

## Inequalities between Quark Densities Elvio Di Salvo

Dipartimento di Fisica and I.N.F.N. - Sez. Genova, Via Dodecaneso, 33  
16146 Genova, Italy

**Abstract** We propose an inequality between the longitudinally polarized density and the transversity of a quark in a nucleon. This inequality, whose validity is limited to very small scales, is based on considerations about Lorentz transformations and on commonly accepted models. Therefore it may be used as a consistency check with other models. It turns out to agree with most model predictions. Moreover it allows to establish, thanks to the positivity constraint, another inequality between the longitudinally polarized and the unpolarized valence quark density. We show that this latter inequality may be extended to any  $Q^2$ , consistently with commonly used factorization schemes and with some nonperturbative evolution models. This inequality finds nontrivial applications to the valence  $d$ -quark densities and is compared with data analyses, with model predictions and with parametrizations of quark densities.

PACS numbers: 13.85.Qk, 13.88.+e

Inequalities may be useful in extracting quark densities from data[1, 2]. In particular, important bounds can be fixed by means of the positivity constraint[3, 1], which gives rise, among other things, to the famous Soffer inequality[4]. This may be employed, for example, for determining the behavior of the polarized densities for  $x \rightarrow 1$ [5]. In the present article we propose two inequalities concerning polarized and unpolarized densities. The first inequality, which regards the transversely and longitudinally polarized densities, may be taken into account only at small  $Q^2$ ; it relies on considerations about Lorentz transformations and on a property shared by commonly accepted models. This inequality may be used as a consistency check with other models and results to be fulfilled by most current model predictions. The second inequality is inherent to unpolarized and longitudinally polarized densities. It is a consequence of the previous one and of the Soffer inequality and, according to some evolution pictures, it may be extended to any  $Q^2$ . This latter inequality finds nontrivial applications to the valence  $d$ -quark densities: it results in accord with model predictions, with available data analyses and with some best fits.

We introduce the transverse momentum (tm) longitudinally polarized density  $\delta q(x, \mathbf{p}_\perp^2)$  and the tm transversity  $\delta_T q(x, \mathbf{p}_\perp)$ , whose integrals over  $\mathbf{p}_\perp$  are, respectively, the functions  $\Delta q(x)$  and  $h_1(x)$  (this latter often denoted as  $\Delta_T q(x)$ ).

Let us consider  $Q^2$  values such that  $Q^2 \ll M^2$ , where  $M$  is the nucleon rest mass. Then a quark can be viewed as a constituent quark[6]. In the rest frame of a polarized nucleon, denote with  $q_0^\pm(\mathbf{p}')$  the probability density for a quark of momentum  $\mathbf{p}' \equiv (\mathbf{p}_\perp, p'_3)$  to have spin parallel (+) or antiparallel (-) to the nucleon spin vector  $\mathbf{S}_0$ . A boost parallel to the nucleon spin, and such that the nucleon momentum becomes infinite, produces a spin dilution in the quark polarization density, which, in the  $Q^2$ -range considered, turns out to coincide with  $\delta q$ , *i. e.*,

$$\delta q(x, \mathbf{p}_\perp^2) = [q_0^+(\mathbf{p}') - q_0^-(\mathbf{p}')] \cos\theta_M. \quad (1)$$

Here  $x = (p'_0 + p'_3)/M$  and  $p'_0 = \sqrt{m^2 + \mathbf{p}'^2}$ . Moreover  $\theta_M$  is the Melosh-Wigner rotation angle[6, 7], *i. e.*,

$$\theta_M = \arccos \left[ \frac{X^2 - \mathbf{p}_\perp^2}{X^2 + \mathbf{p}_\perp^2} \right] \quad (2)$$

and  $X = m + p'_0 + p'_3$ . Under the same conditions, a boost from the nucleon rest system, analogous to the previous one, but in a direction perpendicular to its spin, produces a less drastic spin dilution. Really, in this case the density results in  $\delta_T q$ , the Melosh-Wigner rotation giving[6]

$$\delta_T q(x, \mathbf{p}_\perp) = [q_0^+(\mathbf{p}') - q_0^-(\mathbf{p}')] D_\perp(\theta_M, \phi). \quad (3)$$

Here

$$D_\perp(\theta_M, \phi) = \cos^2 \frac{\theta_M}{2} + \sin^2 \frac{\theta_M}{2} (2 \sin^2 \phi - 1) \quad (4)$$

and  $\phi$  is the azimuthal angle of  $\mathbf{p}_\perp$  with respect to the plane perpendicular to  $\mathbf{S}_0$ . Eqs. (1) and (4) imply

$$\delta_T q(x, \mathbf{p}_\perp) = \delta q(x, \mathbf{p}_\perp^2) \frac{D_\perp(\theta_M, \phi)}{\cos \theta_M}. \quad (5)$$

This, in turn, implies, for  $Q^2 \ll M^2$ , the inequality

$$\frac{\delta_T q(x, \mathbf{p}_\perp)}{\delta q(x, \mathbf{p}_\perp^2)} \geq 1, \quad (6)$$

which reduces to equality for a nonrelativistic bound state. Now we make the following assumption:

*The difference  $q_0^+(\mathbf{p}') - q_0^-(\mathbf{p}')$  is either always positive or always negative, except, at most, in a small neighborhood of  $x = 1$ , where  $q_0^-(\mathbf{p}') \rightarrow 0$ .*

This assumption agrees with the predictions of some models[8, 9, 10, 11], which explain various data concerning the nucleon phenomenology. Moreover it implies that the valence quark density,  $\Delta q_v$ , related to  $q_0^+(\mathbf{p}') - q_0^-(\mathbf{p}')$  through eq. (1) and through integration over  $\mathbf{p}_\perp$ , is either positive (for  $u$ -quarks) or negative (for  $d$ -quarks) for any  $x$  (except, possibly, for  $x \simeq 1$ [5]). As we shall see below, this property may be assumed also at large  $Q^2$ , in accord with best fits to high energy data[1].

Our assumption implies, together with eq. (6),

$$\frac{h_1(x)}{\Delta q(x)} \geq 1 \quad (7)$$

for  $Q^2 \ll M^2$ . Inequality (7) - which again reduces to an equality for a nonrelativistic bound state - is in accord with almost all previous model calculations, based on the

constituent quark model[7, 6], on the bag model[12, 13], on light cone models[14] or on the chiral quark model[15] (see also ref. [16] for a review). It disagrees only with the calculation based on the chiral quark soliton model[17]. It would be interesting to study the origin of this discrepancy.

At increasing  $Q^2$ ,  $h_1$  decreases much more rapidly than  $\Delta q$ , owing to a different evolution kernel, therefore the inequalities (6) and (7) no longer hold true. However, relation (7) gives rise, together with the Soffer inequality[4], to a bound which may be assumed for any  $Q^2$ , as we shall see in a moment. The Soffer inequality reads

$$2|h_1(x)| \leq q(x) + \Delta q(x), \quad (8)$$

where  $q(x)$  is the unpolarized quark density. We get from (7) and (8)

$$2|\Delta q(x)| \leq q(x) + \Delta q(x) \quad (Q^2 \ll M^2), \quad (9)$$

which is nontrivial for negative values of  $\Delta q(x)$ :

$$-3\Delta q(x) \leq q(x) \quad (Q^2 \ll M^2). \quad (10)$$

Since  $q(x)$  is nonnegative, ineq. (10) holds true for *any* value of  $x$ . This inequality is especially interesting as regards the *valence*  $d$ -quark in the nucleon, whose polarized density is found to be negative for almost all  $x$ . In particular, the nonrelativistic  $SU(6)$ -symmetric quark model predicts

$$\Delta d_v(x) = -\frac{1}{3}d_v(x), \quad (11)$$

which saturates ineq. (10).

Now we show that this inequality, if referred to *valence* quarks, *i. e.*

$$-3\Delta q_v(x) \leq q_v(x), \quad (12)$$

may be extended to any  $Q^2$ , consistent with commonly used factorization schemes and nonperturbative evolution models. This amounts to showing that the function

$$\phi(x, t) = q_v(x, t) + 3\Delta q_v(x, t), \quad t = \ln(Q^2/Q_0^2), \quad (13)$$

with  $Q_0^2 \ll M^2$ , is positive for any  $x$  and  $t \geq 0$ , given that  $\phi(x, 0) > 0$ , as follows from (10). To this end, consider evolution equations for  $q_v^\pm(x, t)$ , where

$$q_v^\pm(x, t) = \frac{1}{2}[q_v(x, t) \pm \Delta q_v(x, t)]. \quad (14)$$

Taking into account parity conservation, set

$$\frac{d}{dt}q_v^\pm(x, t) = \int_x^1 \frac{dy}{y} [P_n^v(x, y, t)q_v^\pm(y, t) + P_f^v(x, y, t)q_v^\mp(y, t)]. \quad (15)$$

Here  $P_{n(f)}^v(x, y, t)$  is the probability density for a quark with initial fractional momentum  $y$  to evolve into a quark with fractional momentum  $x \leq y$ , without (with) helicity flip. These equations present a strong analogy with the Boltzmann equation[18, 19]. Positivity of the quark densities  $q_v^\pm(x, t)$  demands[18], for  $x < y$ ,

$$P_n^v > 0, \quad P_f^v > 0, \quad (16)$$

with, at most, singular diagonal terms at  $x = y$ . Moreover eqs. (15) imply

$$\frac{d}{dt}q_v(x, t) = \int_x^1 \frac{dy}{y} P^v(x, y, t)q_v(y, t), \quad (17)$$

$$\frac{d}{dt}\Delta q_v(x, t) = \int_x^1 \frac{dy}{y} \Delta P^v(x, y, t)\Delta q_v(y, t), \quad (18)$$

$$\frac{d}{dt}\phi(x, t) = \int_x^1 \frac{dy}{y} [\Delta P^v(x, y, t)\phi(y, t) + 2P_f^v(x, y, t)q_v(y, t)]. \quad (19)$$

Here  $P^v(x, y, t) = P_n^v(x, y, t) + P_f^v(x, y, t)$  and  $\Delta P^v(x, y, t) = P_n^v(x, y, t) - P_f^v(x, y, t)$ . Then from eq. (19) and from the second condition (16) it follows that  $\phi$  will be positive for  $t > 0$  if, for  $x < y$ ,

$$\Delta P^v > 0, \quad (20)$$

*i. e.*,  $P_n^v > P_f^v$ . As a byproduct, condition (20) implies, together with eq. (18), that, if  $\Delta q_v$  is positive (negative) for any  $x$  at  $t = 0$ , it is positive (negative) for any  $x$  and  $t > 0$ . Now we discuss to what extent condition (20) may be supported by commonly accepted evolution pictures, examining in detail two different situations.

For sufficiently large  $Q^2 (> M^2)$  the evolution of the quarks is governed essentially by perturbative massless QCD. Therefore helicity conservation implies  $P_f^v = 0$  at leading order. At higher orders the evolution is scheme dependent[20], as the parton

densities are not directly observable quantities[18]. However,  $P_f^v$  is still vanishing in the  $\overline{MS}$  chirally conserving scheme[21, 22] (see also [20] and refs. therein), as well as in the JET scheme[23, 1]. Moreover, at least in the  $\overline{MS}$  scheme, it is not completely unrealistic to extend down to  $Q^2 \simeq 0.34 \text{ GeV}^2$  the perturbative evolution picture at next-to-leading order, respecting the positivity conditions (16) (see [24] and refs. therein).

At decreasing  $Q^2$  the perturbative quark-gluon interaction is accompanied, and then gradually supplanted, by more complex, nonperturbative mechanisms of recombination[25]. In particular, owing to spontaneous chiral symmetry breaking, recombination is supposed to give rise, above all, to Goldstone bosons[25], especially to pions[15]. Therefore one may assume the prevalent evolution splitting for a quark at small  $Q^2$  to be  $q \rightarrow q'\pi$ [26]. In this regime, especially at very small  $Q^2$ , it is not clear whether a DGLAP-type evolution picture is applicable at all, since a bound state emission is involved. However, the picture just described seems to account satisfactorily for the evolution of the Gottfried sum from very small to large  $Q^2$ [26]. Therefore it is worth deducing predictions of such a model about the evolution of  $\phi$ . Now each evolution equation (17) to (19) is replaced by a system of two coupled equations, whose unknowns are linear combinations of the *up* and *down* valence quark densities, and whose kernels are  $2 \times 2$  matrices, nondiagonal elements referring to charged pion emission. Conditions (16) are replaced by  $(P_n^v)_{ij} > 0$  and  $(P_f^v)_{ij} > 0$  for all matrix elements; moreover helicity conservation at forward - i. e., in the most favored direction of emission, according to four-momentum and angular momentum conservation - implies  $(P_f^v)_{ij} < (P_n^v)_{ij}$ ,<sup>1</sup> which corresponds to condition (20). Our conclusion is not modified by taking into account also a splitting of the type  $q \rightarrow q'\rho$ , which, although contributing to helicity flip, is much more unlikely than the one involving the pion[27]. Nor is our argument changed by the assumption that the Gottfried rule fails - due to fluctuations of the nucleon into a pion and a baryon[28] - even at very small scales, as results from chiral quark soliton model calculations[29]. Lastly, due to the linearity of the evolution equations, positivity is conserved also in the  $Q^2$  range

---

<sup>1</sup>If we take the model of ref. [26] strictly, we find  $(P_f^v)_{ij} = 0$ , since both the quark and the pion are assumed to be massless.

where both perturbative and nonperturbative evolution mechanisms coexist.

This completes the argument in favor of ineq. (12), which constitutes a nontrivial bound. For example, it implies, together with the positivity constraint,  $-1/3q_v(x) \leq \Delta q_v(x) \leq q_v(x)$ , which is stronger than the inequality  $|\Delta q(x)| \leq q(x)$ , usually taken into account in the fits to data of polarized deep inelastic scattering[1].

Ineq. (12) agrees with the predictions of some models, like the constituent quark model[9] and the Carlitz-Kaur model[10] (see also ref. [30]). Moreover, as regards  $d$ -quarks, integrating over  $x$  from 0 to 1 both sides of that inequality yields

$$\Delta D_v \geq -1/3, \quad (21)$$

where  $\Delta D_v$  is the first moment of  $\Delta d_v(x)$ . Bound (21) is fulfilled by a lattice calculation[31]; it is also in agreement with HERMES[32] and SMC[33] data analyses, where one has assumed a flavor symmetric sea polarization. Concerning best fits, this bound agrees with the one by Leader, Sidorov and Stamenov[1], who assume an  $SU(3)$ -symmetric polarized sea. The accord is even improved, if an asymmetry is introduced in the polarized sea, either by means of an *ad hoc* parameter[34, 35, 36], or as a consequence of a Pauli-blocking ansatz[37]. On the contrary, bound (21) is not respected by the fit in ref. [2], where no constraints are assumed for the sea. Of course, as pointed out in refs. [35, 36], the splitting of a polarized quark density into valence and sea contributions is strongly model dependent. But ineq. (12) may reduce such a dependence, consistently with commonly used factorization schemes - adherent, as far as possible, to a parton model description[38, 21] - and with nonperturbative models at  $Q^2 \ll M^2$ .

To summarize, we have proposed two inequalities, (7) and (12). The former inequality, which relies on general considerations about Lorentz transformations and on a property shared by commonly accepted models, is found to agree with almost all model predictions about  $h_1(x)$  and  $\Delta q(x)$  at very small scales. The latter inequality, deduced from the former one, from Soffer's inequality and successful evolution pictures, agrees with model predictions, with analyses of available data and with some best fits.

## Acknowledgements

The author is grateful to his friend G. Ridolfi for fruitful discussions.

## References

- [1] E. Leader, A.V. Sidorov and D.B. Stamenov, Eur. Phys. J. C **23** (2002) 479;  
Phys. Lett. B **445** (1998) 232
- [2] J. Bartelski and S. Tatur, Phys. Rev. D **65** (2002) 034002; hep-ph/0205089
- [3] J. Soffer, Phys. Rev. Lett. **91** (2003) 092005
- [4] J. Soffer, Phys. Rev. Lett. **74** (1995) 1292
- [5] M. Boglione and E. Leader, Phys. Rev. D **61** (2000) 114001
- [6] I. Schmidt and J. Soffer, Phys. Lett. B **407** (1997) 331
- [7] B.-Q. Ma, J. Phys. G: Nuc. Part. Phys. **17** (1991) L53
- [8] A. Le Yaouanc, L. Oliver, O. Pène and J.C. Reynal, Phys. Rev. **9** (1974) 2636;  
**12** (1975) 2137
- [9] N. Isgur, Phys. Rev. D **59** (1999) 034013
- [10] R. Carlitz and J. Kaur, Phys. Rev. Lett. **38** (1977) 673
- [11] E. Di Salvo, J. Phys. G: Nuclear and Particle Physics **16** L285 (1990)
- [12] R.L. Jaffe and X. Ji, Phys. Rev. Lett. **67** (1991) 552; Nucl. Phys. B **375** (1992)  
527
- [13] S. Scopetta and V. Vento, Phys. Lett. B **424** (1997) 31
- [14] K. Suzuki and W. Weise, Nucl. Phys. A **634** (1998) 141
- [15] A. Manohar and H. Georgi, Nucl. Phys. B **234** (1984) 189



- [16] V. Barone, A. Drago and P. Ratcliffe, Phys. Rept. **359** (2002) 1
- [17] D.I. Diakonov and V.Yu. Petrov, Nucl. Phys. B **272** (1986) 457; D.I. Diakonov, V.Yu. Petrov and P.V. Pobylitsa, Nucl. Phys. B **306** (1988) 809
- [18] C. Bourrely, J. Soffer and O. Teryaev, Phys. Lett. B **420** (1998) 375
- [19] C. Bourrely, E. Leader and O. Teryaev, hep-ph/9803238, Talk given at the VII Workshop on High energy Spin Physics (SPIN-97), Dubna, July 7-12, 1997
- [20] B. Lampe and E. Reya: Phys. Rep. **332** (2000) 1
- [21] W. Vogelsang, Phys. Rev. D **54** (1996) 2023; Nucl. Phys. B **475** (1996) 47
- [22] See refs. [20, 21] and refs. therein
- [23] E. Leader, A.V. Sidorov and D.B. Stamenov, Phys. Rev. D **58** (1998) 14028; see also refs. therein
- [24] M. Glueck, E. Reya, M. Stratmann and W. Vogelsang, Phys. Rev. D **53** (1996) 4775
- [25] M. Lavelle and D. McMullan, Phys. Rep. **279** (1997) 1
- [26] R. Ball and S. Forte, Nucl. Phys. B **425** (1994) 516
- [27] R.J. Fries and A. Shaefer, Phys. Lett. B **443** (1998)
- [28] see, *e. g.*, ref. [27] and refs. therein
- [29] D. Diakonov *et al.*, Nucl. Phys. B **480** (1996) 341
- [30] J. Qiu, G.P. Ramsey, D. Richards and D. Sivers, Phys. Rev. D **41** (1990) 65
- [31] M. Goeckeler *et al.*, Phys. Lett. B **414** (1997) 340
- [32] HERMES coll., K. Ackerstaff *et al.*, Phys. Lett. B **464** (1999) 123
- [33] SMC coll., B. Adeva *et al.*, Phys. Lett. B **420** (1997) 180

- [34] E. Leader, A.V. Sidorov and D.B. Stamenov, Phys. Rev. D **58** (1998) 14028
- [35] S. Tatur, J. Bartelski and M. Kurzela, Acta Phys. Pol. **31** (2000) 647
- [36] J. Bartelski and S. Tatur, Acta Phys. Pol. **32** (2001) 2101
- [37] M. Glueck, E. Reya, M. Stratmann and W. Vogelsang, Phys. Rev. D **63** (2001) 094005
- [38] C. Curci, W. Furmanski and R. Petronzio, Nucl. Phys. B **175** (1980) 27