## Model of Geometric Neutrino Mixing

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#### Abstract

Current neutrino oscillation data from solar, atmospheric, and reactor experiments are consistent with the neutrino mixing matrix elements taking values  $\sin^2 \theta_{12} =$ 1/3,  $\sin^2 \theta_{23} = 1/2$ , and  $\sin^2 \theta_{13} = 0$ . We present a class of renormalizable gauge models which realize such a geometric mixing pattern naturally. These models, which are based on the non–Abelian discrete symmetry  $A_4$ , place significant restrictions on the neutrino mass spectrum, which we analyze. It is shown that baryogenesis via leptogenesis occurs quite naturally, with a single phase (determined from neutrino oscillation data) appearing in leptonic asymmetry and in neutrinoless double beta decay. Such predicted correlations would provide further tests of this class of models.

## Introduction

Our understanding of the fundamental properties of neutrinos has improved dramatically over the last few years. Atmospheric and solar neutrino experiments have by now firmly established occurrences of neutrino flavor oscillations [1]. In order for neutrinos to oscillate, they must have non-degenerate masses. In addition, different neutrino flavor states must mix with one another. When positive evidence for oscillations from solar and atmospheric neutrinos are combined with results from reactor data [2, 3], one obtains the following neutrino mass and mixing pattern (with  $2\sigma$  error bars) [3]:

$$\Delta m_{\odot}^2 = m_2^2 - m_1^2 = 7.92 \times 10^{-5} (1 \pm 0.09) \text{ eV}^2, \tag{1}$$

$$\Delta m_{\rm atm}^2 = m_3^2 - m_2^2 = \pm 2.4 \times 10^{-3} (1^{+0.21}_{-0.61}) \,\,{\rm eV}^2,\tag{2}$$

$$\sin^2 \theta_{12} = 0.314(1^{+0.18}_{-0.15}), \quad \sin^2 \theta_{23} = 0.44(1^{+0.41}_{-0.22}), \quad \sin^2 \theta_{13} = 0.9^{+0.23}_{-0.9} \times 10^{-2}.$$
(3)

Here  $m_i$  are the (positive) neutrino mass eigenvalues, and  $\theta_{ij}$  are the neutrino mixing angles.  $m_2^2 - m_1^2 > 0$  in Eq. (1) is necessary for MSW resonance to occur inside the Sun. The sign of  $\Delta m_{\text{atm}}^2$ , which is physical, is currently unknown.

A remarkable feature of the oscillation data is that they are all consistent with a "geometric" neutrino mixing pattern defined by the parameters  $\sin^2 \theta_{12} = 1/3$ ,  $\sin^2 \theta_{23} = 1/2$ , and  $\sin^2 \theta_{13} = 0$ . In fact, these geometric mixing angles are very close to the central values of Eq. (3). We observe that unlike the quark mixing angles, which are related to the quark mass ratios in many models (eg:  $\theta_C \simeq \sqrt{m_d/m_s}$ ), the neutrino mixing angles seem to be unrelated to the neutrino mass ratios.

The purpose of this Letter is to provide a derivation of such a geometric neutrino mixing based on renormalizable gauge theories. The neutrino mixing matrix (the MNS matrix) that we will derive has the form [4, 5, 6, 7]

$$U_{MNS} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0\\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} P .$$
(4)

Here P is a diagonal phase matrix which is irrelevant for neutrino oscillations, but relevant for neutrinoless double beta decay. Eq. (4) yields the desired values,  $|U_{e2}|^2 = 1/3$ ,  $|U_{\mu3}|^2 = 1/2$ ,  $|U_{e3}|^2 = 0$ .

Our derivation of Eq. (4) will be based on the non-Abelian discrete symmetry  $A_4$ , the symmetry group of a regular tetrahedron. This symmetry group has found application in obtaining maximal atmospheric neutrino mixing [8] and in realizing quasi-degenerate neutrino mass spectrum [9]. No successful derivation of Eq. (4) has been achieved to our knowledge (based on  $A_4$  or other symmetries) in a renormalizable gauge theory context. For attempts along this line see Ref. [4, 10, 11]. In Refs. [4, 6], Eq. (4) was suggested as a phenomenological ansatz. In Ref. [10], a higher dimensional set up is used to motivate Eq. (4). Ref. [11] analyzes special cases of an  $A_4$  derived neutrino mass matrix towards obtaining the structure of Eq. (4). A large number of models in the literature have derived maximal atmospheric mixing based on non-Abelian symmetries [12], but in most models the solar mixing angle is either maximal (now excluded by data) or is a free parameter.

We will see that the derivation of Eq. (4) places strong restrictions on the neutrino mass pattern. We find that the out of equilibrium decay of the lightest right-handed neutrino generates lepton asymmetry at the right level to explain the observed baryon asymmetry of the universe. A single phase appears in leptonic asymmetry as well as in neutrinoless double beta decay, thus providing some hope for testing high scale phenomena via low energy experiments.

## The Model

We work in the context of low energy supersymmetry, which is motivated by a solution to the gauge hierarchy problem as well as by the observed unification of gauge couplings. The gauge group of our model is that of the Standard Model,  $SU(3)_C \times SU(2)_L \times U(1)_Y$ . We augment this symmetry with a non-Abelian discrete symmetry  $A_4$ . This order 12 group is the symmetry group of a regular tetrahedron.  $A_4$  has a unique feature in describing the lepton sector: It has one triplet and three inequivalent singlet representations, thus allowing for assigning the left-handed lepton fields to the triplet and the right-handed charged lepton fields to the three inequivalent singlets.

Denoting the three singlets of  $A_4$  as  $(\mathbf{1}, \mathbf{1}', \mathbf{1}'')$ , with the **1** being the identity representation, we have  $\mathbf{1}' \times \mathbf{1}'' = \mathbf{1}$ ,  $\mathbf{1}' \times \mathbf{1}' = \mathbf{1}''$ , and  $\mathbf{1}'' \times \mathbf{1}'' = \mathbf{1}'$ . Furthermore,  $\mathbf{3} \times \mathbf{3} = \mathbf{1} + \mathbf{1}' + \mathbf{1}'' + \mathbf{3}_s + \mathbf{3}_a$ . Specifically, for the product of two triplets we have  $(a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_1b_1 + a_2b_2 + a_3b_3)$  (**1**);  $(a_1b_1 + \omega^2a_2b_2 + \omega a_3b_3)$  (**1**');  $(a_1b_1 + \omega a_2b_2 + \omega^2a_3b_3)$  (**1**'');  $(a_2b_3 + a_3b_2, a_3b_1 + a_1b_3, a_1b_2 + a_2b_1)$  (**3**<sub>s</sub>). Here  $\omega = e^{2i\pi/3}$ .

In addition to the  $A_4$  symmetry, we assume a  $Z_4 \times Z_3$  discrete symmetry. The  $Z_4$  is an *R*-symmetry under which the superpotential carries 2 units of charge. The  $Z_4$  and  $Z_3$ symmetries are broken softly in the superpotential via the lowest dimensional operators.

The lepton and Higgs fields transform under  $A_4 \times Z_4 \times Z_3$  as follows.

$$L: (3,1,0), e^{c}: (1+1'+1'',3,0), \nu^{c}: (3,0,1), E: (3,1,0), E^{c}: (3,1,0), H_{u}: (1,1,2), H_{d}: (1,0,0), \chi: (3,2,0), \chi': (3,2,1), S_{1,2}: (1,2,1).$$
(5)

Here in the fermion sector we have introduced new vector-like iso-singlet fields E and  $E^c$  transforming under the SM gauge group as (1,1,-1) and (1,1,1), respectively, which will acquire large masses and decouple.  $H_u$  and  $H_d$  are the usual Higgs fields of MSSM, while  $\chi, \chi', S_{1,2}$  are all SM singlet fields needed for achieving symmetry breaking. The quark fields  $(Q, u^c, d^c)$  are all singlets of  $A_4$  with  $Z_4 \times Z_3$  charges of  $Q(1,1); u^c(0,0)$  and  $d^c(1,2)$ , so that the usual quark Yukawa couplings  $Qd^cH_d + Qu^cH_u$  are allowed in the superpotential.<sup>1</sup>

The superpotential terms relevant for lepton masses consistent with the symmetries is

$$W_{\text{Yuk}} = M_E E_i E_i^c + f_e L_i E_i^c H_d + h_{ijk}^e E_i e_j^c \chi_k + \frac{1}{2} f_{S_l} \nu_i^c \nu_i^c S_l + \frac{1}{2} f_{ijk} \nu_i^c \nu_j^c \chi_k' + f_\nu L_i \nu_i^c H_u.(6)$$

Here the flavor structure of the three independent  $h_{ijk}^e$  couplings and the one independent  $f_{ijk}$  coupling can be easily obtained from the  $A_4$  multiplication rules given earlier.

The Higgs superpotential of the model is given by

$$W_{\text{Higs}} = \lambda_{\chi} \chi_{1} \chi_{2} \chi_{3} + \lambda_{\chi's} (\chi_{1}^{\prime 2} + \chi_{2}^{\prime 2} + \chi_{3}^{\prime 2}) S_{1} + \lambda_{\chi'}^{\prime} \chi_{1}^{\prime} \chi_{2}^{\prime} \chi_{3}^{\prime} + \lambda_{s_{11}} S_{1}^{3} + \lambda_{s_{12}} S_{1}^{2} S_{2} + \lambda_{s_{21}} S_{1} S_{2}^{2} + \lambda_{s_{22}} S_{2}^{3} + \mu_{1}^{2} S_{1} + \mu_{2}^{2} S_{2} + \mu_{\chi} (\chi_{1}^{2} + \chi_{2}^{2} + \chi_{3}^{2}).$$
(7)

Here  $\chi = (\chi_1, \chi_2, \chi_3)$ , and  $\chi' = (\chi'_1, \chi'_2, \chi'_3)$ . The last three terms in the last line of Eq. (7) break the  $Z_4$  and  $Z_3$  symmetries softly. The  $\mu^2_{1,2}$  terms are the lowest dimensional terms that break the  $Z_3$  symmetry softly, while leaving  $Z_4$  unbroken. The  $\mu_{\chi}$  term is the lowest dimensional term that breaks the  $Z_4$  symmetry softly. Such soft breaking can be

<sup>&</sup>lt;sup>1</sup>A bare mass term  $\mu H_u H_d$  is not allowed in the superpotential by the symmetries, but the Kahler potential, which is assumed to not respect these symmetries, allows a Planck mass suppressed term,  $\mathcal{L} \supset \int H_u H_d Z^* d^4 \theta / M_{\rm Pl}$ , generating the required  $\mu$  term.

understood as spontaneous breaking occurring at a higher scale. We have chosen without loss of generality the combination of  $S_1$  and  $S_2$  that couples to  $\chi'$  as simply  $S_1$  in Eq. (7).

Minimizing the potential derived from Eq. (7) in the supersymmetric limit, we obtain the following vacuum structure:

$$\langle S_2 \rangle = v_s, \ \langle S_1 \rangle = 0; \ \langle \chi_1 \rangle = \langle \chi_2 \rangle = \langle \chi_3 \rangle = v_{\chi}; \ \langle \chi'_2 \rangle = v_{\chi'}, \ \langle \chi'_1 \rangle = 0; \ \langle \chi'_3 \rangle = 0.$$
(8)

with  $v_{\chi} = -2\mu_{\chi}/\lambda_{\chi}$ ,  $v_s^2 = -\mu_2^2/(3\lambda_{s_{22}})$ , and  $v_{\chi'}^2 = (\lambda_{s_{21}}\mu_2^2 - 3\lambda_{s_{22}}\mu_1^2)/(3\lambda_{s_{22}}\lambda_{\chi's})$ . Electroweak symmetry breaking is achieved in the usual way by  $\langle H_u \rangle = v_u$ ,  $\langle H_d \rangle = v_d$ . We emphasize that vanishing of certain VEVs is a stable result, owing to the discrete symmetries present in the model. This is important for deriving the MNS matrix of Eq. (4). We observe that there are no pseudo-Goldstone modes, as can be seen by directly computing the masses of the Higsinos from Eq. (7).

The mass matrices  $M_{eE}$  for the charged leptons and  $M_{\nu\nu^c}$  for the neutral leptons resulting from Eqs. (6) and (8) are given by (in the notation  $\mathcal{L} = (e, E) M_{eE} (e^c, E^c)^T$ )

$$M_{eE} = \begin{pmatrix} 0 & 0 & 0 & f_e v_d & 0 & 0 \\ 0 & 0 & 0 & 0 & f_e v_d & 0 \\ 0 & 0 & 0 & 0 & 0 & f_e v_d \\ h_1^e v_{\chi} & h_2^e v_{\chi} & h_3^e v_{\chi} & M_E & 0 & 0 \\ h_1^e v_{\chi} & h_2^e \omega v_{\chi} & h_3^e \omega^2 v_{\chi} & 0 & M_E & 0 \\ h_1^e v_{\chi} & h_2^e \omega^2 v_{\chi} & h_3^e \omega v_{\chi} & 0 & 0 & M_E \end{pmatrix},$$

$$M_{\nu\nu^e} = \begin{pmatrix} 0 & 0 & 0 & f_{\nu} v_u & 0 & 0 \\ 0 & 0 & 0 & 0 & f_{\nu} v_u & 0 \\ 0 & 0 & 0 & 0 & f_{\nu} v_u & 0 \\ f_{\nu} v_u & 0 & 0 & f_{s_2} v_s & 0 & f_{\chi'} v_{\chi'} \\ 0 & f_{\nu} v_u & 0 & 0 & f_{s_2} v_s & 0 \\ 0 & 0 & f_{\nu} v_u & f_{\chi'} v_{\chi'} & 0 & f_{s_2} v_s \end{pmatrix}.$$
(9)

Since the E and the  $E^c$  fields acquire large masses, of order the GUT scale, they can be readily integrated out. The reduced  $3 \times 3$  mass matrices for the light charged leptons is given by

$$M_e = U_L \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}, \quad U_L = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$
(10)

where  $m_i = \sqrt{3} (f_e v_d v_\chi/M_E) h_i^e (1 + (h_i v_\chi)^2)/M_E^2)^{-1/2}$ . The light neutrino mass matrix is found to be

$$M_{\nu}^{\text{light}} = m_0 \begin{pmatrix} 1 & 0 & x \\ 0 & 1 - x^2 & 0 \\ x & 0 & 1 \end{pmatrix},$$
(11)

where  $m_0 = f_{\nu}^2 v_u^2 f_{s_2} v_s / (f_{s_2}^2 v_s^2 - f_{\chi'}^2 v_{\chi'}^2)$ , and  $x = -f_{\chi'} v_{\chi'} / (f_{s_2} v_s)$ . We define  $x = |x|e^{i\psi}$ .  $M_{\nu}^{\text{light}}$  can be diagonalized by the transformation  $M_{\nu}^{\text{light}} = U_{\nu}^* D_{\nu} U_{\nu}^{\dagger}$  with

$$U_{\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix} P^*; \quad D_{\nu} = m_0 \begin{pmatrix} |1+x| & & \\ & |1-x^2| & \\ & & |1-x| \end{pmatrix}.$$
(12)

 $P^*$  is a diagonal phase matrix given by

$$P^* = diag\{e^{-i\phi_1/2}, e^{-i(\phi_1 + \phi_2)/2}, e^{-i(\phi_2 + \pi)/2}\}, \phi_1 = arg(1+x), \phi_2 = arg(1-x) .$$
(13)

These Majorana phases will not be relevant for neutrino oscillations, but they will appear in neutrinoless double beta deacy and in leptogenesis. The MNS matricx is given by  $U_{MNS} = U_L^T U_{\nu}^*$  which has the form given in Eq. (4). (In making this identification, we make a field redefinition of the left-handed charged lepton fields,  $e_L = diag.\{1, \omega, \omega^2\}e'_L.$ ) This is the main result of this paper.

## Constraints on neutrino masses

From Eq. (12), the expressions for the mass eigenvalues can be inverted to obtain the following relations for the parameters  $|m_0|$ , |x| and  $\psi = \arg(x)$ :

$$|m_{0}| = \frac{m_{1}m_{3}}{m_{2}}$$

$$|x| = \frac{1}{\sqrt{2}m_{1}m_{3}} \left[ m_{1}^{2}m_{2}^{2} + m_{2}^{2}m_{3}^{2} - 2m_{1}^{2}m_{3}^{2} \right]^{1/2},$$

$$\cos \psi = \frac{-(m_{3}^{2} - m_{1}^{2})m_{2}^{2}}{2\sqrt{2}m_{1}m_{3} \left[ m_{1}^{2}m_{2}^{2} + m_{2}^{2}m_{3}^{2} - 2m_{1}^{2}m_{3}^{2} \right]^{1/2}}.$$
(14)

Here  $m_1 = |m_0(1+x)|$ ,  $m_2 = |m_0(1-x^2)|$ ,  $m_3 = |m_0(1-x)|$ . There are restrictions arising from the conditions that |x| be real and  $|\cos \psi| \leq 1$ , which we analyze now.

Because of the observed hierarchy  $\Delta m_{\rm atm}^2 \gg \Delta m_{\odot}^2$ , and the requirement of MSW resonance for solar neutrinos, two possible neutrino mass ordering are allowed. (i)  $m_1 < m_2 < m_3$  (normal mass ordering) and (ii)  $m_3 < m_1 < m_2$  (inverted mass ordering).

If the neutrino masses are strongly hierarchical,  $m_1 \ll m_2 \ll m_3$ , then from Eq. (14) one sees that  $|\cos \psi| \leq 1$  cannot be satisfied, since  $|\cos \psi| \simeq |m_2/(2\sqrt{2}m_1)| \gg 1$  in this case. Similarly  $m_3 \ll m_1 \ll m_2$  is also not allowed. We find that only two possibilities can arise, depending on (a)  $m_2^2 - m_1^2 \sim m_2^2 + m_1^2$  (normal ordering), and (b)  $m_2^2 - m_1^2 \ll m_2^2 + m_1^2$ (inverted ordering). We consider these cases in turn.

These conclusions can also be arrived at by analyzing the neutrino mass matrix in the flavor basis, i.e., in a basis where the charged lepton mass matrix is diagonal:

$$M_{\nu}^{\text{flavor}} = \frac{m_0}{3} \begin{pmatrix} 3+2x-x^2 & -x-x^2 & -x-x^2 \\ -x-x^2 & 2x-x^2 & 3-x-x^2 \\ -x-x^2 & 3-x-x^2 & 2x-x^2 \end{pmatrix}.$$
 (15)

When x = -1 + q with  $|q| \ll 1$ , we see that entries in the first row and column of Eq. (15) become small. This will be the case of normal ordering of masses.

### (a) Normal ordering

This case is realized when  $m_2^2 - m_1^2 \sim m_2^2 + m_1^2$ . The condition  $|\cos \psi| \leq 1$  can be satisfied only if  $m_1/m_2 \simeq 1/2$ . Making expansions in small  $m_1/m_3$  and small  $(2m_1/m_2 - 1)$ , we find

$$|x| \simeq 1 + \frac{m_1^2}{m_3^2} - 2\left(\frac{2m_1}{m_2} - 1\right),$$
  

$$\cos\psi \simeq -\left[1 - 2\left\{\frac{m_1^2}{m_3^2} - \left(\frac{2m_1}{m_2} - 1\right)^2\right\}\right].$$
(16)

We see a further restriction that  $m_1/m_3 \ge |2m_1/m_2 - 1|$ .

Neutrinoless double beta decay is sensitive to the effective mass

$$m_{\beta\beta} = \left| \sum_{i} U_{ei}^2 m_i \right| = \frac{|m_0|}{3} |3 - x| |1 + x|.$$
(17)

In the case under study this takes the value

$$m_{\beta\beta} \simeq \frac{4}{3} m_1 \simeq \frac{4}{3} \left(\frac{\Delta m_{\odot}^2}{3}\right)^{1/2} \simeq 0.0068 \text{ eV} .$$
 (18)

The effective neutrino mass measurable in tritium beta decay  $m_{\nu_e}$  is given by

$$m_{\nu_e} = \left[\sum_i |U_{ei}|^2 m_i^2\right]^{1/2}$$
(19)

which in this case takes the value

$$m_{\nu_e} \simeq \left[\frac{2}{3}\Delta m_{\odot}^2\right]^{1/2} \simeq 0.0073 \text{ eV}.$$
 (20)

Here we made use of the fact that  $3m_1^2 \simeq \Delta m_{\odot}^2$ . Finally, the sum of neutrino masses, sensitive to cosmological measurements, is given by

$$\sum_{i} m_{i} \simeq 3m_{1} + m_{3} \simeq 3 \left(\frac{\Delta m_{\odot}^{2}}{3}\right)^{1/2} + \left|\Delta m_{\rm atm}^{2}\right|^{1/2} \simeq 0.064 \text{ eV} .$$
(21)

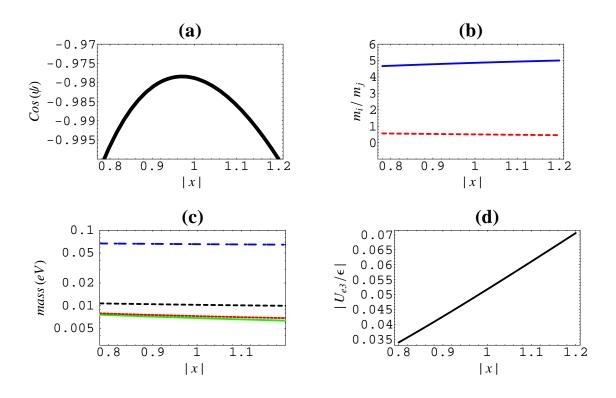


Figure 1: Various quantities as functions of |x| for normal mass hierarchy case. (a)  $\cos \psi$  vs. |x|; (b)  $m_1/m_2$  (dashed) and  $m_3/m_2$  (solid) vs. |x|; (c)  $m_{ee}$  (solid (green)),  $m_{\nu_e}$  (dotted (red)),  $m_2$  (dashed (black)) and  $\sum m_i$  (dot-dashed (blue)) (in eV unit) vs. |x|; (d) RG running correction to  $|U_{e3}/\epsilon|$  vs. |x|.

We have plotted in Fig. 1a - 1c various masses and mass ratios as functions of |x| using the exact expressions with central values for  $\Delta m_{\odot}^2$  and  $\Delta m_{atm}^2$ . These results confirm our analytical solutions.

#### (b) Inverted mass ordering

In this case,  $m_2^2 - m_1^2 \ll m_2^2 + m_1^2$ . The small parameter expansion is different from (a), since the denominator of Eq. (16) in  $\cos \psi$  becomes small. Here  $m_1$  and  $m_2$  are nearly equal, and  $m_3$  will turn out to be much smaller than  $m_1$ . In order to satisfy  $\Delta m_{\odot}^2 \ll \Delta m_{\rm atm}^2$  it is necessary that  $\cos \psi \simeq |x|/2$ , which then requires  $|x| \leq 2$ . Writing

$$\cos\psi = \frac{|x|}{2}(1+q), \ q \ll 1,$$
(22)

we find

$$\Delta m_{\odot}^{2} = m_{2}^{2} - m_{1}^{2} \simeq -|m_{0}|^{2}|x|^{2}(1+2|x|^{2})q$$
  
$$\Delta m_{\rm atm}^{2} = m_{3}^{2} - m_{2}^{2} \simeq -2|m_{0}|^{2}|x|^{2}.$$
 (23)

It becomes clear that  $m_3^2 < m_2^2$  (and thus  $m_3^2 < m_1^2$ ), that is, this case corresponds to an inverted hierarchy. With negative  $q \ll 1$ , the hierarchy in the two oscillation parameters can be accommodated.

In this case the effective mass for double beta decay is given by

$$m_{\beta\beta} \simeq \frac{m_3}{3} \left[ 9 + \frac{\Delta m_{\rm atm}^2}{m_3^2} \right]^{1/2} \left[ 1 - \frac{\Delta m_{\rm atm}^2}{m_3^2} \right]^{1/2}$$
(24)

with  $\Delta m_{\rm atm}^2$  being negative. Here we used the fact that  $|m_0| \simeq m_3$ . The value of  $m_3$  is not determined by oscillation data. If  $m_3^2 \gg |\Delta m_{\rm atm}^2|$ , we have three-fold degeneracy of masses and  $m_{\beta\beta} \simeq m_3$ . As  $m_3^2 \to (1/9)|\Delta m_{\rm atm}^2|$ , the double beta decay amplitude vanishes. Furthermore we have for this case

$$m_{\nu_e} \simeq m_1 \simeq \sqrt{|\Delta m_{\rm atm}^2| + m_3^2}, \quad \sum_i m_i \simeq 2m_1 + m_3 \simeq 2\sqrt{|\Delta m_{\rm atm}^2| + m_3^2} + m_3.$$
 (25)

The three-fold degenerate case is obtained from this case by setting  $m_3^2$  much larger than  $|\Delta m_{\rm atm}^2|$  (or equivalently,  $|x| \ll 1$ ). This case also coincides with the leading results of Ref. [9].

The exact results for various masses and mass ratios are plotted in Fig. 2a - 2c as functions of |x|. These results confirm the analytical approximations presented here.

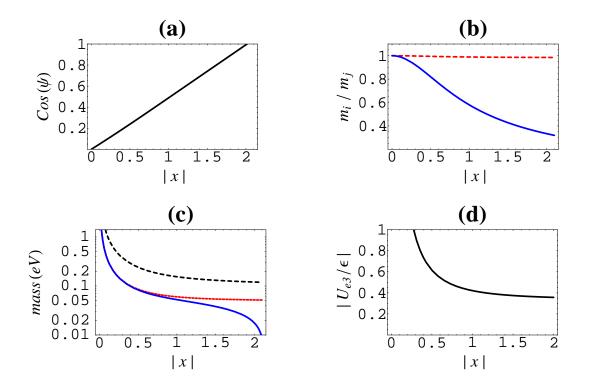


Figure 2: Various quantities as functions of |x| for inverted mass hierarchy case. (a)  $\cos \psi$  vs. |x|; (b)  $m_1/m_2$  (dashed) and  $m_3/m_2$  (solid) vs. |x|; (c)  $m_{ee}$  (solid (blue)),  $m_{\nu_e} \simeq m_2$  (dashed (red)) and  $\sum m_i$  (dot-dashed (black)) (in eV unit) vs. |x|; (d) RG running correction to  $|U_{e3}/\epsilon|$  vs. |x|.

# Stability of $U_{e3}$

A distinctive feature of the geometric mixing pattern is that  $U_{e3} = 0$  at the scale of  $A_4$ symmetry breaking, which we have taken to be of order the GUT scale. When running from this high scale to low energy scale  $(M_{EW})$ , the mixing matrix may change, in particular  $U_{e3}$ may not be zero any more. One should ensure that the pattern of Eq. (4) is not destabilized, which can happen if the induced  $U_{e3}$  is too large. We demonstrate this stability now.

The leading flavor-dependent effect of the running from high scale to low scale is given by the one–loop RGE [13]

$$\frac{dM_{\nu}^{e}}{d\ln t} = \frac{1}{32\pi^{2}} [M_{\nu}^{e} Y_{e}^{\dagger} Y_{e} + (Y_{e}^{\dagger} Y_{e})^{T} M_{\nu}^{e}] + \dots$$
(26)

This leads to correction, to the leading order, to the entries  $M_{13,23}(1-\epsilon)$  and  $M_{33}(1-2\epsilon)$ 

with  $\epsilon \simeq Y_{\tau}^2 \ln(M_{\rm GUT}/M_{EW})/32\pi^2$ . One obtains to order  $\epsilon$ ,

$$|U_{e3}| \approx \frac{|\epsilon x| \, ||x| + \cos\psi + i\sin\psi|}{3\sqrt{2}|\cos\psi(|x| + 2\cos\psi)|}.$$
(27)

One obtains the induced  $|U_{e3}|$  for the normal and inverted hierarchies by inserting the corresponding expressions for  $\cos \psi$  and |x| given earlier.

The results are shown in Figs 1d (normal mass ordering) and in 2d (inverted ordering) where we plot  $|U_{e3}/\epsilon|$  as a function of |x|. We see that the induced  $|U_{e3}|$  is small, too small to be measured by near future experiments for the normal mass hierarchy case in the whole allowed |x| range. For the inverted mass hierarchy case for |x| larger than about 0.2,  $|U_{e3}|$ remains small. For smaller values of |x|, with  $\epsilon$  of order one (corresponding to  $Y_{\tau} \approx 1$ ),  $|U_{e3}|$  can be as large as 0.1 which may be measured in the future. In this case, all three neutrinos are nearly degenerate and the cosmological mass limit on neutrinos will be nearly saturated. We conclude that the structure of the mixing matrix derived is not upset by radiative corrections.

#### Leptogenesis

Leptogenesis occurs in a simple way in this model via the decay of the right-handed neutrinos [14]. The heavy Majorana mass matrix of  $\nu^c$  is given in the model as (see Eq. (9))

$$M_{\nu^{c}} = M_{R} \begin{bmatrix} 1 & 0 & -x \\ 0 & 1 & 0 \\ -x & 0 & 1 \end{bmatrix} .$$
(28)

The Dirac neutrino Yukawa coupling matrix is proportional to an identity matrix at the scale of  $A_4$  symmetry breaking which we take to be near the GUT scale. The  $\nu^c$  fields will remain light below that scale, down to the scale  $M_R$ . Renormalization group effects in the momentum range  $M_R < \mu < M_{\rm GUT}$  will induce non-universal corrections to the Dirac neutrino Yukawa coupling matrix. Without such non-universality no lepton asymmetry will be induced in the decay of right-handed neutrinos. The effective theory in this momentum range is the MSSM with the  $\nu^c$  fields.

From the renormalization group equation

$$\frac{dY_{\nu}}{dt} = \frac{1}{16\pi^2} Y_{\nu}(Y_{\ell}^{\dagger}Y_{\ell}) + \dots$$
(29)

where  $W = eY_{\ell}LH_d + \nu^c Y_{\nu}LH_u + ...$ , we obtain at the scale  $M_R$ ,  $Y_{\nu} = Y_{\nu}^0 \times diag(1, 1, 1 - \delta)$ with  $\delta \simeq (Y_{\tau}^2/16\pi^2)\ln(M_{\rm GUT}/M_R)$ .  $Y_{\nu}^0$  is the value of the universal Dirac Yukawa coupling at the GUT scale.

We diagonalize  $M_{\nu^c}$  by the rotation  $\nu^c = OQN$ , where

$$O = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1\\ 0 & \sqrt{2} & 0\\ -1 & 0 & 1 \end{pmatrix}, \quad Q = diag\{e^{-i\phi_1/2}, \ 1, \ e^{-i\phi_2/2}\}$$
(30)

so that the N fields are the mass eigenstates with real and positive mass eigenvalues:  $M_N = M_R \times diag(|1 + x|, 1, |1 - x|).$ 

In the basis where the heavy  $\nu^c$  fields have been diagonalized, the Dirac neutrino Yukawa coupling matrix takes the form  $\hat{Y}_{\nu} = QO^T Y_{\nu}$ , so that

$$\hat{Y}_{\nu}\hat{Y}_{\nu}^{\dagger} = \frac{|Y_{\nu}^{0}|^{2}}{2} \begin{bmatrix} 1 + (1-\delta)^{2} & 0 & -e^{i(\phi_{2}-\phi_{1})/2}\{(1-\delta)^{2}-1\} \\ 0 & 2 & 0 \\ -e^{i(\phi_{1}-\phi_{2})/2}\{(1-\delta)^{2}-1\} & 0 & 1 + (1-\delta)^{2} \end{bmatrix} .$$
(31)

The CP asymmetry arising from the decay of the field  $N_i$  is given by

$$\epsilon_{i} = \frac{-1}{8\pi} \frac{1}{[\hat{Y}_{\nu} \hat{Y}_{\nu}^{\dagger}]_{ii}} \sum_{j} Im\{[\hat{Y}_{\nu} \hat{Y}_{\nu}^{\dagger}]_{ij}^{2}\} f\left(\frac{M_{j}^{2}}{M_{i}^{2}}\right)$$
(32)

where

$$f(y) = \sqrt{y} \left(\frac{2}{y-1} + \log\frac{1+y}{y}\right) . \tag{33}$$

As  $y \gg 1$ ,  $f(y) \to 3/\sqrt{y}$ .

In the normal hierarchy case,  $m_1/m_3 = M_1/M_3$ , so the lightest N field is  $N_1$ . In this case we have

$$\epsilon_{1} = \frac{-3|Y_{\nu}^{0}|^{2}}{8\pi} \delta^{2} \left(\frac{m_{1}}{m_{3}}\right) \sin(\phi_{2} - \phi_{1})$$

$$\simeq \frac{\pm 3|Y_{\nu}^{0}|^{2}}{8\pi} \delta^{2} \left(\frac{m_{1}}{m_{3}}\right) \left[1 - \frac{(2m_{1}/m_{2} - 1)^{2}}{(m_{1}/m_{3})^{2}}\right]^{1/2} .$$
(34)

To see the numerical value of  $\epsilon_1$ , we note that  $|Y_{\nu}^0|$  can be of order one,  $\delta \simeq (0.1Y_{\tau}^2)$ , and  $m_1/m_3 \simeq [\Delta m_{\rm solar}^2/3\Delta m_{\rm atm}^2]^{1/2} \simeq 0.1$ . For very large value of  $\tan\beta$ ,  $Y_{\tau} \simeq 1$ , and we find  $\epsilon_1 \simeq 10^{-4}$ . Even for moderate values of  $\tan\beta \sim 20$ , we find that  $\epsilon_1 \simeq 10^{-6}$ is possible. The negative sign will also ensure the correct sign of baryon asymmetry. The induced lepton asymmetry is converted to baryon asymmetry through electroweak sphaleron processes. The baryon asymmetry is given by  $Y_B \simeq -Y_L/2$ , where  $Y_L = \kappa \epsilon_1/g^*$ , where  $g^* \sim 200$  is the effective number of degrees of freedom in equilibrium during leptogenesis, and  $\kappa$  is the efficiency factor obtained by solving the Boltzman's equations. A simple approximate formula for  $\kappa$  is [15]

$$\kappa \simeq 10^{-2} \left[ \frac{0.01}{\tilde{m}_1 \text{ eV}} \right]^{1.1}$$
(35)

where

$$\tilde{m}_1 = \frac{v_u^2}{M_1} [\hat{Y}_\nu \hat{Y}_\nu^\dagger]_{11}$$
(36)

For  $\delta \sim 0.1$  and  $M_1 \sim 10^{14}$  GeV, we obtain  $Y_B \sim 7 \times 10^{-11}$ , in good agreement with observations.

For the case of inverted mass hierarchy,  $N_3$  is lighter than  $N_1$ , so we focus on  $\epsilon_3$ . It is given by

$$\epsilon_3 \simeq \frac{\mp 3 |Y_{\nu}^0|^2}{4\pi} \delta^2 \left(\frac{m_3}{m_1}\right) \frac{|x|\sqrt{1 - |x|^2/4}}{\sqrt{1 + 2|x|^2}} . \tag{37}$$

Again we see that reasonable lepton asymmetry is generated.

In summary, we have presented a class of renormalizable gauge models based on the non-Abelian discrete symmetry  $A_4$  which realize the geometric neutrino mixing pattern of Eq. (4) naturally. The resulting constraints on the neutrino masses have been outlined. We have also highlighted an intriguing connection between high scale leptogenesis and low energy neutrino experiments.

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