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# Deformed Spectral Representation of the BFKL Kernel and the Bootstrap for Gluon Reggeization

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#### Abstract

We investigate the space of functions in which the BFKL kernel acts. For the amplitudes which describe the scattering of colorless projectiles it is convenient to define, in transverse coordinates, the Möbius space in which the solutions to the BFKL equation vanish as the coordinates of the two reggeized gluons coincide. However, in order to fulfill the bootstrap relation for the BFKL kernel it is necessary to modify the space of functions. We define and investigate a new space of functions and show explicitly that the bootstrap relation is valid for the corresponding spectral form of the kernel. We calculate the generators of the resulting deformed representation of the  $\mathfrak{sl}(2,\mathbb{C})$ algebra.

# 1 Introduction

The leading order BFKL kernel [1], derived from Feynman diagrams in momentum space, has been investigated in much detail. As a function of the transverse momenta, it is a meromorphic function. In the color singlet exchange channel, it describes the Pomeron contribution in pQCD, consisting of ladder diagrams with reggeized gluons. This BFKL Pomeron couples to the impact factors of colourless particles which, because of gauge invariance, vanish as one of the two reggeized gluons carries a zero transverse momentum. This property allows us to modify the space of functions to which the Pomeron wave function belongs. On the other hand, in the color octet exchange channel the BFKL equation has the bootstrap solution, which represents a fundamental consistency property derived from the s-channel unitarity. A manifestation of this bootstrap property also takes place for the color singlet state of two gluons and inside the coupling of three or more gluons to colorless particle impact factors. In particular, it plays an important role in the Odderon solution [2] which appears as a bound state of three reggeized gluons. An important virtue of the color singlet kernel is its invariance under the Möbius transformations in the space of transverse coordinates. Exploiting the gauge invariance of the colorless impact factor to which the BFKL Pomeron couples, the Möbius symmetry allows to search solutions of the BFKL eigenvalues equation in the space of Möbius functions which vanish as the coordinates of the two reggeized gluons coincide, and to define a spectral representation of the BFKL kernel in terms of conformal eigenfunctions. In this space of functions, the BFKL kernel also enjoys the property of holomorphic separability. Translating back to the momentum representation, the kernel in the corresponding spectral form acquires  $\delta$  function - like pieces which, because of the special properties of the impact factors, do not contribute to physical amplitudes.

As mentioned above, the BFKL kernel, when applied to the color octet wave function in momentum space, satisfies the bootstrap condition, which reflects the s-channel consistency of the BFKL calculation. When transforming to transverse coordinates one finds that, in the space of Möbius functions, this bootstrap condition cannot be satisfied, i.e. the colour octet wave function lies outside this space of functions. In this paper we will define a similarity transform,  $\Phi$ , which takes us from the Möbius space of functions (M space) to another space of functions (named 'analytic Feynman diagram' (AF) space) in which the bootstrap holds. With the same transformation we can also define 'deformed' Möbius transformations and a 'deformed' spectral representation of the BFKL kernel.

The paper is organized as follows. In the following section we review the properties of the BFKL kernel, both in momentum space and in the space of transverse coordinates. In section 3 we introduce the deformed representation, and we define the similarity transform which takes us from the Möbius space (M space) to the new deformed space (AF space) of functions. In section 4 we explicitly show that, in the AF representation, the bootstrap properties are fulfilled. Section 5 is devoted to the deformed representation of the conformal algebra which follows from the particular choice of the scalar product; in particular we compute the transformed generators of the Möbius group  $\mathfrak{sl}(2, \mathbb{C})$ . Some details of our calculations are collected in two appendices.

### 2 The BFKL equation and bootstrap relation in LLA

In this section we give a brief review of the BFKL approach describing the dynamics of the reggeized gluons in LLA of perturbative QCD. Let us start from the Schrödinger-like BFKL equation [1] describing the compound state of two reggeized gluons,

$$K_2^{(R)} \otimes \psi_E = E \,\psi_E, \quad K_2^{(R)} = -\frac{N_c}{2} \left(\tilde{\omega}_1 + \tilde{\omega}_2\right) - \lambda_R \bar{V}_{12}.$$
 (1)

Here R labels the colour representation of the two gluon state and in the singlet and octet channel one has respectively  $\lambda_1 = N_c$  and  $\lambda_8 = N_c/2$ . The symbol  $\otimes$  denotes an integration in the transverse space in the case of the integral operator  $\bar{V}_{12}$ , while  $E = -\omega = 1 - j$ where j is the t-channel angular momentum. The eigenvalues of  $K_2^R$  give the positions of singularities of the t channel partial waves, related to the scattering amplitude by the Mellin transformation in the variable  $\ln s$ , where s is the squared total energy of colliding particles. In LLA, in the momentum representation, the gluon trajectory (scaled by  $N_c/2$ ) is given by the well known expression

$$\tilde{\omega}_i = \tilde{\omega}(\boldsymbol{k}_i) = -c \int d^2 \boldsymbol{l} \, \frac{\boldsymbol{k}_i^2}{\boldsymbol{l}^2 (\boldsymbol{k}_i - \boldsymbol{l})^2}, \quad c = \frac{g^2}{(2\pi)^3}$$
(2)

and the interaction term is defined by its action on the wave functions

$$\bar{V}_{12} \otimes \phi(\boldsymbol{k}_1, \boldsymbol{q} - \boldsymbol{k}_1) = \int d^2 \boldsymbol{k}_1' \, \bar{V}(\boldsymbol{k}_1, \boldsymbol{q} - \boldsymbol{k}_1 | \boldsymbol{k}_1', \boldsymbol{q} - \boldsymbol{k}_1') \phi(\boldsymbol{k}_1', \boldsymbol{q} - \boldsymbol{k}_1')$$
(3)

$$= c \int d^2 \mathbf{k}_1' \left[ \frac{\mathbf{k}_1^2}{(\mathbf{k}_1')^2 (\mathbf{k}_1 - \mathbf{k}_1')^2} + \frac{(\mathbf{q} - \mathbf{k}_1)^2}{(\mathbf{q} - \mathbf{k}_1')^2 (\mathbf{k}_1 - \mathbf{k}_1')^2} - \frac{\mathbf{q}^2}{(\mathbf{k}_1')^2 (\mathbf{q} - \mathbf{k}_1')^2} \right] \phi(\mathbf{k}_1', \mathbf{q} - \mathbf{k}_1')$$

and  $\boldsymbol{q} = \boldsymbol{k}_1 + \boldsymbol{k}_2$ . Let us note that this form is proper when the kernel acts on amputated (without gluon propagators) impact factors, otherwise the propagators should be removed from the kernel, as for example is the case for the expression in eq. (9).

We stress that this form of the LL BFKL kernel is obtained directly from the Feynman diagrams analysis in perturbative QCD and has a simple analytic behavior in the transverse momentum space. In particular it does not contain  $\delta$ -functions  $\delta^2(k'_1)$  and  $\delta^2(q-k'_1)$ .

In the construction of the BFKL kernels a very important property is the gluon reggeization which can be verified with the bootstrap relation for the amplitude with the octet quantum numbers [1]. The bootstrap relation is a consequence of the *s*-channel unitarity. It claims, that the scattering amplitude with octet quantum numbers obtained as a solution of the corresponding BFKL equation should coincide with the Born term multiplied by the Regge factor  $s^{\tilde{\omega}(\boldsymbol{q})}$ .

Because the BFKL equation was obtained by summing contributions from the multiparticle production described by the multi-Regge amplitudes, the bootstrap relation valid in the leading and next-to-leading orders allows to connect the gluon trajectory and interaction terms. In LLA the bootstrap condition for the BFKL kernel can be written as

$$\omega(\boldsymbol{q}) - \omega(\boldsymbol{k}_1) - \omega(\boldsymbol{k}_2) = \bar{V}_{12} \otimes 1 \quad \text{or} \quad \bar{K}_{12}^{(8)} \otimes 1 = -\omega(\boldsymbol{q}), \qquad (4)$$

where the constant 1 is the wave function which can be conveniently obtained after rescaling any function depending only on  $\boldsymbol{q}$ , and the kernel  $K_{12}^{(8)}$  acts on the amputated amplitudes. Let us note that it is crucial in the bootstrap relation in LLA that the two gluons are located at the same point in the transverse coordinate plane. It has played a crucial role in the discovery of the leading Odderon solution in the LLA of perturbative QCD [2] and also some general relation [3] between bound states of n and n + 1 reggeized gluons.

In the following we shall use its equivalent form [3], which has the virtue of being infrared finite and which uses the standard BFKL singlet kernel:

$$\bar{K}_{12}^{(1)} \otimes 1 = -2\omega(\boldsymbol{q}) + \omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2) \,. \tag{5}$$

On using an infrared mass regularization  $(m \to 0)$  one may write

$$\omega(\mathbf{k}) = -\frac{1}{2}\bar{\alpha}_s \log\left(\frac{\mathbf{k}^2}{m^2}\right), \qquad (6)$$

where  $\bar{\alpha}_s = \alpha_s N_c / \pi$  and  $\alpha_s = g^2 / (4\pi)$ , so that a relation equivalent to the bootstrap, but explicitly infraredly finite, is given by

$$\bar{K}_{12}^{(1)} \otimes 1 = \frac{1}{2} \bar{\alpha}_s \log \left( \frac{\boldsymbol{q}^4}{\boldsymbol{k}_1^2 \boldsymbol{k}_2^2} \right) \,. \tag{7}$$

The 2-gluon kernel (1) in the singlet channel has been investigated in great details in the coordinate representation [4]. The amplitude for the scattering of colorless objects is factorized, in the high energy limit, in the product of the Green's function (which exponentiates the BFKL kernel) and two impact factors. The impact factors vanish as the momentum of one of the attached reggeized gluons goes to zero. This permits to choose a special representation for the two-gluon propagator and for the full BFKL kernel acting in the space of the so called Möbius functions (for two gluon states these are functions  $f(\rho_1, \rho_2)$  such that  $f(\rho, \rho) = 0$ ) [5]. This is a special choice among infinite others compatible with the gauge freedom of integrating a colorless impact factor with any function belonging to the set defined by the equivalence relation

$$f(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \sim \tilde{f}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = f(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) + f^{(1)}(\boldsymbol{\rho}_1) + f^{(2)}(\boldsymbol{\rho}_2), \qquad (8)$$

since the corresponding shift in the momentum representation is proportional to  $\delta^{(2)}(\boldsymbol{p}_i)$ .

In such a representation the operators are Möbius (conformal) invariant [4]. Using complex coordinates, the BFKL hamiltonian  $H_{12} = K_{12}^{(1)} 2/\bar{\alpha}_s$  acting on functions with propagators included can be written in the operator form [6]

$$H_{12} = \ln |p_1|^2 + \ln |p_2|^2 + \frac{1}{p_1 p_2^*} \ln |\rho_{12}|^2 p_1 p_2^* + \frac{1}{p_1^* p_2} \ln |\rho_{12}|^2 p_1^* p_2 - 4\Psi(1), \qquad (9)$$

where  $\Psi(x) = d \ln \Gamma(x)/dx$ , and we introduced the gluon holomorphic momenta  $p_r = i\partial/\partial \rho_r$ . On the Möbius space of functions the holomorphic separability applies:

$$H_{12} = h_{12} + h_{12}^*, \ h_{12} = \sum_{r=1}^2 \left( \ln p_r + \frac{1}{p_r} \ln(\rho_{12}) p_r - \Psi(1) \right), \tag{10}$$

and it is possible to find more easily the solutions of the homogeneous BFKL pomeron equation which belong to irreducible unitary representations of the Möbius group. In particular the symmetry generators in this representation are

$$M_r^3 = \rho_r \partial_r , \ M_r^+ = \partial_r , \ M_r^- = -\rho_r^2 \partial_r .$$
 (11)

For two reggeized gluons one has  $M^k = \sum_{r=1}^2 M_r^k$  and the Casimir operator is defined as follows

$$M^{2} = |\vec{M}|^{2} = -\rho_{12}^{2} \partial_{1} \partial_{2}, \qquad (12)$$

where  $\vec{M} = \sum_{r=1}^{2} \vec{M}_r$  and  $\vec{M}_r \equiv (M_r^+, M_r^-, M_r^3)$ . The eigenfunctions of the BFKL kernel are also eigenstates of two Casimir operators and

The eigenfunctions of the BFKL kernel are also eigenstates of two Casimir operators and given by [4]

$$E_{h,\bar{h}}(\boldsymbol{\rho}_{10},\,\boldsymbol{\rho}_{20}) \equiv \langle \rho | h \rangle = \left(\frac{\rho_{12}}{\rho_{10}\rho_{20}}\right)^h \left(\frac{\rho_{12}^*}{\rho_{10}^*\rho_{20}^*}\right)^h \,, \tag{13}$$

where  $h = \frac{1+n}{2} + i\nu$ ,  $\bar{h} = \frac{1-n}{2} + i\nu$  are conformal weights for the principal series of the unitary representations of the Möbius group, n is the conformal spin and  $d = 1 - 2i\nu$  is the anomalous dimension of the operator  $O_{h,\bar{h}}(\rho_0)$  describing the compound state of two reggeized gluons (note, that here and below we use other notations for conformal weights  $m, \tilde{m}$  in Refs. [4, 7]). The corresponding eigenvalues of the BFKL kernel are given by

$$\chi_h \equiv \chi(\nu, n) = \bar{\alpha}_s \left( \psi(\frac{1+|n|}{2}+i\nu) + \psi(\frac{1+|n|}{2}-i\nu) - 2\psi(1) \right) = \bar{\alpha}_s \,\epsilon_h \,. \tag{14}$$

The action of the BFKL kernel can also be written, again on the space of Möbius functions, after a duality transformation [6, 8, 5], in terms of integral operator appearing in the dipole picture [9]

$$H_{12} f_{\omega}(\boldsymbol{\rho}_{1}, \boldsymbol{\rho}_{2}) = \int \frac{d^{2} \boldsymbol{\rho}_{3}}{\pi} \frac{|\boldsymbol{\rho}_{12}|^{2}}{|\boldsymbol{\rho}_{13}|^{2} |\boldsymbol{\rho}_{23}|^{2}} \left( f_{\omega}(\boldsymbol{\rho}_{1}, \boldsymbol{\rho}_{2}) - f_{\omega}(\boldsymbol{\rho}_{1}, \boldsymbol{\rho}_{3}) - f_{\omega}(\boldsymbol{\rho}_{2}, \boldsymbol{\rho}_{3}) \right) .$$
(15)

We now consider the pomeron eigenstates in the momentum representation. Different forms of the Fourier transform of the function in eq. (13) have been given in the most general case. One finds a sum of an analytic term  $\tilde{E}^{A}_{h,\bar{h}}$ , which has been given in an explicit way or in a integral form, and a  $\delta$ -like one  $\tilde{E}^{\delta}_{h,\bar{h}}$ . More precisely we obtain

$$\tilde{E}_{h,\bar{h}}(\boldsymbol{k}_{1},\boldsymbol{k}_{2}) = \int \frac{d^{2}\boldsymbol{r}_{1}}{(2\pi)^{2}} \frac{d^{2}\boldsymbol{r}_{2}}{(2\pi)^{2}} \left(\frac{r_{12}}{r_{1}r_{2}}\right)^{h} \left(\frac{r_{12}^{*}}{r_{1}^{*}r_{2}^{*}}\right)^{\bar{h}} e^{i(\boldsymbol{k}_{1}\cdot\boldsymbol{r}_{1}+\boldsymbol{k}_{2}\cdot\boldsymbol{r}_{2})} = \tilde{E}_{h,\bar{h}}^{A}(\boldsymbol{k}_{1},\boldsymbol{k}_{2}) + \tilde{E}_{h,\bar{h}}^{\delta}(\boldsymbol{k}_{1},\boldsymbol{k}_{2}) \\
= \langle k|h\rangle = \langle k|h^{A}\rangle + \langle k|h^{\delta}\rangle,$$
(16)

where a bra-ket compact notation is also introduced. By explicit computation one finds

$$\tilde{E}_{h,\bar{h}}^{\delta}(\boldsymbol{k}_{1},\boldsymbol{k}_{2}) = \left[\delta^{(2)}(\boldsymbol{k}_{1}) + (-1)^{n}\delta^{(2)}(\boldsymbol{k}_{2})\right]\frac{i^{n}}{2\pi}2^{1-h-\bar{h}}\frac{\Gamma(1-h)}{\Gamma(h)}q^{\bar{h}-1}q^{*h-1}, \quad (17)$$

where the complex form  $v = v_x + iv_y$  of any bidimensional transverse vector  $\boldsymbol{v} = (v_x, v_y)$ (coordinate or momentum) is used. In the calculation of a physical cross section one performs integrations with two-gluon colourless impact factors and this  $\delta$ -like term gives zero contribution, so that all the physics is related to the analytic term. In fact the  $\delta$ -terms have been introduced when one has chosen to move to the Möbius representation which is characterized by the simple and beautiful form in the coordinate representation.

The analytic part can be written, for example, as given in [10] wherein one finds

$$\tilde{E}^{A}_{h\bar{h}}(\boldsymbol{k}_{1},\boldsymbol{k}_{2}) = C_{v}\Big(X(\boldsymbol{k}_{1},\boldsymbol{k}_{2}) + (-1)^{n}X(\boldsymbol{k}_{2},\boldsymbol{k}_{1})\Big),$$
(18)

with the coefficient  $C_v$  given by

$$C_v = \frac{(-i)^n}{(4\pi)^2} h\bar{h}(1-h)(1-\bar{h})\Gamma(1-h)\Gamma(1-\bar{h}).$$
(19)

The expression for X in complex notation was written in terms of the hypergeometric functions

$$X(\boldsymbol{k}_1, \boldsymbol{k}_2) = \left(\frac{k_1}{2}\right)^{\bar{h}-2} \left(\frac{k_2^*}{2}\right)^{h-2} {}_2F_1\left(\begin{array}{c}1-h, 2-h\\2\end{array}\right) - \frac{k_1^*}{k_2^*} {}_2F_1\left(\begin{array}{c}1-\bar{h}, 2-\bar{h}\\2\end{array}\right) - \frac{k_2}{k_1} {}_2(20)$$

Another expression for  $\tilde{E}^{A}_{h\bar{h}}(\boldsymbol{k}_{1},\boldsymbol{k}_{2})$  which will be very useful for our purposes has been given in [11] in an integral form:

$$\tilde{E}_{h\bar{h}}^{A}(\boldsymbol{k}_{1},\boldsymbol{k}_{2}) = C_{l} \frac{1}{\boldsymbol{k}_{1}^{2} \boldsymbol{k}_{2}^{2}} \int d^{2}\boldsymbol{p} \left[ \frac{p\left(q-p\right)}{k_{1}-p} \right]^{h-1} \left[ \frac{p^{*}(q^{*}-p^{*})}{k_{1}^{*}-p^{*}} \right]^{h-1}, \qquad (21)$$

where

$$C_{l} = -(-1)^{n} \frac{i^{\bar{h}-h}h\,\bar{h}\,}{2^{h+\bar{h}}\pi^{3}} \frac{\Gamma(1-h)}{\Gamma(\bar{h})}.$$
(22)

Below we shall consider the completeness relation and the spectral representation for the BFKL kernel (cf. Ref. [7]).

Let us before discuss the scalar product in the space of functions where the kernel is acting on. For the kernel in the momentum space associated to the Feynman diagram derivation there is only one scalar product available. If we consider non amputated functions of two momenta  $\mathbf{k}_1, \mathbf{k}_2$ , i.e. with 2-dimensional propagators  $1/(\mathbf{k}_1^2 \mathbf{k}_2^2)$  included, the scalar product is defined as

$$\langle f|g\rangle \equiv \int d\mu(\boldsymbol{k}) f^*(\boldsymbol{k}_1, \boldsymbol{k}_2) g(\boldsymbol{k}_1, \boldsymbol{k}_2) , \qquad (23)$$

with the integration measure

$$d\mu(\boldsymbol{k}) = \boldsymbol{k}_1^2 \boldsymbol{k}_2^2 \, \delta^2(\boldsymbol{q} - \boldsymbol{k}_1 - \boldsymbol{k}_2) \, d^2 \boldsymbol{k}_1 d^2 \boldsymbol{k}_2 \tag{24}$$

which kills the extra propagators.

In the Möbius space of functions (in coordinate space) for the principal series of the Möbius group representation, whenever functions with conformal weight h = 0 or h = 1 are not considered, two possible choices are available [6]. Since the Möbius functions contain propagators, one possibility is to remove, as before, the propagators in one function, acting with the operator  $\partial_1^2 \partial_2^2$ . The other possibility, related to the form of the Casimir operator of the Möbius group, is to use instead the measure  $d^2 \rho_1 d^2 \rho_2 / |\rho_{12}|^4$ . The latter is not equivalent to the former when we consider the conformal wheights h = 0 or h = 1. Note, that, providing that we go from the Regge kinematics to the deep-inelastic scattering in the Bjorken regime, the additional series of the unitary representations of the Möbius group should be also used [11] (cf. Ref. [12]).

We now consider the completeness relation in coordinate space [4], in the space of Möbius functions, where the eigenfunctions of the Casimir operator of the Möbius group, defined in (13), constitute a suitable spectral basis for the operators acting on the space of the colourless impact factors, which, we stress, never "feel" the presence of the terms with a  $\delta$ -distribution behavior in momentum space, i.e. those of eq. (17). For the operators amputated (spatial propagators removed) to the left (with notation  $\hat{1}_L$  and  $\hat{K}_{12}^{(1)}$ ) one can write [4]

$$\langle \rho | \hat{1}_{L} | \rho' \rangle \equiv (2\pi)^{4} \delta^{2}(\boldsymbol{\rho}_{11'}) \, \delta^{2}(\boldsymbol{\rho}_{22'}) = \int d^{2} \boldsymbol{\rho}_{0} \sum_{h} \frac{N_{h}}{|\boldsymbol{\rho}_{12}|^{4}} \, E_{h,\bar{h}}(\boldsymbol{\rho}_{10},\,\boldsymbol{\rho}_{20}) \, E_{h,\bar{h}}^{*}(\boldsymbol{\rho}_{1'0},\,\boldsymbol{\rho}_{2'0}) \,,$$
  
$$\langle \rho | \hat{\bar{K}}_{12}^{(1)} | \rho' \rangle = \int d^{2} \boldsymbol{\rho}_{0} \sum_{h} \frac{N_{h}}{|\boldsymbol{\rho}_{12}|^{4}} \, E_{h,\bar{h}}(\boldsymbol{\rho}_{10},\,\boldsymbol{\rho}_{20}) \, \chi_{h} \, E_{h,\bar{h}}^{*}(\boldsymbol{\rho}_{1'0},\,\boldsymbol{\rho}_{2'0}) \,, \qquad (25)$$

where we have defined the weights  $N_h$ , as well as  $\tilde{N}_h$  for later use,

$$N_h = 16(\nu^2 + n^2/4), \quad \tilde{N}_h = (2\pi)^2 \frac{\nu^2 + n^2/4}{[\nu^2 + (n-1)^2/4][\nu^2 + (n+1)^2/4]}$$
(26)

and  $\sum_{h} \equiv \sum_{n} \int d\nu$ . If we try to extrapolate these operators to a wider domain which contains functions which no longer vanish for zero momenta, contrary to the colorless impact factor case, one needs some care. For example, for the function, which in momentum space depends only on the total momentum, just the case we meet in the bootstrap relation in LLA, one obtains the the result  $\delta^2(\rho_{12})$  in coordinate space (for the amputated impact factor). But such generalized function is orthogonal to the basis constituted by the functions in eq. (13) since  $\int d^2 \rho_{1'} E_{h,\bar{h}}^*(\rho_{1'0}, \rho_{2'0}) \delta^2(\rho_{1'2'}) = 0$  and therefore the application of the operators in eq. (25) on this function will give zero. In other words the bootstrap relation cannot be fullfilled in such a framework with the spectral representations given in eq. (25).

But this should not be surprising, since the choice done thanks to the equivalence relation in eq. (8), is based on a restriction of the action of the kernel to the functions similar to colorless impact factors.

# **3** Deformed representation

Moving to the momentum representation, one may write the Fourier transform of the relations in (25) and the same considerations done above may be repeated in this case, leading to the same conclusions. In particular one cannot write a bootstrap relation in a spectral basis built upon the states of eq. (16). On the other hand one notices that the BFKL kernel  $K_{12}^{(1)}$ in momentum space, as constructed from the Feynman diagram analysis, has an analytic behavior which must be reflected also in its spectral decomposition. It can also be noted that the behavior in  $|\mathbf{q}|$  of the  $\delta$ -like term (17) is  $|\mathbf{q}|^{-1+2i\nu}$ , which is singular in the forward limit where analytic behavior is expected for fixed  $\vec{k}_1 = -\vec{k}_2$ .

Therefore it is natural to suggest the modified relations for the eigenfunction completeness and for the spectral representation of the BFKL kernel:

$$\langle k | \hat{1}_{L} | k' \rangle \equiv 1 = \sum_{h} \tilde{N}_{h} \tilde{E}_{h\bar{h}}^{A}(\boldsymbol{k}_{1}, \boldsymbol{k}_{2}) \tilde{E}_{h\bar{h}}^{A*}(\boldsymbol{k}_{1}', \boldsymbol{k}_{2}') , \langle k | \hat{K}_{12}^{(1)} | k' \rangle = \sum_{h} \tilde{N}_{h} \tilde{E}_{h\bar{h}}^{A}(\boldsymbol{k}_{1}, \boldsymbol{k}_{2}) \chi_{h} \tilde{E}_{h\bar{h}}^{A*}(\boldsymbol{k}_{1}', \boldsymbol{k}_{2}') , \langle k | \hat{G}_{12}^{(1)}(y) | k' \rangle = \sum_{h} \tilde{N}_{h} \tilde{E}_{h\bar{h}}^{A}(\boldsymbol{k}_{1}, \boldsymbol{k}_{2}) e^{y\chi_{h}} \tilde{E}_{h\bar{h}}^{A*}(\boldsymbol{k}_{1}', \boldsymbol{k}_{2}') ,$$

$$(27)$$

where only the analytic contribution from each state of the spectral basis is used and the measure for integration is defined in eq. (24), according to our choice of considering functions with propagators included. Below we shall verify this anzatz.

Firstly we consider a space of functions (with removed propagators) which includes functions corresponding to the colorless impact factors and at least one function depending only on the total momentum, which is associated to a particular colored impact factor. Secondly we modify the spectral representation of the LL BFKL kernel, wherein the eigenvalues are assumed to be the same expressions of the conformal weights as for the restriction on the Möbius space of function, and the eigenfunctions are deformed in order to be analytic in the momentum space. A similar idea was already considered in the literature in an attempt to couple the LL BFKL pomeron to a quark [13, 14].

We proceed now with the definition of a transformation between the Möbius (M) basis and the analytic Feynman (AF) basis. Let us observe that one can go from one basis to the other by a simple transformation, which reads in coordinate representation as follows

$$\Phi^{-1}: \mathcal{M} \to \mathcal{AF}, \ E_{h}^{A}(\boldsymbol{\rho}_{10}, \boldsymbol{\rho}_{20}) = E_{h}^{M}(\boldsymbol{\rho}_{10}, \boldsymbol{\rho}_{20}) - \lim_{\rho_{1} \to \infty} E_{h}^{M}(\boldsymbol{\rho}_{10}, \boldsymbol{\rho}_{20}) - \lim_{\rho_{2} \to \infty} E_{h}^{M}(\boldsymbol{\rho}_{10}, \boldsymbol{\rho}_{20}) = E_{h}^{M}(\boldsymbol{\rho}_{10}, \boldsymbol{\rho}_{20}) - 2^{-h-\bar{h}} \left( E_{h}^{M}(-\boldsymbol{\rho}_{20}, \boldsymbol{\rho}_{20}) + (-1)^{n} E_{h}^{M}(-\boldsymbol{\rho}_{10}, \boldsymbol{\rho}_{10}) \right),$$
(28)  
$$\Phi: \mathcal{AF} \to \mathcal{M}: \ E_{h}^{M}(\boldsymbol{\rho}_{10}, \boldsymbol{\rho}_{20}) = E_{h}^{A}(\boldsymbol{\rho}_{10}, \boldsymbol{\rho}_{20}) + \frac{1}{2} \left( E_{h}^{A}(-\boldsymbol{\rho}_{10}, \boldsymbol{\rho}_{20}) + (-1)^{n} E_{h}^{A}(-\boldsymbol{\rho}_{10}, \boldsymbol{\rho}_{10}) \right),$$
(28)

$$\Phi: AF \to M:, E_h^M(\boldsymbol{\rho}_{10}, \boldsymbol{\rho}_{20}) = E_h^A(\boldsymbol{\rho}_{10}, \boldsymbol{\rho}_{20}) + \frac{1}{2^{h+\bar{h}}-2} \left( E_h^A(-\boldsymbol{\rho}_{20}, \boldsymbol{\rho}_{20}) + (-1)^n E_h^A(-\boldsymbol{\rho}_{10}, \boldsymbol{\rho}_{10}) \right)$$

These relations are clearly non local and have been obtained by noting that

$$E_{h}^{M}(-\boldsymbol{\rho},\boldsymbol{\rho}) = 2^{h+\bar{h}} \left(\frac{1}{\rho}\right)^{h} \left(\frac{1}{\rho^{*}}\right)^{\bar{h}}, \quad E_{h}^{A}(-\boldsymbol{\rho},\boldsymbol{\rho}) = (2^{h+\bar{h}}-2) \left(\frac{1}{\rho}\right)^{h} \left(\frac{1}{\rho^{*}}\right)^{\bar{h}}.$$
 (29)

For the case of even conformal spins (n) the second transformation can be written in a simple local form on observing that

$$\frac{1}{2^{h+\bar{h}}-2} \left( E_h^A(-\boldsymbol{\rho}_{20},\boldsymbol{\rho}_{20}) + (-1)^n E_h^A(-\boldsymbol{\rho}_{10},\boldsymbol{\rho}_{10}) \right) = -\frac{1}{2} \left( E_h^A(\boldsymbol{\rho}_{10},\boldsymbol{\rho}_{10}) + (-1)^n E_h^A(\boldsymbol{\rho}_{20},\boldsymbol{\rho}_{20}) \right)$$
(30)

while for odd conformal spin such a simple transformation is not possible since  $E_h^A(\rho, \rho) = 0$ . It is related to the fact, that in the last case both functions  $E_h^M(\rho_{10}, \rho_{20})$  and  $E_h^A(\rho_{10}, \rho_{20})$  satisfy the colour transparancy property  $E_h^{M,A}(\rho, \rho) = 0$ , but only  $E_h^M(\rho_{10}, \rho_{20})$  has the simple conformal properties.

It can be easily checked that

$$\Phi \Phi^{-1} \equiv I_M \,, \quad \Phi^{-1} \Phi \equiv I_{AF} \tag{31}$$

so that  $\Phi$  is a 1-1 mapping with inverse really given by  $\Phi^{-1}$ .

The Green's function for the evolution in the rapidity y based on the deformed analytic spectral basis was also written above (see (27)). In the next section we shall show that with this prescription one is recovering the correct relation compatible with the bootstrap requirement for the gluon reggeization in LLA. Since the spectrum in both representations is the same it is natural to expect that the BFKL kernel satisfying the bootstrap relation is conformal invariant, in the new deformed analytic Feynman basis obtained by a similarity transformation given by the operator  $\Phi$ . Namely, the well known generators of the Möbius group  $\vec{M_r}$  could be extended to this basis as follows

$$\vec{M}_r^{AF} = \Phi^{-1} \vec{M}_r \Phi \,. \tag{32}$$

Let us summarize the result. If one considers the impact factors derived from Feynman diagrams calculations, they are the functions belonging to the AF-space and not to the M-space. We remind that we have two different completeness relations based on the two spectral basis  $E_h^{AF}$  and  $E_h^M$  and in these two spaces there are different definitions for scalar products and normalizations of wave functions. Nevertheless the cardinality of the two basis is the same. The two spaces are related by the 1-1 mapping  $\Phi$ : AF-space  $\rightarrow$  M-space as previously discussed (see also Fig. 1) and one space is *shifted* with respect to the other. Moreover any operator  $O^{AF}$  in the AF-space which can be decomposed on the spectral basis can be defined on the M-space as  $O^M = \Phi O^{AF} \Phi^{-1}$  and viceversa  $O^{AF} = \Phi^{-1} O^M \Phi$ . This can be seen as an isometry between the Möbius space and the analytical Feynman space with the different scalar products in these spaces.



Figure 1: There is a 1-1 mapping between the AF-space and the M-space.

The next interesting question is therefore if it is possible to construct explicitly the new representation defined in eq. (32). We shall study this problem in the section 5, starting from analyzing the Fourier transform of the Möbius algebra in the M-space and later on moving to the AF-space. In the next section we will show that with the meromorphic prescription we indeed recover the bootstrap properties of the BFKL kernel.

### 4 AF representation and the bootstrap relation

For the sake of using a compact notation, let us introduce for some functions of transverse gluon momenta the corresponding wave functions for a color singlet state. These are the power  $|P_{\lambda}\rangle$ , the unity  $|U\rangle$  and the log  $|L\rangle$  states (modulus the propagators), defined by their

representations in the momentum space:

$$\langle k | P_{\lambda} \rangle \equiv \frac{1}{\boldsymbol{k}_{1}^{2} \boldsymbol{k}_{2}^{2}} \left( \frac{\boldsymbol{q}^{4}}{\boldsymbol{k}_{1}^{2} \boldsymbol{k}_{2}^{2}} \right)^{\lambda} ,$$

$$\langle k | U \rangle \equiv \frac{1}{\boldsymbol{k}_{1}^{2} \boldsymbol{k}_{2}^{2}} = (\langle k | P_{\lambda} \rangle)_{\lambda=0} ,$$

$$\langle k | L \rangle \equiv \frac{1}{\boldsymbol{k}_{1}^{2} \boldsymbol{k}_{2}^{2}} \log \left( \frac{\boldsymbol{q}^{4}}{\boldsymbol{k}_{1}^{2} \boldsymbol{k}_{2}^{2}} \right) = \frac{d}{d\lambda} \left( \langle k | P_{\lambda} \rangle \right)_{\lambda=0} .$$

$$(33)$$

Since, as previoulsy discussed, one has for the Möbius-invariant wave function h the equality  $\langle h|U\rangle = 0$  we obtain

$$\langle h^A | U \rangle = -\langle h^\delta | U \rangle \,. \tag{34}$$

Let us recall the completeness relation and the BFKL kernel written as

$$\hat{1}_{L} = \sum_{h} \tilde{N}_{h} |h^{A}\rangle \langle h^{A}|,$$
  

$$\hat{K}_{12}^{(1)} = \sum_{h} \tilde{N}_{h} |h^{A}\rangle \chi_{h} \langle h^{A}|.$$
(35)

On using the previous relations we write the bootstrap relation in the form of eq. (7) as

$$\bar{\alpha}_s \sum_h \tilde{N}_h |h^A\rangle \,\epsilon_h \,\langle h^A |U\rangle = \frac{\alpha_s}{2} \sum_h \tilde{N}_h |h^A\rangle \,\langle h^A |L\rangle \,, \tag{36}$$

where eq. (14) was used. In order to show that our definitions (35) are correct, we should prove that the coefficients of the conformal basis  $|h^A\rangle$  on both the l.h.s. and the r.h.s. do coincide, i.e.

$$\epsilon_h \langle h^A | U \rangle = \frac{1}{2} \langle h^A | L \rangle .$$
(37)

It is therefore enough to calculate the integral given by  $\langle h^A | P_\lambda \rangle$ . Verifying that this expression is finite it is sufficient also to calculate the first two terms of the Taylor expansion in  $\lambda$  to verify the eq. (37), thanks to eq. (33).

We also note that the left hand side of eq. (37) can be calculated easily in an independent way using the relations in eq. (34) and (17), which give:

$$\langle h^A | U \rangle = -\delta_{n,\text{even}} \frac{i^n}{2\pi} 2^{h+\bar{h}} \frac{\Gamma(h)}{\Gamma(1-\bar{h})} q^{*-h} q^{-\bar{h}} , \qquad (38)$$

where  $\delta_{n,\text{even}} = [1 + (-1)^n]/2$  has support (and value 1) for any even integer number.

Let us now find explicitly  $\langle h^A | P_\lambda \rangle$ . It is convenient to perform the calculations in the momentum space where we have defined the analytic element  $|h^A\rangle$  of the conformal basis. Taking for it the expression given in eq. (21) we write:

$$\langle h^{A} | P_{\lambda} \rangle = \int d^{2} \boldsymbol{k}_{1} d^{2} \boldsymbol{k}_{2} \, \delta^{(2)}(\boldsymbol{q} - \boldsymbol{k}_{1} - \boldsymbol{k}_{2}) \tilde{E}_{h\bar{h}}^{A*}(\boldsymbol{k}_{1}, \boldsymbol{k}_{2}) \left(\frac{\boldsymbol{q}^{4}}{\boldsymbol{k}_{1}^{2} \boldsymbol{k}_{2}^{2}}\right)^{\lambda}$$

$$= C_{l}^{*} |\boldsymbol{q}|^{4\lambda} \int d^{2} \boldsymbol{k} \int d^{2} \boldsymbol{p} \, p^{-\bar{h}} (\boldsymbol{q} - \boldsymbol{p})^{-\bar{h}} (\boldsymbol{k} - \boldsymbol{p})^{\bar{h}} \, \boldsymbol{k}^{-1-\lambda} (\boldsymbol{q} - \boldsymbol{k})^{-1-\lambda} \times (\text{h.c.}) \,.$$
(39)

Using new integration variables induced by the change x = p/q and y = q/k with analogous relations for the complex conjugated ones, we obtains

$$\langle h^A | P_\lambda \rangle = C_l^* q^{-\bar{h}} q^{*-h} I_\lambda(1) , \qquad (40)$$

where

$$I_{\lambda}(z) = \int d^2 \boldsymbol{x} \, d^2 \boldsymbol{y} \, x^{-\bar{h}} (1-x)^{-\bar{h}} y^{2\lambda-\bar{h}} (1-y)^{-1-\lambda} (1-xyz)^{\bar{h}} \times (h.c.) \,. \tag{41}$$

It is convenient to define the function  $I_{\lambda}(z)$  because one can find an explicit expression for it in terms of the generalized hypergeometric functions  $_{3}F_{2}$ . We illustrate some details of the calculation in appendix A.

It is sufficient to represent the result of the integration in power series of  $\lambda$  for the point z = 1:

$$I_{\lambda}(1) = \sum_{m} \frac{1}{m!} I_{\lambda}^{(m)}(1)|_{\lambda=0} \ \lambda^{m} \,.$$
(42)

Recalling the first two terms in the expansion (see eq. (104) in appendix A) one can see, on using the definitions of eq. (33) and (40), which imply  $\langle h^A | U \rangle = (\langle h^A | P_\lambda \rangle)_{\lambda=0}$  and  $\langle h^A | L \rangle = (d \langle h^A | P_\lambda \rangle / d\lambda)_{\lambda=0}$ , that the relation in eq. (37) is verified and therefore also the bootstrap relation. As a check one may evaluate explicitly the quantity

$$(\langle h^A | P_\lambda \rangle)_{\lambda=0} = C_l^* q^{-\bar{h}} q^{*-h} \frac{2\pi^2}{(1-h)(1-\bar{h})} \,\delta_{n,\text{even}} \,, \tag{43}$$

using the eq. (22). It is easy to show that it coincides with the expression of eq. (38), computed in a completely different way.

In order to give an alternative check of the representation proposed, we have also calculated explicitly the third term of the expansion of the BFKL Green's function  $G_{12}^{(1)}(y) = e^{y\bar{K}_{12}^{(1)}} \otimes 1$  (the first term is trivial and the second corresponds to the bootstrap relation), which contains the double iteration of the kernel acting on the unity (momentum space). The details are in the appendix B.

Using the spectral representation one can compute an arbitrary number of iterations of the BFKL kernel or the action of the BFKL Green's function on the impact factor of a quark or gluon line.

# 5 The deformed representation of the Möbius (conformal) algebra

In this section we will examine the representation of the  $\mathfrak{sl}(2,\mathbb{C})$  algebra in the AF space of functions. To this end we start from the familiar Möbius algebra in the M space and transform it into momentum space. We then switch to the AF space: in order to retain the correct structure of the algebra we have to construct a new representation of the generator  $M_{-}$ . Let us reconsider the generators of the conformal Möbius group in 2 dimensions in coordinate space given in eq. (11) and the corresponding Casimir operator (12). We will be interested in writing similar relations in momentum space. We start from the conformal eigenfunctions (13) which from now on will carry the superscript 'M', and we define their Fourier transforms:

$$E_{h\bar{h}}^{M}(\boldsymbol{\rho}_{0};\boldsymbol{\rho}_{1},\boldsymbol{\rho}_{2}) = \left(\frac{\rho_{12}}{\rho_{10}\rho_{20}}\right)^{h} \left(\frac{\rho_{12}^{*}}{\rho_{10}^{*}\rho_{20}^{*}}\right)^{h}, \qquad (44)$$

$$E_{h\bar{h}}^{M}(\boldsymbol{q};\boldsymbol{k}_{1},\boldsymbol{k}_{2}) = \mathcal{F}[E_{h\bar{h}}^{M}](\boldsymbol{q};\boldsymbol{k}_{1},\boldsymbol{k}_{2}) = \int d\mu \ e^{-i\boldsymbol{q}\cdot\boldsymbol{\rho}_{0}+i\boldsymbol{k}_{1}\cdot\boldsymbol{\rho}_{1}+i\boldsymbol{k}_{2}\cdot\boldsymbol{\rho}_{2}} E_{h\bar{h}}^{M}(\boldsymbol{\rho}_{0};\boldsymbol{\rho}_{1},\boldsymbol{\rho}_{2}) = \\ = (2\pi)^{2}\delta^{(2)}(\boldsymbol{q}-\boldsymbol{k}_{1}-\boldsymbol{k}_{2})\tilde{E}_{h\bar{h}}^{M}(\boldsymbol{k}_{1},\boldsymbol{k}_{2}), \qquad (45)$$

$$\tilde{E}_{h\bar{h}}^{M}(\boldsymbol{\rho}_{0};\boldsymbol{k}_{1},\boldsymbol{k}_{2}) = \tilde{\mathcal{F}}[E_{h\bar{h}}^{M}](\boldsymbol{\rho}_{0};\boldsymbol{k}_{1},\boldsymbol{k}_{2}) = \int d\tilde{\mu} \ e^{i\boldsymbol{k}_{1}\cdot\boldsymbol{\rho}_{1}+i\boldsymbol{k}_{2}\cdot\boldsymbol{\rho}_{2}}E_{h\bar{h}}^{M}(\boldsymbol{\rho}_{0};\boldsymbol{\rho}_{1},\boldsymbol{\rho}_{2}) = \\
= \ e^{i(\boldsymbol{k}_{1}+\boldsymbol{k}_{2})\cdot\boldsymbol{\rho}_{0}}\tilde{E}_{h\bar{h}}^{M}(\boldsymbol{k}_{1},\boldsymbol{k}_{2}),$$
(46)

$$\tilde{E}_{h\bar{h}}^{M}(\boldsymbol{k}_{1},\boldsymbol{k}_{2}) = \tilde{\mathcal{F}}[E_{h\bar{h}}^{M}](\boldsymbol{\rho}_{0}=0;\boldsymbol{k}_{1},\boldsymbol{k}_{2}) = \int d\tilde{\mu} \ e^{i\boldsymbol{k}_{1}\cdot\boldsymbol{\rho}_{1}+i\boldsymbol{k}_{2}\cdot\boldsymbol{\rho}_{2}} E_{h\bar{h}}^{M}(0;\boldsymbol{\rho}_{1},\boldsymbol{\rho}_{2}), \qquad (47)$$

where  $d\mu = d^2 \rho_0 d^2 \rho_1 d^2 \rho_2$ ,  $d\tilde{\mu} = d^2 \rho_1 d^2 \rho_2$  and  $\mathcal{F}$ ,  $\tilde{\mathcal{F}}$  are the Fourier transform operators in three and two variables respectively. In coordinate space the action of the generators on the eigenfunctions  $E_{h\bar{h}}^M(\rho_{10}, \rho_{20})$  is:

$$M^{+}E^{M}_{h\bar{h}}(\boldsymbol{\rho}_{0};\boldsymbol{\rho}_{1},\boldsymbol{\rho}_{2}) = -\partial_{0}E^{M}_{h\bar{h}}(\boldsymbol{\rho}_{0};\boldsymbol{\rho}_{1},\boldsymbol{\rho}_{2}), \qquad (48)$$

$$M^{3}E_{h\bar{h}}^{M}(\boldsymbol{\rho}_{0};\boldsymbol{\rho}_{1},\boldsymbol{\rho}_{2}) = (-h-\rho_{0}\partial_{0})E_{h\bar{h}}^{M}(\boldsymbol{\rho}_{0};\boldsymbol{\rho}_{1},\boldsymbol{\rho}_{2}), \qquad (49)$$

$$M^{-}E^{M}_{h\bar{h}}(\boldsymbol{\rho}_{0};\boldsymbol{\rho}_{1},\boldsymbol{\rho}_{2}) = (2h\rho_{0}+\rho_{0}^{2}\partial_{0})E^{M}_{h\bar{h}}(\boldsymbol{\rho}_{0};\boldsymbol{\rho}_{1},\boldsymbol{\rho}_{2}).$$
(50)

Note, that the functions  $E_{h\bar{h}}^{M}(\boldsymbol{\rho}_{0};\boldsymbol{\rho}_{1},\boldsymbol{\rho}_{2})$  can be considered as the Clebsch-Gordon coefficients in the expansion of the product of the Reggeon wave functions  $\varphi(\boldsymbol{\rho}_{r})$  in the sum of the irreducible representations

$$\varphi(\boldsymbol{\rho}_1)\,\varphi(\boldsymbol{\rho}_2) = \int_{-\infty}^{\infty} d\nu \sum_{n=-\infty}^{\infty} a_{\nu n} \int d^2 \rho_0 E_{h\bar{h}}^M(\boldsymbol{\rho}_0; \boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \,O_{h\bar{h}}(\boldsymbol{\rho}_0)\,, \tag{51}$$

where the conformal weights for the principal series of unitary representations are

$$h = \frac{1}{2} + i\nu + \frac{n}{2}, \ \bar{h} = \frac{1}{2} + i\nu - \frac{n}{2}$$
(52)

and the coefficients  $a_{\nu n}$  are fixed with the use of the normalization conditions for the wave functions  $\varphi(\boldsymbol{\rho}_r)$  and  $O_{h\bar{h}}(\boldsymbol{\rho}_0)$ . In the irreducible representations  $O_{h\bar{h}}(\boldsymbol{\rho}_0)$  the Möbius group generators are given below

$$M^+ O_{h\bar{h}}(\boldsymbol{\rho}_0) = \partial_0 O_{h\bar{h}}(\boldsymbol{\rho}_0), \qquad (53)$$

$$M^{3}O_{h\bar{h}}(\boldsymbol{\rho}_{0}) = (1 - h + \rho_{0}\partial_{0})O_{h\bar{h}}(\boldsymbol{\rho}_{0}) , \qquad (54)$$

$$M^{-}O_{h\bar{h}}(\boldsymbol{\rho}_{0}) = (-2(1-h)\rho_{0} - \rho_{0}^{2}\partial_{0})O_{h\bar{h}}(\boldsymbol{\rho}_{0}).$$
(55)

The generators of the Möbius group can be translated into the momentum space, using the relation  $\boldsymbol{\rho} \cdot \boldsymbol{k} = (\rho^* k + \rho k^*)/2$ :

$$M^{+}\tilde{E}_{h\bar{h}}^{M}(\boldsymbol{\rho}_{0};\boldsymbol{k}_{1},\boldsymbol{k}_{2}) = e^{i(\boldsymbol{k}_{1}+\boldsymbol{k}_{2})\cdot\boldsymbol{\rho}_{0}}\tilde{M}^{+}(\boldsymbol{k})\tilde{E}_{h\bar{h}}^{M}(\boldsymbol{k}_{1},\boldsymbol{k}_{2}), \qquad (56)$$

$$M^{3}\tilde{E}_{h\bar{h}}^{M}(\boldsymbol{\rho}_{0};\boldsymbol{k}_{1},\boldsymbol{k}_{2}) = e^{i(\boldsymbol{k}_{1}+\boldsymbol{k}_{2})\cdot\boldsymbol{\rho}_{0}}\left(\tilde{M}^{3}(\boldsymbol{k})+\rho_{0}\tilde{M}^{+}(\boldsymbol{k})\right)\tilde{E}_{h\bar{h}}^{M}(\boldsymbol{k}_{1},\boldsymbol{k}_{2}), \qquad (57)$$

$$M^{-}\tilde{E}_{h\bar{h}}^{M}(\boldsymbol{\rho}_{0};\boldsymbol{k}_{1},\boldsymbol{k}_{2}) = e^{i(\boldsymbol{k}_{1}+\boldsymbol{k}_{2})\cdot\boldsymbol{\rho}_{0}} \left(\tilde{M}^{-}(\boldsymbol{k}) - 2\rho_{0}\tilde{M}^{3}(\boldsymbol{k}) - \rho_{0}^{2}\tilde{M}^{+}(\boldsymbol{k})\right)\tilde{E}_{h\bar{h}}^{M}(\boldsymbol{k}_{1},\boldsymbol{k}_{2}), (58)$$

where

$$\tilde{M}^{+}(\boldsymbol{k}) = -\frac{i}{2}(k_{1}^{*}+k_{2}^{*}), \qquad (59)$$

$$\tilde{M}^{3}(\boldsymbol{k}) = -(\partial_{k_{1}^{*}}k_{1}^{*} + \partial_{k_{2}^{*}}k_{2}^{*}), \qquad (60)$$

$$\tilde{M}^{-}(\boldsymbol{k}) = -2i(\partial_{k_{1}^{*}}^{2}k_{1}^{*} + \partial_{k_{2}^{*}}^{2}k_{2}^{*}).$$
(61)

One can calculate also the action of some bilinear combinations of the generators in the momentum representation

$$\begin{split} M^{+}M^{-}\tilde{E}_{h\bar{h}}^{M}(\boldsymbol{\rho}_{0};\boldsymbol{k}_{1},\boldsymbol{k}_{2}) &= e^{i(\boldsymbol{k}_{1}+\boldsymbol{k}_{2})\cdot\boldsymbol{\rho}_{0}}\tilde{M}^{+}(\boldsymbol{k})\left(\tilde{M}^{-}(\boldsymbol{k})-2\rho_{0}\tilde{M}^{3}(\boldsymbol{k})-\rho_{0}^{2}\tilde{M}^{+}(\boldsymbol{k})\right)\tilde{E}_{h\bar{h}}^{M}(\boldsymbol{k}_{1},\boldsymbol{k}_{2}),\\ &\left(M^{3}(M^{3}+1)+M^{+}M^{-}\right)\tilde{E}_{h\bar{h}}^{M}(\boldsymbol{\rho}_{0};\boldsymbol{k}_{1},\boldsymbol{k}_{2}) = e^{i(\boldsymbol{k}_{1}+\boldsymbol{k}_{2})\cdot\boldsymbol{\rho}_{0}}\tilde{M}^{2}\tilde{E}_{h\bar{h}}^{M}(\boldsymbol{k}_{1},\boldsymbol{k}_{2})\,,\end{split}$$

where the Casimir operator in the "short" momentum representation is

$$\tilde{M}^{2} = \tilde{M}^{3}(\boldsymbol{k}) \left( \tilde{M}^{3}(\boldsymbol{k}) + 1 \right) + \tilde{M}^{+}(\boldsymbol{k}) \tilde{M}^{-}(\boldsymbol{k}) = -(\partial_{k_{1}^{*}} - \partial_{k_{2}^{*}})^{2} k_{1}^{*} k_{2}^{*} \,.$$
(62)

Note that in our notations  $\tilde{M}^3$  is a generator while  $\tilde{M}^2$  is the Casimir operator. The algebra of these generators in momentum space is the same as in coordinate space:

$$[\tilde{M}_r^+, \tilde{M}_r^3] = \tilde{M}_r^+, \quad [\tilde{M}_r^+, \tilde{M}_r^-] = -2\tilde{M}_r^3, \quad [\tilde{M}_r^-, \tilde{M}_r^3] = -\tilde{M}_r^-,$$

While the action of the generator  $\tilde{M}^+$  does not need special comments, it is interesting to look at others. One finds

$$\tilde{M}^{3}\tilde{E}_{h}^{M}(\boldsymbol{k}_{1},\boldsymbol{k}_{2}) = -h\,\tilde{E}_{h}^{M}(\boldsymbol{k}_{1},\boldsymbol{k}_{2}), \qquad (63)$$

$$\tilde{M}^{-}\tilde{E}_{h}^{M}(\boldsymbol{k}_{1},\boldsymbol{k}_{2}) = 0.$$
 (64)

The last relation follows from the action of the Casimir operator

$$\tilde{M}^2 \tilde{E}_h^M = h(h-1)\tilde{E}_h^M \tag{65}$$

and from relation (63). In particular, given that

$$\tilde{M}^{2} = \frac{1}{2} \left[ \tilde{M}^{+} \tilde{M}^{-} + \tilde{M}^{-} \tilde{M}^{+} \right] + \left( \tilde{M}^{3} \right)^{2} = \tilde{M}^{+} \tilde{M}^{-} + \tilde{M}^{3} \left( \tilde{M}^{3} + 1 \right) , \qquad (66)$$

where the commutation relation  $[\tilde{M}^+, \tilde{M}^-] = -2\tilde{M}^3$  has been used, one can see that indeed the relation (64) is satisfied.

These relations mean that in the M-space in momentum representation the Möbius algebra is projected to

$$[\tilde{M}^+, \tilde{M}^3]\tilde{E}^M = \tilde{M}^+\tilde{E}^M, \quad \tilde{M}^-\tilde{M}^+\tilde{E}^M = 2\tilde{M}^3\tilde{E}^M, \quad 0\,\tilde{E}^M = 0\,\tilde{E}^M.$$
 (67)

Let us now move to the AF space. We start, in coordinate space, from the definition

$$E_h^A = E_h^M - E_h^\delta, (68)$$

where, according to eq. (28),

$$E_{h}^{\delta}(\boldsymbol{\rho}_{10},\boldsymbol{\rho}_{20}) = 2^{1-h-\bar{h}}PE_{h}^{M}(\boldsymbol{\rho}_{10},\boldsymbol{\rho}_{20}) = \left(\frac{1}{\rho_{20}}\right)^{h} \left(\frac{1}{\rho_{20}^{*}}\right)^{\bar{h}} + \left(\frac{1}{-\rho_{10}}\right)^{h} \left(\frac{1}{(-\rho_{10})^{*}}\right)^{\bar{h}}$$
(69)

and P is a projector  $(P^2 = P)$  defined by its action

$$Pf(x_1, x_2) = \frac{1}{2} \left( f(x_1, -x_1) + f(-x_2, x_2) \right) .$$
(70)

In order to compute the action of the Möbius generators on the analytic part,  $E_h^A$ , of the conformal eigenfunction we first examine the singular piece,  $E_h^{\delta}$ .

First of all let us observe that the Casimir operator acting on  $E_h^{\delta}$  gives zero, as it is trivially checked in the coordinate representation. This means that h in such a case can be obtained with the action of a function of the Casimir (solving the eq. (65) for h) only before applying the projector P. After the application of P one needs to define a new operator to extract the scaling parameter h. It is related to the fact, that  $E_h^A$  is not an eigenfunction of the usual Casimir operator. We should construct a new Casimir operator  $\tilde{M}_{AF}^2$  for this representation.

In coordinate representation we can deduce the following relations for generators

$$M^{+}E_{h}^{\delta}(\boldsymbol{\rho}_{10}, \boldsymbol{\rho}_{20}) = -\partial_{0}E_{h}^{\delta}(\boldsymbol{\rho}_{10}, \boldsymbol{\rho}_{20}), \qquad (71)$$

$$M^{3}E_{h}^{\delta}(\boldsymbol{\rho}_{10}, \boldsymbol{\rho}_{20}) = (-h - \rho_{0}\partial_{0})E_{h}^{\delta}(\boldsymbol{\rho}_{10}, \boldsymbol{\rho}_{20}), \qquad (72)$$

which coincides with their action for  $E_h^M$  while the action of  $M^-$  is different. According to this fact the action of Möbius generators on  $E_h^A$  can be written as

$$M^{+}E_{h}^{A}(\boldsymbol{\rho}_{10}, \boldsymbol{\rho}_{20}) = -\partial_{0}E_{h}^{A}(\boldsymbol{\rho}_{10}, \boldsymbol{\rho}_{20}), \qquad (73)$$

$$M^{3}E_{h}^{A}(\boldsymbol{\rho}_{10}, \boldsymbol{\rho}_{20}) = (-h - \rho_{0}\partial_{0})E_{h}^{A}(\boldsymbol{\rho}_{10}, \boldsymbol{\rho}_{20})$$
(74)

and

$$M^{-}E_{h}^{A}(\boldsymbol{\rho}_{10},\boldsymbol{\rho}_{20}) = (\rho_{0}^{2}\partial_{0} + 2h\rho_{0})E_{h}^{A}(\boldsymbol{\rho}_{10},\boldsymbol{\rho}_{20}) - \left[h\rho_{10}E_{h}^{M}(\boldsymbol{\rho}_{10},-\boldsymbol{\rho}_{10}) + h\rho_{20}E_{h}^{M}(-\boldsymbol{\rho}_{20},\boldsymbol{\rho}_{20})\right]2^{-h-\bar{h}}.$$
(75)

Therefore only the last generator  $M^-$ , needs to be redefined in the AF space.

Let us consider now again the momentum space for these functions. First we analyze the action of the  $\tilde{M}^3$  generator on  $\tilde{E}_h^A(\mathbf{k}_1, \mathbf{k}_2)$ , which we have written in terms of hypergeometric functions. One can immediately see that

$$\tilde{M}^{3}\tilde{E}_{h}^{A}(\boldsymbol{k}_{1},\boldsymbol{k}_{2}) = -h\,\tilde{E}_{h}^{A}(\boldsymbol{k}_{1},\boldsymbol{k}_{2})$$
(76)

for any  $\mathbf{k}_1 + \mathbf{k}_2$ . This means again that we are not forced to use a non local operator  $\hat{h}$  (solving an equation with the corresponding Casimir  $\tilde{M}_{AF}^2$ ) to extract the dimension h of an eigenstate, as we had to do in the coordinate representation.

The  $E_h^{\delta}$  function is instead very peculiar since we know that the Casimir gives zero. This fact is also verified directly in momentum space, due to the deltas present in  $\tilde{E}_h^{\delta}$ . Later we will show it explicitly. Instead we note that the action of the  $\tilde{M}^3$  operator is good. Indeed, from the explicit form given in Eq. (17), we obtain

$$\hat{M}^{3}E_{h}^{\delta}(\boldsymbol{k}_{1},\boldsymbol{k}_{2}) = -h E_{h}^{\delta}(\boldsymbol{k}_{1},\boldsymbol{k}_{2}).$$
 (77)

This is compatible with the fact that the  $\tilde{M}^3$  operator has the same action on the  $\tilde{E}_h^M(\mathbf{k}_1, \mathbf{k}_2)$ and  $\tilde{E}_h^A(\mathbf{k}_1, \mathbf{k}_2)$  functions.

The fact that we may choose to extract the dimension h in momentum space using the  $\tilde{M}^3$  operator may be useful to simplify the construction of the mappings  $\Phi$  and  $\Phi^{-1}$ . On defining

$$F = 2^{1-\hat{h}-\hat{\bar{h}}} \tag{78}$$

one may write the mapping introduced in eq. (28) as

$$\Phi^{-1}: M \to AF, \quad \Phi^{-1} = 1 - F^{\delta}P = 1 - PF^{M},$$
(79)

$$\Phi: AF \to M, \quad \Phi = 1 + \frac{F^{\circ}}{1 - F^{\delta}}P = 1 + P\frac{F^{AF}}{1 - F^{AF}},$$
(80)

where in general PFP = FP and which imply  $\Phi\Phi^{-1} = \Phi^{-1}\Phi = 1$  and we have denoted  $F^{\delta} = F(\hat{h}(\tilde{M}^3)), F^M = F(\hat{h}(\tilde{M}^2))$  and similarly  $F^{AF} = F(\hat{h}(\tilde{M}^2_{AF}))$ . Clearly in the relation for the mappings one can use different forms for the operator F depending on which space it acts.

In momentum space  $\hat{h} = -\tilde{M}^3$  then one can write [P, F] = 0 (with some abuse of notation) and things are further simplified.

Proceeding along this line one can give a deformed representation  $\tilde{M}_{AF}^-$  which lives on the AF-space. In order to do this, let us first study

$$\tilde{M}^{-}\tilde{E}_{h}^{A} = \tilde{M}^{-}(\tilde{E}_{h}^{M} - \tilde{E}_{h}^{\delta}) = -\tilde{M}^{-}\tilde{E}_{h}^{\delta}$$

$$(81)$$

and

$$\tilde{M}^{-}\tilde{E}_{h}^{\delta} = -2i(\partial_{k_{1}^{*}}^{2}k_{1}^{*} + \partial_{k_{2}^{*}}^{2}k_{2}^{*}) \left[\delta^{(2)}(\boldsymbol{k}_{1}) + (-1)^{n}\delta^{(2)}(\boldsymbol{k}_{2})\right] c_{h}(k_{1} + k_{2})^{\bar{h}-1}(k_{1}^{*} + k_{2}^{*})^{h-1} \\
= 2ih(h-1)\frac{1}{k_{1}^{*} + k_{2}^{*}}\tilde{E}_{h}^{\delta} = -h(h-1)\left(\tilde{M}^{+}\right)^{-1}\tilde{E}_{h}^{\delta} \\
= -\left(\tilde{M}^{+}\right)^{-1}h(h-1)\frac{F}{1-F}P\tilde{E}_{h}^{A}.$$
(82)

Note that  $\tilde{M}^+$  has the same form on any of the spaces of functions we consider. The resulting action of  $\tilde{M}^-$  on  $\tilde{E}_h^A$  has to be studied in order to see if it is possible to define a new representation for  $\tilde{M}^-$  on AF-space which satisfies the conformal algebra. But before let us note that from the previous relation we have

$$\tilde{M}^{+}\tilde{M}^{-}\tilde{E}_{h}^{\delta} = -h(h-1)\tilde{E}_{h}^{\delta}, 
\tilde{M}^{-}\tilde{M}^{+}\tilde{E}_{h}^{\delta} = -h(h+1)\tilde{E}_{h}^{\delta},$$
(83)

so that we reobtain, recalling eq. (66),

$$\tilde{M}^2 \tilde{E}_h^\delta = 0. aga{84}$$

In general we have defined the Möbius generators on the AF space (spanned by the Basis functions  $\tilde{E}_h^A$ ) using the isometry with the M-space:

$$\tilde{M}_{AF} = \Phi^{-1}\tilde{M}\Phi.$$
(85)

Let us look at each of the three generators. From the previous discussion, also in the coordinate representation, we observe that

$$\tilde{M}_{AF}^{+} = \tilde{M}^{+} , \quad \tilde{M}_{AF}^{3} = \tilde{M}^{3} ,$$
(86)

which means that in momentum space the actions of these two operators is the same in both M-space and AF-space.

We therefore define for the AF-space a new generator in momentum representation such that

$$\tilde{M}_{AF}^{-}\tilde{E}_{h}^{A}=0, \qquad (87)$$

which puts into relation, as before, the action of the Casimir operator such that

$$\tilde{M}_{AF}^2 \tilde{E}_h^A = h(h-1)\tilde{E}_h^A, \qquad (88)$$

together with the piece of the Möbius algebra  $[\tilde{M}_{AF}^+, \tilde{M}_{AF}^-] = -2\tilde{M}_{AF}^3$  and the action of  $\tilde{M}_{AF}^3$ . According to the previous result (see eqs. (81) and (82)), by construction the choice is

$$\tilde{M}_{AF}^{-} = \tilde{M}^{-} - \left(\tilde{M}^{+}\right)^{-1} h(h-1) \frac{F}{1-F} P = \tilde{M}^{-} - \left(\tilde{M}^{+}\right)^{-1} P G(\tilde{M}_{AF}^{2}), \qquad (89)$$

where, in the last form, G is a non local operator such that

$$G\tilde{E}_{h}^{A} = h(h-1)\frac{F(h)}{1-F(h)}\tilde{E}_{h}^{A}.$$
 (90)

Let us finally verify that, with this new representation of  $\tilde{M}_{AF}^{-}$ , the algebra of the Möbius generators is correct. It is easy to see that

$$[\tilde{M}_{AF}^{-}, \tilde{M}_{AF}^{3}] = -\tilde{M}_{AF}^{-}, \qquad (91)$$

which is a relation that is projected to zero when applied to the function  $\tilde{E}_h^A$ . In order to check that the Möbius algebra is closed we need to examine the relation

$$[\tilde{M}_{AF}^{-}, \tilde{M}_{AF}^{+}] = +2\tilde{M}_{AF}^{3}$$
(92)

understood to act on  $\tilde{E}_h^A$ . One then has to verify that

$$\tilde{M}_{AF}^{-}\tilde{M}_{AF}^{+}\tilde{E}_{h}^{A} = -2h\tilde{E}_{h}^{A}.$$
(93)

Using eq. (83) we derive also

$$\tilde{M}^{-}\tilde{M}^{+}\tilde{E}_{h}^{A} = \left(\tilde{M}^{+}\tilde{M}^{-} + 2\tilde{M}^{3}\right)\left(\tilde{E}_{h}^{M} - \tilde{E}_{h}^{\delta}\right) = -2h\tilde{E}_{h}^{A} + h(h-1)\tilde{E}_{h}^{\delta}$$
(94)

and making the difference between eqs. (93) and (94) we therefore obtain

$$\left(\tilde{M}_{AF}^{-}-\tilde{M}^{-}\right)\tilde{M}^{+}\tilde{E}_{h}^{A}=-h(h-1)\tilde{E}_{h}^{\delta}.$$
(95)

If  $\tilde{M}^+$  commutes with  $(\tilde{M}^-_{AF} - \tilde{M}^-)$  this is equivalent to

$$\left(\tilde{M}_{AF}^{-} - \tilde{M}^{-}\right)\tilde{E}_{h}^{A} = -h(h-1)\left(\tilde{M}^{+}\right)^{-1}\tilde{E}_{h}^{\delta} = -\left(\tilde{M}^{+}\right)^{-1}PG(\tilde{M}_{AF}^{2})\tilde{E}_{h}^{A},\qquad(96)$$

which coincides with the starting definition of the generator. The commutation relation is fullfilled since the operator  $\tilde{M}^+$  is just a multiplicative operator associated to the total momentum and therefore commutes with the action of the projector P and moreover it commutes with the function of the Casimir  $G(\tilde{M}_{AF}^2)$ .

Finally let us look at the relation between the two Casimir operators in M-space and AF-space: using  $\tilde{M}_{AF}^2 = \frac{1}{2} \left[ \tilde{M}_{AF}^+ \tilde{M}_{AF}^- + \tilde{M}_{AF}^- \tilde{M}_{AF}^+ \right] + \left( \tilde{M}_{AF}^3 \right)^2$  we find from the previous relations

$$\tilde{M}_{AF}^2 + P G(\tilde{M}_{AF}^2) = \tilde{M}^2$$
 (97)

This last relation, applied to  $\tilde{E}_h^A$ , gives

$$h(h-1)\tilde{E}_{h}^{A} + h(h-1)\tilde{E}_{h}^{\delta} = \tilde{M}^{2}\tilde{E}_{h}^{A}$$
, (98)

which indeed is compatible with  $\tilde{M}^2 \tilde{E}_h^M = h(h-1)\tilde{E}_h^M$ .

# 6 Conclusions

Among the properties of the LL BFKL kernel the bootstrap relation connected to the gluon reggeization is a fundamental consistency condition. This bootstrap property leads to an important feature of the BFKL kernel also in the color singlet state: this can be demonstrated most easily in momentum space where the BFKL kernel is meromorphic. Many previous investigations, exploiting the conformal symmetry of the kernel, have been carried out in the space of Möbius functions (M). In this space of functions, however, the bootstrap relation for the BFKL kernel is not satisfied.

In this paper we have defined a modified space of functions (AF), in which the bootstrap property is valid. In particular, we have derived a spectral representation of the kernel which is also useful in evaluating the coupling of the BFKL Green's function to colored impact factors, and we have verified explicitly that the bootstrap relation is fulfilled. We also have derived the corresponding representation of the Möbius algebra; in order to act in this modified space of functions, one of the Möbius generators has to be deformed.

Our discussion has been limited to the the case of two-gluon Green's functions. Since in both spaces, M and AF, the eigenvalues of the Casimir operator,  $\tilde{M}^2$  and  $\tilde{M}_{AF}^2$ , resp., are the same, and since the hamiltonian for the pairwise interaction of two reggeized gluons can be expressed in terms of this Casimir operator, one can expect that, for states consisting of more than two gluons all remarkable properties of the multi-colour BFKL dynamics derived in the Möbius picture can be generalized to the AF picture. This includes, in particular, the holomorphic separability and integrability [6]. We are going to return to these interesting problems in future publications.

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# Appendix A

The calculation of the integral in eq. (41) is carried on noting that one can find a third order differential operator in z and another in  $z^*$ , related to the  ${}_3F_2$  functions, which, when applied to the integrand, gives a total derivative in an integration variable so that, on applying the Stokes theorem, one finds that  $I_{\lambda}(z)$  satisfies the related differential equation in both the holomorphic and antiholomorphic sectors. This allows to write it as a sum of products of the linearly independent solutions of the two differential equations and on imposing the single valueness and the correct normalization (at some convenient point) the full expression is found [15]. In particular this has the following, so called, conformal block structure

$$I_{\lambda}(z) = \sum_{i=0}^{2} \lambda_{i} u_{i}(z) \bar{u}_{i}(z^{*}), \qquad (99)$$

where  $u_i$  and  $\bar{u}_i$  are three independent solutions of the generalized hypergeometric differential equations in z and  $z^*$ . The coefficients  $\lambda_i$  depend through  $\Gamma$  functions on the conformal weights. The general form of  $I_{\lambda}(z)$  is pretty complicated but it simplifies considerably for z = 1 using the relations [16]

$${}_{3}F_{2}\left(\begin{array}{c}a,b,c\\\frac{a+b+1}{2},2c\end{array}\middle|1\right) = \pi^{1/2}\frac{\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1+a+b}{2}\right)\Gamma\left(c+\frac{1-a-b}{2}\right)}{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{1+b}{2}\right)\Gamma\left(c+\frac{1-a}{2}\right)\Gamma\left(c+\frac{1-a-b}{2}\right)},$$

$${}_{3}F_{2}\left(\begin{array}{c}a,1-a,c\\f,2c+1-f\end{array}\middle|1\right) = \pi\frac{\Gamma\left(f\right)\Gamma\left(2c+1-f\right)2^{1-2c}}{\Gamma\left(\frac{a+f}{2}\right)\Gamma\left(\frac{1-a+f}{2}\right)\Gamma\left(c+\frac{1+a-f}{2}\right)\Gamma\left(1+c-\frac{a+f}{2}\right)}.$$
(100)

On using the above relation, after algebraic simplifications, we find

$$I_{\lambda}(1) = I^{(0)} + I^{(1)} + I^{(2)}$$
(101)

with

$$I^{(0)} = \frac{2^{-3+4\lambda}\Gamma\left(1-\frac{h}{2}\right)\Gamma\left(1-\frac{\bar{h}}{2}\right)\Gamma^{2}\left(1-\lambda\right)\Gamma^{2}\left(-\lambda\right)\Gamma\left(\frac{1-h}{2}+\lambda\right)\Gamma\left(\frac{1-\bar{h}}{2}+\lambda\right)}{\Gamma\left(\frac{3-h}{2}\right)\Gamma\left(\frac{3-\bar{h}}{2}\right)\Gamma\left(1-\frac{h}{2}-\lambda\right)\Gamma\left(1-\frac{\bar{h}}{2}-\lambda\right)} \times \frac{\sin\pi(\bar{h}-2\lambda)\sin\pi\lambda\tan\pi\bar{h}}{\sin\pi(\bar{h}-\lambda)},$$

$$I^{(1)} = -\frac{2^{-3+4\lambda}\Gamma\left(\frac{h-1}{2}\right)\Gamma\left(\frac{\bar{h}-1}{2}\right)\Gamma^{2}\left(1-\lambda\right)\Gamma^{2}\left(-\lambda\right)\Gamma\left(\frac{h}{2}+\lambda\right)\Gamma\left(\frac{\bar{h}}{2}+\lambda\right)}{\Gamma\left(\frac{h}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1+h}{2}-\lambda\right)\Gamma\left(\frac{1+\bar{h}}{2}-\lambda\right)} \times \frac{\sin\pi(\bar{h}+2\lambda)\sin\pi\lambda\tan\pi\bar{h}}{\sin\pi(\bar{h}+\lambda)},$$

$$I^{(2)} = \frac{2^{4\lambda}\pi^{6}\Gamma^{2}\left(1-\lambda\right)}{\Gamma\left(\frac{3-h}{2}\right)\Gamma\left(\frac{h}{2}\right)\Gamma\left(\frac{3-\bar{h}}{2}\right)\Gamma\left(\frac{h}{2}\right)\Gamma\left(\frac{1+h}{2}-\lambda\right)\Gamma\left(\frac{1+\bar{h}}{2}-\lambda\right)} \times \frac{1}{\Gamma\left(1-\frac{h}{2}-\lambda\right)\Gamma\left(1-\frac{\bar{h}}{2}-\lambda\right)\Gamma^{2}\left(1+\lambda\right)\sin\pi(\bar{h}+\lambda)}.$$
(102)

The expressions of  $I^{(0)}$  and  $I^{(1)}$  have a simple pole at  $\lambda = 0$  and opposite residues for this pole, while  $I^{(2)}$  is holomorphic at  $\lambda = 0$ , therefore the full function in eq. (101) is not singular at  $\lambda = 0$ . Using the transformation properties of the Gamma function, each of three terms can be reduced to a common multiplicative factor and expressions involving only trigonometric functions. These terms are combined together to give the following compact form for  $I_{\lambda}(1)$ 

$$I_{\lambda}(1) = \frac{2}{(1-h)(1-\bar{h})} \delta_{n,\text{even}} \times \frac{\Gamma\left(\frac{h}{2}+\lambda\right) \Gamma\left(\frac{\bar{h}}{2}+\lambda\right) \Gamma\left(\frac{1-h}{2}+\lambda\right) \Gamma\left(\frac{1-\bar{h}}{2}+\lambda\right)}{\Gamma\left(\frac{h}{2}\right) \Gamma\left(\frac{\bar{h}}{2}\right) \Gamma\left(\frac{1-h}{2}\right) \Gamma\left(\frac{1-\bar{h}}{2}\right)} \times 16^{\lambda} \left(1+i^{n} \frac{\sin 2\pi\lambda}{\sin \pi\left(\frac{h+\bar{h}}{2}\right)}\right) \Gamma^{2}(1-\lambda) \Gamma^{2}(-\lambda) \sin^{2}\pi\lambda.$$
(103)

Now the first few derivatives with respect to  $\lambda$  can be easily calculated. In particular we obtain, after some manipulations,

$$c_{0} \equiv I_{\lambda}(1)|_{\lambda=0} = \frac{2\pi^{2}}{(1-h)(1-\bar{h})} \,\delta_{n,\text{even}},$$

$$I_{\lambda}'(1)|_{\lambda=0} = 2 \,\epsilon_{h} \,c_{0},$$

$$I_{\lambda}''(1)|_{\lambda=0} = \left[ (2 \,\epsilon_{h})^{2} - (-1)^{h-\bar{h}} \,4\pi^{2} \,\csc^{2}\frac{h+\bar{h}}{2} + \psi'\left(\frac{h}{2}\right) + \psi'\left(\frac{\bar{h}}{2}\right) + \psi'\left(\frac{\bar{h}}{2}\right) + \psi'\left(\frac{1-\bar{h}}{2}\right) \right] c_{0}. \quad (104)$$

# Appendix B

We recompute here the double iteration of the BFKL kernel in momentum space acting on a constant function [17] and compare it with the expressions obtained with the use of the above spectral representation of the kernel (and therefore of the Green's function). Let us write before the result of the direct integration in momentum space:

$$(\bar{K}_{12}^{(1)})^2 \otimes 1 = \frac{1}{2} \bar{\alpha}_s \, \bar{K}_{12}^{(1)} \otimes \log\left(\frac{\boldsymbol{q}^4}{\boldsymbol{k}_1^2 \boldsymbol{k}_2^2}\right) = \\ = \left(\frac{1}{2} \bar{\alpha}_s\right)^2 \left[\log^2\left(\frac{\boldsymbol{q}^2}{\boldsymbol{k}_1^2}\right) + \log^2\left(\frac{\boldsymbol{q}^2}{\boldsymbol{k}_2^2}\right)\right]. \tag{105}$$

Before giving some details of the computation let us consider what we obtain from the spectral approach. On using the spectral representation for the kernel, the relation (105) can be written as

$$\langle k | \left( \hat{\bar{K}}_{12}^{(1)} \right)^2 | U \rangle = \left( \bar{\alpha}_s \right)^2 \sum_h \tilde{N}_h \langle k | h^A \rangle \epsilon_h^2 \langle h^A | U \rangle , \qquad (106)$$

a relation which we have checked numerically. This check has been also performed in the case of the single iteration (bootstrap relation). The argument of the integration over  $\nu$  oscillates with a very slow decay for fixed conformal spin, but summing over some tens of n gives a good suppression of the tails and the integral can be computed.

Finally we sketch the derivation of eq. (105). There are three kinds of integral involved; they can be calculated with the dimensional regularization and the Feynman parameterization, in the case  $\mathbf{k}_{1,2} \neq 0$ ,  $\mathbf{k}_1 \neq \mathbf{k}_2$ . The needed integrals are:

The needed integrals are:

$$\omega(\mathbf{k}_{1}) = \int d^{d}\mathbf{k}' \frac{\mathbf{k}_{1}^{2}}{(\mathbf{k}')^{2}(\mathbf{k}_{1} - \mathbf{k}')^{2}}, 
J_{1}(\mathbf{k}_{1}, \mathbf{k}_{2}) = \int d^{d}\mathbf{k}' \frac{\mathbf{k}_{1}^{2}}{(\mathbf{k}')^{2}(\mathbf{k}_{1} - \mathbf{k}')^{2}} \log \frac{\mathbf{q}^{2}}{(\mathbf{k}')^{2}}, 
J_{2}(\mathbf{k}_{1}, \mathbf{k}_{2}) = \int d^{d}\mathbf{k}' \frac{\mathbf{k}_{1}^{2}}{(\mathbf{k}')^{2}(\mathbf{k}_{1} - \mathbf{k}')^{2}} \log \frac{\mathbf{q}^{2}}{(\mathbf{q} - \mathbf{k}')^{2}},$$
(107)

where  $d = 2 + 2\epsilon$ ,  $q = k_1 + k_2$  and  $\omega(k_1)$  is the gluon trajectory (2) rescaled by  $-N_c/2c$ . The first two integral can be easily calculated exactly, giving:

$$\omega(\mathbf{k}_{1}) = \pi^{1+\epsilon} \frac{\Gamma(1-\epsilon)\Gamma^{2}(\epsilon)}{\Gamma(2\epsilon)} (\mathbf{k}_{1}^{2})^{\epsilon} =$$

$$= \pi^{1+\epsilon}\Gamma(1-\epsilon)(\mathbf{q}^{2})^{\epsilon} \left[\frac{2}{\epsilon} - 2\log\frac{\mathbf{q}^{2}}{\mathbf{k}_{1}^{2}} + \mathcal{O}(\epsilon)\right],$$

$$J_{1}(\mathbf{k}_{1},\mathbf{k}_{2}) = \pi^{1+\epsilon}\frac{\Gamma(1-\epsilon)\Gamma^{2}(\epsilon)}{\Gamma(2\epsilon)} \left[\pi\cot\pi\epsilon + \log\frac{\mathbf{q}^{2}}{\mathbf{k}_{1}^{2}} + \psi(2\epsilon) - \psi(1)\right] (\mathbf{k}_{1}^{2})^{\epsilon} =$$

$$= \pi^{1+\epsilon}\Gamma(1-\epsilon)(\mathbf{q}^{2})^{\epsilon} \left[\frac{1}{\epsilon^{2}} + \frac{1}{\epsilon}\log\frac{\mathbf{q}^{2}}{\mathbf{k}_{1}^{2}} - \frac{\pi^{2}}{6} - \frac{3}{2}\log^{2}\frac{\mathbf{q}^{2}}{\mathbf{k}_{1}^{2}} + \mathcal{O}(\epsilon)\right]. \quad (108)$$

The third integral is calculated to the order  $\mathcal{O}(\epsilon)$ :

$$J_2(\mathbf{k}_1, \mathbf{k}_2) = \pi^{1+\epsilon} \Gamma(1-\epsilon) (\mathbf{q}^2)^{\epsilon} \left[ \frac{1}{\epsilon} \log \frac{\mathbf{q}^2}{\mathbf{k}_2^2} - \frac{1}{2} \log \frac{\mathbf{q}^2}{\mathbf{k}_2^2} - \log \frac{\mathbf{q}^2}{\mathbf{k}_1^2} \log \frac{\mathbf{q}^2}{\mathbf{k}_2^2} + \mathcal{O}(\epsilon) \right] .$$
(109)

Putting together all the pieces to construct the complete action of the kernel, all the singulatities cancel and the result, which is finite in the limit  $\epsilon \to 0$ , correspond to the expression given in eq. (105).

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