -meson couples (at least approximately) with the same strength to pions and photons constitutes one important aspect of the vector meson dominance picture (e.g. [29, 32, 33], see also comment in [7]). We note in passing that the relations (19) have also been obtained recently in a more general framework by demanding perturbative renormalizability in the sense of effective field theories of a Lagrangian involving pions,  $\rho$ -mesons, nucleons and photons [34, 35].

### III. EXTENSION TO THE VECTOR MESON NONET

In the previous section we have determined vector meson properties in the combined chiral and large- $N_c$  limit. In the present section we go beyond the chiral limit by systematically including terms in linear order in the quark masses (quadratic order in the Goldstone boson masses). Still we restrict ourselves to the large- $N_c$  limit to keep things manageable. In that framework the mass splitting of the lowest-lying vector meson nonet (and of other multiplets) was studied in [21] in the resonance saturation approach. Here we go beyond that work as we also include the splitting of the coupling constants of the vector nonet.

We shall first review the results from [21] as far as they concern the vector mesons: As already pointed out, in the large- $N_c$  limit the vector mesons can be collected in a nonet (for simplicity we drop Lorentz indices as long as possible):

$$R = \frac{1}{\sqrt{3}}R_0\mathbb{1} + \frac{1}{\sqrt{2}}R_i\lambda_i. \quad (20)$$

In the chiral limit this nonet would be completely degenerate. Splitting effects in linear order in the quark masses can be systematically included. Concerning the vector meson masses the free resonance Lagrangian is extended in the following way:

$$\mathcal{L}_{\text{free}} = \frac{1}{2}\text{tr}(\nabla R \cdot \nabla R - M_V^2 R^2) + e_m^V \text{tr}(\chi_+ R^2). \quad (21)$$

For our purposes  $\chi_+$  reduces to<sup>3</sup>

$$\chi_+ \rightarrow 4B_0\mathcal{M} = 4B_0 \begin{pmatrix} m_q & 0 & 0 \\ 0 & m_q & 0 \\ 0 & 0 & m_s \end{pmatrix} \quad (22)$$

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<sup>3</sup> We assume perfect isospin symmetry.

with [1, 2]

$$B_0 = -\frac{\langle \bar{q}q \rangle}{F_0^2}. \quad (23)$$

In the large- $N_c$  limit one obtains ideal mixing

$$R_\omega = R_{\text{non-strange}} = \frac{1}{\sqrt{3}}(R_8 + \sqrt{2}R_0) \quad (24a)$$

$$R_\phi = -R_{\text{strange}} = \frac{1}{\sqrt{3}}(\sqrt{2}R_8 - R_0) \quad (24b)$$

and the following mass splitting pattern for the vector meson masses:

$$M_\rho^2 = M_\omega^2 = M_V^2 - 4e_m^V M_\pi^2, \quad (25a)$$

$$M_\phi^2 = M_V^2 - 4e_m^V (2M_K^2 - M_\pi^2), \quad (25b)$$

$$M_{K^*}^2 = M_V^2 - 4e_m^V M_K^2. \quad (25c)$$

For more details we refer to [21]. Note that the  $\rho$ - and the  $\omega$ -meson are still degenerate. This degeneracy will also hold for the coupling constants to which we will turn next. It is only lifted beyond the large- $N_c$  limit [21].

So far we have only reviewed the results of [21]. Now we extend this approach by including also splitting terms for the couplings of the vector mesons to Goldstone bosons and external currents, i.e. we extend (7) to

$$\begin{aligned} \mathcal{L}_{\text{int}} = & \frac{1}{2\sqrt{2}} \left( F_V \text{tr}(V_{\mu\nu} f_+^{\mu\nu}) + \frac{1}{2} d_F \text{tr}(V_{\mu\nu} \{f_+^{\mu\nu}, \chi_+\}) + \frac{1}{2} f_F \text{tr}(V_{\mu\nu} [f_+^{\mu\nu}, \chi_+]) + i G_V \text{tr}(V_{\mu\nu} [u^\mu, u^\nu]) \right. \\ & \left. + \frac{i}{2} d_G \text{tr}(V_{\mu\nu} \{\chi_+, [u^\mu, u^\nu]\}) + \frac{i}{2} f_G \text{tr}(V_{\mu\nu} [\chi_+, [u^\mu, u^\nu]]) + 2i e_G \text{tr}(u^\mu V_{\mu\nu} u^\nu \chi_+) \right). \end{aligned} \quad (26)$$

This leads to a plenty of different coupling constants for various processes. In the following we will be concerned with the couplings of vector mesons to photons and the hadronic decays of vector mesons. This leads to

$$F_{\rho\gamma} = F_{\omega\gamma} = F_V + 4B_0 m_q d_F = F_V + 2M_\pi^2 d_F, \quad (27a)$$

$$F_{\phi\gamma} = F_V + 4B_0 m_s d_F = F_V + 2(2M_K^2 - M_\pi^2) d_F \quad (27b)$$

instead of the chiral limit value  $F_V$  and

$$G_{\rho\pi\pi} = G_V + 4B_0 m_q (d_G - e_G) = G_V + 2M_\pi^2 (d_G - e_G), \quad (28a)$$

$$G_{\rho KK} = G_{\omega KK} = G_V + 4B_0 m_q d_G - 4B_0 m_s e_G = G_V + 2M_\pi^2 (d_G + e_G) - 4M_K^2 e_G, \quad (28b)$$

$$G_{\phi KK} = G_V + 4B_0 m_s d_G - 4B_0 m_q e_G = G_V + 4M_K^2 d_G - 2M_\pi^2 (d_G + e_G), \quad (28c)$$

instead of  $G_V$ . Note that the explicit isospin factors, e.g. the fact that  $\rho$ -mesons couple differently to kaons and pions are not included in the definitions of the coupling constants. They have to be taken into account for the calculation of the respective process of interest (see below).

In addition, we will study the coupling  $A_{VG}$  of a neutral (photon-like) vector meson  $V$  to a Goldstone boson  $G$  and an axial-vector current. Such coupling constants are given by

$$A_{\rho\pi} = F_V - 2G_V + 4B_0 m_q (d_F - 2d_G + 2e_G) = F_V - 2G_V + 2M_\pi^2 (d_F - 2d_G + 2e_G), \quad (29a)$$

$$\begin{aligned} A_{\rho K} = A_{\omega K} &= F_V - 2G_V + 4B_0 m_q (d_F - 2d_G) + 8B_0 m_s e_G \\ &= F_V - 2G_V + 2M_\pi^2 (d_F - 2d_G - 2e_G) + 8M_K^2 e_G, \end{aligned} \quad (29b)$$

$$\begin{aligned} A_{\phi K} &= F_V - 2G_V + 4B_0 m_s (d_F - 2d_G) + 8B_0 m_q e_G \\ &= F_V - 2G_V + 4M_K^2 (d_F - 2d_G) - 2M_\pi^2 (d_F - 2d_G - 2e_G). \end{aligned} \quad (29c)$$

Above we have used the Gell-Mann–Oakes–Renner relations [2, 36]

$$-m_q \langle \bar{q}q \rangle = \frac{1}{2} F_0^2 M_\pi^2, \quad (30a)$$

$$-m_s \langle \bar{q}q \rangle = \frac{1}{2} F_0^2 (2M_K^2 - M_\pi^2), \quad (30b)$$

$$-\frac{1}{2} (m_s + m_q) \langle \bar{q}q \rangle = \frac{1}{2} F_0^2 M_K^2. \quad (30c)$$

Note that consistent with our approximation we have used

$$\langle \bar{q}q \rangle := \langle \bar{u}u \rangle = \langle \bar{d}d \rangle \approx \langle \bar{s}s \rangle \quad (31)$$

and

$$\frac{M_K^2}{M_\pi^2} \approx \frac{m_s + m_q}{2m_q}. \quad (32)$$

Like for the chiral limit discussed above, we can utilize the high-energy behavior of form factors to relate various coupling constants. We start our discussion with the pion. Its coupling to real and virtual photons is mediated by the  $\rho$ -meson only. (In contrast, the kaons involve  $\rho$ ,  $\omega$  and  $\phi$ , see below.) We study first the electromagnetic form factor. Generalizing (10) it is now given by

$$F(t) = 1 + \frac{F_{\rho\gamma} G_{\rho\pi\pi}}{F_\pi^2} \frac{t}{M_\rho^2 - t} \quad (33)$$

with the pion decay constant  $F_\pi$ . In the large- $N_c$  approximation the latter is related to its chiral limit value  $F_0$  by [2]

$$F_\pi^2 = F_0^2 + 8L_5 M_\pi^2 + o(M_\pi^4). \quad (34)$$

Demanding again that the form factor (33) vanishes at high energies we obtain

$$d_F G_V + (d_G - e_G) F_V = 4L_5 \quad (35)$$

in addition to the chiral limit relation (11).

For the axial form factor of the pion the contribution of the  $\rho$ -meson — which is a constant in energy and should therefore vanish — is proportional to  $A_{\rho\pi}$  as given in (29a). Thus we get

$$d_F - 2d_G + 2e_G = 0 \quad (36)$$

in addition to the chiral limit relation (13). Hence we can already relate some of the coupling constants to the low-energy parameter  $L_5$ :

$$d_F = \frac{2\sqrt{2}L_5}{F_0}, \quad d_G - e_G = \frac{\sqrt{2}L_5}{F_0}. \quad (37)$$

Next we turn to the kaons. Here the discussion becomes more subtle since  $\rho$ -,  $\omega$ - and  $\phi$ -mesons are involved. The demand for a proper high-energy behavior of the form factors constrains only the sum of the vector meson contributions. Apparently the relative weight of the vector meson contributions becomes an issue. Here the quark charges come into play and in particular their  $N_c$ -dependence. We have to work out first how the different flavor currents contribute to the electromagnetic current for an arbitrary number of colors. We introduce currents which correspond to  $\rho$ ,  $\omega$  and  $\phi$ :

isovector current

$$j_\mu^\rho := \frac{1}{2} (\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d), \quad (38)$$

non-strange isoscalar current

$$j_\mu^\omega := \frac{1}{2} (\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d), \quad (39)$$

strange isoscalar current

$$j_\mu^\phi := \frac{1}{\sqrt{2}} \bar{s}\gamma_\mu s. \quad (40)$$

We decompose the quark charges into weak isospin and hypercharge. In the standard model the quarks belong to the fundamental representation of weak isospin. This yields

$$Q_u = Y_q + \frac{1}{2}, \quad Q_d = Q_s = Y_q - \frac{1}{2}. \quad (41)$$

According to [23, 24, 25, 26] the weak hypercharge of the quarks is given by

$$Y_q = \frac{1}{2N_c}. \quad (42)$$

This assignment ensures the cancellation of anomalies in an electroweak theory with an arbitrary number of quark colors. One purpose of the present work is to demonstrate that one gets a similar (somewhat weaker) relation between hypercharge and number of colors within the vector meson framework outlined here. Therefore we shall not make use of (42) in the following. The electromagnetic current can now be decomposed:

$$j_\mu^{\text{el}} = Q_u \bar{u}\gamma_\mu u + Q_d \bar{d}\gamma_\mu d + Q_s \bar{s}\gamma_\mu s = (Q_u - Q_d) j_\mu^\rho + (Q_u + Q_d) j_\mu^\omega + \sqrt{2} Q_s j_\mu^\phi = j_\mu^\rho + 2Y_q j_\mu^\omega + \frac{1}{\sqrt{2}} (2Y_q - 1) j_\mu^\phi. \quad (43)$$

Demanding that the electromagnetic form factor of the charged kaon vanishes at high energies leads to

$$F_{\rho\gamma} G_{\rho KK} + 2Y_q F_{\omega\gamma} G_{\omega KK} + \frac{1}{\sqrt{2}} (2Y_q - 1) F_{\phi\gamma} (-\sqrt{2}) G_{\phi KK} = 2F_K^2 \quad (44)$$

with the kaon decay constant  $F_K$ . The factor  $(-\sqrt{2})$  in front of  $G_{\phi KK}$  accounts for the different coupling of the kaon pair to  $\rho$  and  $\phi$ , respectively (isospin). Note that relation (44) is only correct in leading order of the  $1/N_c$ -expansion. In this limit the kaon decay constant is given by (cf. (34)) [2]

$$F_K^2 = F_0^2 + 8L_5 M_K^2. \quad (45)$$

Using all knowledge obtained so far we get

$$Y_q B_0 (m_s - m_q) (G_V d_F + F_V d_G + F_V e_G) = 0. \quad (46)$$

In particular, all terms which are not proportional to  $Y_q$  have vanished on account of the corresponding relation (35) for the pion. There are two possible solutions to equation (46) — if one disregards  $B_0 = 0$  and  $m_s = m_q$ . Either the terms in the brackets vanish,

$$G_V d_F + F_V d_G + F_V e_G \stackrel{?}{=} 0, \quad (47)$$

or  $Y_q$ . Since we work in leading order of the  $1/N_c$ -expansion the latter possibility implies

$$Y_q \stackrel{?}{=} o(1/N_c). \quad (48)$$

Obviously, this would be in agreement with (42), but a somewhat weaker statement. In the following, we will collect further evidence that (48) is right and (47) wrong.

It is tempting to discuss next the electromagnetic form factor of the *uncharged* kaon. However, we refrain from involving it. The reason is the following: From the point of view of chiral perturbation theory all orders contribute to the form factors for the charged Goldstone bosons whereas the leading order does not contribute to the form factor of the uncharged kaon. The constraints which we derive for the vector meson coupling constants emerge from a cancellation between the leading and the next-to-leading order contribution.<sup>4</sup> To derive further relations from the electromagnetic form factor of the uncharged kaon one should involve also at least two orders in the chiral expansion which is beyond the scope of the present work.

We turn to the axial form factor of the charged kaon. In the by now usual way we get

$$\frac{1}{M_\rho^2} F_{\rho\gamma} A_{\rho K} + \frac{1}{M_\omega^2} 2Y_q F_{\omega\gamma} A_{\omega K} + \frac{1}{M_\phi^2} \frac{1}{\sqrt{2}} (2Y_q - 1) F_{\phi\gamma} (-\sqrt{2}) A_{\phi K} = 0. \quad (49)$$

Using (13) and (29) we observe that all the coupling constants  $A_{\dots}$  are already of linear order in the current quark masses. Therefore, the vector meson masses and the coupling constants  $F_{\dots}$  in (49) can be taken in the chiral limit. Finally we obtain

$$Y_q B_0 (m_s - m_q) (d_F - 2d_G - 2e_G) = 0. \quad (50)$$

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<sup>4</sup> Of course, the constraints emerge for high energies, i.e. outside the realm of strict chiral perturbation theory. This is accounted for by using the full propagator structure of the vector mesons instead of a heavy vector meson mass expansion.

Again, it is the corresponding condition (36) for the pion which causes all terms to vanish which are not proportional to  $Y_q$ . Again we conclude that (48) holds or

$$d_F - 2d_G - 2e_G \stackrel{?}{=} 0. \quad (51)$$

Suppose for a moment that (48) does *not* hold. In this case, we can use (14), (37), (47) and (51) to obtain

$$L_5 = d_F = d_G = e_G = 0 \quad (\text{presumably wrong!}) \quad (52)$$

From a principal point of view this possibility cannot be excluded. On the other hand, there is up to now no QCD-motivated reason known why the low-energy constant  $L_5$  should vanish. In fact, the experimental facts suggest that  $L_5 \neq 0$  and has the same order of magnitude as all other  $L_i$ 's given in (5) [2]. Thus we conclude that either (47) or (51) is wrong (or both) which inevitably leads to

$$Y_q = o(1/N_c). \quad (53)$$

The situation can be summarized as following: In the chiral limit, (44) and (49) are always fulfilled, irrespective of the  $N_c$ -dependence of the weak hypercharge. Beyond the chiral limit we could deduce the scaling (53) but not the exact relation (42). It would be interesting to extend the approach presented here beyond the leading order of  $1/N_c$ . It might then be possible to connect the weak hypercharge to hadronic quantities, e.g. to  $M_V^2/F_0^2$ . In this way the weak hypercharge, or to phrase it differently: the number of colors, could be determined within a hadronic framework (cf. the discussions in [23, 37, 38]). Such an extension, however, is beyond the scope of the present work.

We still have one form factor to further constrain our coupling constants, namely the axial form factor for the neutral kaons. We get

$$\frac{1}{M_\rho^2} F_{\rho\gamma} A_{\rho K} + \frac{1}{M_\omega^2} 2Y_q F_{\omega\gamma} (-1) A_{\omega K} + \frac{1}{M_\phi^2} \frac{1}{\sqrt{2}} (2Y_q - 1) F_{\phi\gamma} \sqrt{2} A_{\phi K} = 0. \quad (54)$$

Note the sign changes relative to (49). This leads to

$$(1 - 2Y_q)(d_F - 2d_G - 2e_G) = 0. \quad (55)$$

We ignore the possibility  $Y_q = 1/2$  and conclude

$$d_F - 2d_G - 2e_G = 0. \quad (56)$$

Together with (37) we get

$$d_F = \frac{2\sqrt{2}L_5}{F_0}, \quad d_G = \frac{\sqrt{2}L_5}{F_0}, \quad e_G = 0. \quad (57)$$

To summarize we have obtained a consistent picture by adopting the arguments of [7] concerning the form factors and extending them beyond the chiral limit. It was important to introduce the correct dependence (53) of the hypercharge on  $N_c$ . Only in this way the equations for the kaon form factors produce the same results as the ones for the pion form factors.

Starting with five initially unknown constants  $d_F$ ,  $d_G$ ,  $e_G$ ,  $f_F$  and  $f_G$  in (26) we have managed to determine the first three of them. It is actually not surprising that we got no constraints on the other two constants: All form factors involved only the neutral vector mesons (due to their coupling to photons). If the flavor matrix  $V_{\mu\nu}$  has only diagonal entries, it commutes with the mass matrix in (22). Therefore, the  $f_F$ - and  $f_G$ -terms in (26) vanish in such cases. In principle,  $f_G$  influences the hadronic width of the  $K^*$ , while  $f_F$  appears in the coupling of the  $K^*$  to the quark current  $\bar{u}\gamma_\mu s$  which in turn is important e.g. for the  $\tau$ -decay and for QCD sum rules [39, 40]. We postpone the determination of these parameters to future work.

In the next section we will determine the coupling constants (27) and (28) as far as they are easily accessible from experiment. We will confront the obtained results with our relations (57).

#### IV. EXPERIMENTAL RESULTS FOR VECTOR MESON MASSES AND DECAYS

In (25) the masses of the vector mesons are expressed in terms of the two parameters  $M_V$  and  $e_m^V$ . The former is the vector meson mass in the chiral limit and the latter parameterizes the splitting pattern. We fit these two parameters

to the experimental values of the vector meson masses. The results are shown in table I. Obviously a very good fit is obtained with an average deviation of the squared masses from the fit of only 1.5%.

Again we stress that this is by no means a new result [21]. The new aspects concern the decay properties of the vector mesons to which we turn now. Indeed, in terms of only two parameters the relations (57), (27), (28), (34) and (45) determine the hadronic and electromagnetic coupling constants of the neutral vector mesons and the pion and kaon decay constant, in total — as we will see — seven quantities. In the following, we will present the formulae for the vector meson decays. Subsequently we will fit our two free parameters to two of these decay widths and predict the other ones.

The dilepton decay widths are given by

$$\Gamma(\rho \rightarrow e^+e^-) = \frac{4\pi\alpha^2}{3} \frac{F_{\rho\gamma}^2}{M_\rho}, \quad (58a)$$

$$\Gamma(\omega \rightarrow e^+e^-) = \frac{4\pi\alpha^2}{3} \frac{1}{9} \frac{F_{\omega\gamma}^2}{M_\omega}, \quad (58b)$$

$$\Gamma(\phi \rightarrow e^+e^-) = \frac{4\pi\alpha^2}{3} \frac{2}{9} \frac{F_{\phi\gamma}^2}{M_\phi}. \quad (58c)$$

with the electromagnetic fine structure constant  $\alpha \approx 1/137$ . The different factors in (58) are caused by the different contributions of the currents (38)-(40) to the electromagnetic current (43). Recall that the weight factors are  $\frac{1}{N_c} = \frac{1}{3}$  for the  $\omega$  and  $\frac{1}{\sqrt{2}}(\frac{1}{N_c} - 1) = -\frac{\sqrt{2}}{3}$  for the  $\phi$ . Our strategy will be to calculate  $F_{i\gamma}$  for  $i = \rho, \omega, \phi$  in our approach and compare these values to the experimental ones obtained from (58).

Next we turn to the hadronic decay widths. We consider the decays of the vector mesons into two Goldstone bosons. The  $\omega$  decays dominantly into three pions due to an anomalous coupling. We do not consider this effect here. The  $\omega$ -decay into two pions caused by  $\rho$ - $\omega$  mixing is beyond the leading  $1/N_c$  effects and therefore also not included. We get

$$\Gamma(\rho^0 \rightarrow \pi^+\pi^-) = \frac{1}{48\pi} \frac{G_{\rho\pi\pi}^2 M_\rho^2}{F_\pi^4} M_\rho \left(1 - \frac{4M_\pi^2}{M_\rho^2}\right)^{3/2}, \quad (59a)$$

$$\Gamma(\phi \rightarrow K^+K^-) = \frac{1}{96\pi} \frac{G_{\phi KK}^2 M_\phi^2}{F_K^4} M_\phi \left(1 - \frac{4M_K^2}{M_\phi^2}\right)^{3/2}. \quad (59b)$$

Again, we will calculate the coupling constants  $G_{\dots}$  in our approach and compare the results to the experimental ones obtained from (59).

Note that the hadronic decay widths determined above are  $O(1/N_c)$ , i.e. suppressed, as it should be [27]. This can be most easily seen by recalling that the vector meson masses are  $O(1)$  and the pion and kaon decay constants  $O(\sqrt{N_c})$ . Suppressed decay widths mean that the vector mesons are stable in the large- $N_c$  limit [27]. In turn, this indicates that our large- $N_c$  description should work best for narrow states and less accurate for broad resonances. Therefore we determine our free parameters from fits to partial decay widths of the most narrow states.

As free parameters we take the pion decay constant in the chiral limit,  $F_0$ , and the splitting parameter  $d_F$  and fit them to the dilepton decay widths of the narrow states  $\omega$  and  $\phi$ . Results are shown in table I. Obviously, the results show an overall very satisfying agreement. In the following, we shall discuss these results in more detail.

Concerning the value of the pion decay constant in the chiral limit  $F_0$  we observe that it is on the upper end of the (rather large) allowed range. Note that the chiral limit value obtained in [1] concerns the  $SU(2)$  chiral limit where the up and down quark masses are sent to zero while the strange quark mass is kept fixed. Here we deal with the  $SU(3)$  chiral limit studied also in [2].

From (27a) it is obvious that  $F_{\rho\gamma}$  and  $F_{\omega\gamma}$  agree in the large- $N_c$  limit. On the other hand, the experimental value of  $F_{\rho\gamma}$  deviates from  $F_{\omega\gamma}$  by about 10%. One can take different points of view concerning this deviation: In principle, one could be satisfied with a 10% accuracy of a large- $N_c$  approach. On the other hand, we see that on average our fit is even much better. We recall that one can expect to find a better description of narrow resonances within a large- $N_c$  treatment. In total, concerning the decay widths of the vector mesons, we observe that our values for the coupling constants of the  $\rho$ -meson (which is the broadest resonance!) show the largest deviations from experiment (about 10%). All the other coupling constants are reproduced extremely well.

Our value for  $L_5$  is somewhat low. On the other hand, the value for  $L_5$  depends on the renormalization scale. This dependence is subleading in  $N_c$ . From the practical point of view, however, the dependence is rather large: The central value given in table I varies from 2.4 to 0.8, if the renormalization point changes from 0.5 to 1 GeV. More

quantity	theory	experiment	ref.
$M_V$ [GeV]	0.766	–	
$e_m^V$	-0.233	–	
$M_\rho$ [GeV]	0.778	0.776	[28]
$M_\omega$ [GeV]	0.778	0.783	[28]
$M_{K^*}$ [GeV]	0.902	0.892	[28]
$M_\phi$ [GeV]	1.012	1.019	[28]
$F_0$ [GeV]	0.0971	$0.086 \pm 0.010$	[2]
$d_F$ [1/GeV]	0.0295	–	
$F_{\rho\gamma}$ [GeV]	0.138	0.154	[28]
$F_{\omega\gamma}$ [GeV]	0.138	0.138	[28]
$F_{\phi\gamma}$ [GeV]	0.162	0.162	[28]
$G_{\rho\pi\pi}$ [GeV]	0.0692	0.0650	[28]
$G_{\phi KK}$ [GeV]	0.0808	0.0793	[28]
$L_1$ [ $10^{-3}$ ]	1.0	$0.37 \pm 0.23$	[41]
$L_2$ [ $10^{-3}$ ]	2.0	$1.35 \pm 0.23$	[41]
$L_5$ [ $10^{-3}$ ]	0.89	$1.4 \pm 0.5$	[2]
$F_\pi$ [GeV]	0.0978	0.0924	[28]
$F_K$ [GeV]	0.106	0.113	[28]

TABLE I: Properties of the vector meson nonet. The free parameters  $F_0$  and  $d_F$  were fitted to  $F_{\omega\gamma}$  and  $F_{\phi\gamma}$ . See main text for more details.

general, the values for the calculated  $L_i$ 's can easily change by a factor of two or more if the renormalization point changes from 0.5 to 1 GeV [6]. Therefore, our somewhat large disagreement is not really significant.

Our values for the pion and kaon decay constants deviate less than 10% from the experimental value. We have obtained the correct sign for the splitting between the decay constants (which is already not trivial). Quantitatively, however, the splitting between  $F_\pi$  and  $F_K$  is underestimated. This can be traced back to the too small value for  $L_5$  which we have already discussed. Obviously, the physical values for the pion and kaon decay constants are significantly influenced by chiral logs which are formally subleading in  $1/N_c$ .

In general, we can be rather satisfied with our approach based on chiral symmetry and the large- $N_c$  approximation. We close this section with some comments concerning the comparison to some other approaches on the properties of vector mesons. First of all, it is important to stress that the splitting pattern of the coupling constant is experimentally significant: E.g. the deviations between the experimental values of the different  $G_i$ 's are larger than 10%. The same is true for the different  $F_i$ 's. In approaches where  $e_f^V$  and  $e_g^V$  are neglected (e.g. [42]) these experimental differences must be generated by Goldstone boson loops, i.e. by subleading orders in  $1/N_c$ . In principle, we see no reason why there should be no splitting in leading  $N_c$  order.

There is one case where a rather specific splitting pattern is introduced, namely if the vector mesons are introduced as gauge bosons in one or the other way [43, 44, 45]. (Note that this is not the approach used here.) In this case the following combinations would all be equal to the coupling constant  $g^2$  (cf. (17)):

$$\frac{G_{\rho\pi\pi}^2 M_\rho^2}{F_\pi^4}, \frac{G_{\phi KK}^2 M_\phi^2}{F_K^4}, \quad (60)$$

and the following combinations would all be equal to the coupling constant  $g_\gamma^2$  (cf. (18)):

$$\frac{M_\rho^2}{F_{\rho\gamma}^2}, \frac{M_\omega^2}{F_{\omega\gamma}^2}, \frac{M_\phi^2}{F_{\phi\gamma}^2}. \quad (61)$$

Note that these combinations appear as coefficients of the various decay widths (58) and (59). The experimental values for these coefficients are not really all equal as can be seen e.g. in the tables of [29]. Our splitting pattern is parametrically very different from these gauge boson approaches. We conclude that our successful description of the splitting pattern of the vector meson coupling constants casts some doubts on the introduction of vector mesons as gauge bosons. In principle, one might argue that a universal coupling constant is more economical than our approach

or, to argue the other way around, that it is not surprising that we get a better description of the data since we have more free parameters. This, however, is not true: Any approach has to introduce  $F_0$  and  $L_5$  as free parameters. Since our splitting parameters for the coupling constants are related to each other and to  $L_5$  and  $F_0$  by (57) we do not have more parameters than other approaches.

## V. SUMMARY

We have used the resonance saturation approach to determine properties of the whole lowest-lying vector meson nonet. For that purpose the approach was extended here beyond the chiral limit [6, 7] not only for the vector meson masses [21] but also for their coupling constants.

From the conceptual point of view we have found that the correct assignment of the  $N_c$ -dependence of quark charges is mandatory to obtain a consistent picture within a basically purely hadronic approach. For the future this might open the possibility to determine the number of colors in a hadronic framework. In principle, such a perspective does not come unexpected: In vector meson dominance approaches with universal vector meson coupling constants the coupling of the  $\rho$ -meson e.g. to nucleons is suppressed by  $1/N_c$  as compared to the  $\omega$ -meson (see e.g. [43] and also the appendix of [46]). This is just the factor  $Q_u + Q_d = 2Y_q$  which appeared in (43).

From the quantitative point of view the following was achieved in the present work: With four input parameters we determined the vector meson nonet masses, the decay constants of pion and kaon, and the coupling constants for the decays of the neutral vector mesons into dileptons and two Goldstone bosons. In general, we obtained very satisfying results within our approach. In no case the deviation was larger than 10%; usually it turned out to be much lower.

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