Generalized S-matrix in Mixed Representation

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A generalized scattering amplitude where momenta of incoming-particles and outgoing-particles as well as positions of incoming-particles and outgoing-particles are specified is formulated. Idealistic beams and idealistic measuring instruments where momenta and positions satisfy minimum uncertainty are studied with a use of minimum wave packets, coherent states. In the present work, we show general features of the generalized scattering amplitudes based on ϕ^4 theory. We give a proof of completeness of many body states, asymptotic behaviors in the large distance region, and factorization of the amplitudes. Despite of the non-orthogonal properties of wave packets, we found that the probability interpretation is verified. A differential probability depends upon the wave packet size but a total probability that is integrated in the final states is independent from the size of final state wave packet and becomes universal. Few body amplitudes are studied as examples.

§1. Introduction

We study scattering processes in which both of momenta and positions are simultaneously measured. Observations of both variables are made in recent neutrino experiments where a distance between a source region and a detector is fixed in a macroscopic length and the momenta of the particles are measured. Scattering amplitudes of both variables are formulated in the present paper.

In quantum mechanics, precise values of momentum and coordinate of a particle can not be measured simultaneously from Heisenberg uncertainty relation. So in scattering processes, one of the variables is selected and a dependence of the scattering amplitude on this variable is studied. Normally momentum variable is selected and the momentum dependence of the transition probability is studied and is compared with a theoretical calculation. Since momenta are commuting variables, it is possible to determine them precisely. In real experiments, however, the exact value of momentum is difficult to measure and the momentum is measured with finite uncertainty due to a finite resolution of a detector. When the momentum is measured with uncertainties, simultaneous measurements of positions and momenta become possible in experiments. The positions are measured also within finite uncertainties due to spatial resolution of detectors. When momenta and positions are measured with finite uncertainties, total number of information could be equivalent with the standard case where the exact values of momenta are measured.

Since a wave function with a definite momentum is a plane wave and is invariant under any translations, any particular position is undefined. But a wave function with a finite uncertainty of the momentum can have a finite spatial extension and is localized around certain position, so in this case the spatial position is defined. So it is necessary to introduce finite uncertainty of momenta in order to define the positions. As a price of a finite uncertainty on the momentum, it became possible to introduce a position where the particle is measured or is produced. The position has also finite uncertainty. In this work, we introduce such amplitude that is defined with finite uncertainties of momenta and positions.

The wave functions with finite spatial extensions are necessary actually for asymptotic conditions of scattering processes to be satisfied[1),2)] However in many situations of high energy physics, effects of finite wave functions have been ignorable and it has been sufficient to use propagators of momentum variables with $i\epsilon$ prescription. We clarify these points and we study situations where the effects of finite wave functions are important. To investigate these problems we define the generalized S-matrix of both variables using mixed representation and we show several features of the scattering amplitudes in which the momentum and the positions are measured. Some implications and applications are also studied.

We formulate a position-dependent and momentum-dependent S-matrix in an idealistic limit allowed from Heisenberg uncertainty relation. In our formalism, each momentum and position of the initial particle and final particle satisfy minimum uncertainty relation. This corresponds to a scattering process where coordinates as well as momenta of the beam satisfy the minimum uncertainty relation and the measurement are made with minimum uncertainty allowed from Heisenberg uncertainty relation.

In real experiments the uncertainty of the momentum and coordinate may be larger than those of the present work. But our idealistic scattering matrix should be realized in a suitable method and should give new insights in quantum mechanical scattering processes. Also extensions to non-minimum wave packets is straightforward.

Uncertainties of momenta and positions are actually determined in experiments by resolutions of beam sources and of detectors. The resolutions depend on each accelerator and detector. So we leave the uncertainties unfixed and study the scattering matrix of this situation.

Because the momentum is conjugate to the coordinate, they satisfy the commutation relation,

$$[x, p] = i, \tag{1·1}$$

in a unit $h/2\pi = 1$ and a momentum resolution and a spatial resolution of a detector satisfy Heisenberg's uncertainty relation,

$$\delta p \times \delta x \geq \frac{1}{2}. \tag{1·2}$$

A product of uncertainties becomes minimum in coherent states. The coherent state of a variable, x,

$$\langle x|P_0, X_0\rangle = N_1 e^{iP_0(x-X_0)-\frac{1}{2\sigma}(x-X_0)^2} \tag{1·3}$$

$$N_1^2 = (\pi\sigma)^{-\frac{1}{2}} \tag{1·4}$$

has expectation values

$$\langle |x| \rangle = X_0 \tag{1·5}$$

$$\langle|x^2|\rangle = {X_0}^2 + \frac{1}{2}\sigma \tag{1·6}$$
$$\langle|p|\rangle = P_0 \tag{1·7}$$
$$\langle|p^2|\rangle = {P_0}^2 + \frac{1}{2\sigma}. \tag{1·8}$$

The product between the variances of the momentum and coordinate,

$$(\delta x)^2 \times (\delta p)^2 = \frac{1}{4} \tag{1·9}$$
$$(\delta x)^2 = \langle|x^2|\rangle - (\langle|x|\rangle)^2 \tag{1·10}$$
$$(\delta p)^2 = \langle|p^2|\rangle - (\langle|p|\rangle)^2 \tag{1·11}$$

is independent from σ and satisfies the minimum uncertainty condition. The uncertainty becomes minimum with the coherent state.

The coherent state satisfies also completeness condition,

$$\begin{aligned}&\int \frac{dP_0 dX_0}{2\pi}\langle x|P_0,X_0\rangle\langle P_0,X_0|y\rangle \\ &= \int \frac{dP_0 dX_0}{2\pi} N_1^2 e^{iP_0(x-y)} e^{-\frac{1}{2\sigma}(x-X_0)^2-\frac{1}{2\sigma}(y-X_0)^2} \\ &= \delta(x-y).\end{aligned} \tag{1·12}$$

We use these wave functions for expressing in-states and out-states. In the former, the state corresponds to the idealistic beam and in the latter the state corresponds to the idealistic measuring apparatus. A transition element between the states of idealistic beams and the states of the idealistic measurement is computed from these matrix elements. Using them, we define the idealistic scattering matrix in mixed representation and study its properties and applications.

Our formalism will be applied to long line experiments such as, solar neutrino experiments, atmospheric neutrino experiments, long base line neutrino experiments, reactor neutrino experiments, and others where systems have huge scales and positions of detectors play important roles,[3),4),8)9)10).13)] In addition to neutrino, scattering of other weakly interacting particles where position dependence in addition to momentum dependence are measured and give important physical informations will be studied.

The paper is organized in the following manner. In Section 2, mathematics on wave packets are given. Wave packet wave functions are explicitly given and the overlap integrals and time dependent behaviors are analyzed. New uncertainty relations between the velocity of expansions in asymptotic region and the initial sizes are obtained. Although some of the materials in this section may be known to experts, they are necessary and useful for later arguments. In Section 3, generalized scattering amplitude is defined and general properties are studied. We find suitable integration measures for both variables in which the probability interpretation is verified. A differential probability depends upon the wave packet size but a total probability that is integrated in the final states is independent from the size of final state wave packet and becomes universal. Explicit examples are given in Section 4 and summary is given in Section 5.

§2. Wave packets

2.1. *Mathematical preliminaries*

2.1.1. Complete sets of minimum wave packets :time independent wave packets

Let momentum eigenstates and position eigenstates be $|\vec{p}\rangle$ and $|\vec{x}\rangle$. Transformation from a momentum eigenstate to a position eigenstate is made from

$$\langle\vec{x}|\vec{p}\rangle = (2\pi)^{-\frac{3}{2}} e^{i\vec{p}\cdot\vec{x}}, \tag{2·1}$$

$$\langle\vec{x}|\vec{p}\rangle^{*} = \langle\vec{p}|\vec{x}\rangle = \langle\vec{x}|\vec{-p}\rangle. \tag{2·2}$$

Both sets of functions are complete sets and satisfy

$$\int d\vec{x}\langle\vec{p}_1|\vec{x}\rangle\langle\vec{x}|\vec{p}_2\rangle = \delta(\vec{p}_1 - \vec{p}_2), \tag{2·3}$$

$$\int d\vec{p}\langle\vec{x}_1|\vec{p}\rangle\langle\vec{p}|\vec{x}_2\rangle = \delta(\vec{x}_1 - \vec{x}_2), \tag{2·4}$$

and

$$\int d\vec{x}|\vec{x}\rangle\langle\vec{x}| = 1, \tag{2·5}$$

$$\int d\vec{p}|\vec{p}\rangle\langle\vec{p}| = 1. \tag{2·6}$$

The normalized coherent states in three spatial dimensions[11)] are defined as

$$\langle\vec{x}|\vec{P}_0,\vec{X}_0\rangle = N_3 e^{i\vec{P}_0(\vec{x}-\vec{X}_0)-\frac{1}{2\sigma}(\vec{x}-\vec{X}_0)^2} \tag{2·7}$$

$$N_3^2 = (\pi\sigma)^{-\frac{3}{2}},$$

$$\langle\vec{x}|\vec{P}_0,\vec{X}_0\rangle^{*} = \langle\vec{P}_0,\vec{X}_0|\vec{x}\rangle = \langle\vec{x}|-\vec{P}_0,\vec{X}_0\rangle. \tag{2·8}$$

Due to the Gaussian factor the wave function is localized around the center position $\vec{x}_0$ with a width $\sqrt{2\sigma}$ and is an approximate eigenfunction of the momentum operator. A constant N_3 is determined from the normalization condition

$$\begin{aligned}&\int d^3x|\langle\vec{x}|\vec{P}_0,\vec{X}_0\rangle|^2\\ &= N_3^2\int d^3x e^{-\frac{1}{\sigma}(\vec{x}-\vec{X}_0)^2}\\ &= N_3^2(\pi\sigma)^{\frac{3}{2}}\\ &= 1.\end{aligned} \tag{2·9}$$

The same states are expressed in the momentum representation as

$$\langle\vec{p}|\vec{P}_0,\vec{X}_0\rangle = \int d\vec{x}\langle\vec{p}|\vec{x}\rangle\langle\vec{x}|\vec{P}_0,\vec{X}_0\rangle$$

$$= N_3\sigma^{3/2} e^{-i\vec{p}\cdot\vec{X}_0-\frac{\sigma}{2}(\vec{P}_0-\vec{p})^2}, \tag{2·10}$$

$$\langle\vec{p}|\vec{P}_0,\vec{X}_0\rangle^{*} = \langle\vec{P}_0,\vec{X}_0|\vec{p}\rangle = \langle\vec{p}|-\vec{P}_0,\vec{X}_0\rangle \tag{2·11}$$

and satisfy the normalization condition in the momentum representation.

The set of functions for an arbitrary value of σ satisfy the completeness condition in the coordinates representation,

$$\int \frac{d\vec{P}_0 d\vec{X}_0}{(2\pi)^3} \langle \vec{x}_1 | \vec{P}_0, \vec{X}_0 \rangle \langle \vec{P}_0, \vec{X}_0 | \vec{x}_2 \rangle = \delta(\vec{x}_1 - \vec{x}_2), \tag{2·12}$$

and the completeness condition in the momentum representation,

$$\int \frac{d\vec{P}_0 d\vec{X}_0}{(2\pi)^3} \langle \vec{p}_1 | \vec{P}_0, \vec{X}_0 \rangle \langle \vec{P}_0, \vec{X}_0 | \vec{p}_2 \rangle = \delta(\vec{p}_1 - \vec{p}_2). \tag{2·13}$$

The completeness condition is simply expressed as

$$\int \frac{d\vec{P}_0 d\vec{X}_0}{(2\pi)^3} | \vec{P}_0, \vec{X}_0 \rangle \langle \vec{P}_0, \vec{X}_0 | = 1. \tag{2·14}$$

Overlap between two coherent states are computed from the above definition

$$\begin{aligned}
&\langle \vec{P}_1, \vec{X}_1 | \vec{P}_2, \vec{X}_2 \rangle \\
&= \int d^3x \langle \vec{P}_1, \vec{X}_1 | \vec{x} \rangle \langle \vec{x} | \vec{P}_2, \vec{X}_2 \rangle \\
&= N_3^2 \int d^3x e^{i\frac{(P_1-P_2)}{\hbar}x + i(\frac{P_1}{\hbar}X_1 - P_2 X_2) - \frac{1}{2\sigma^2}\left((\vec{x}-\vec{X}_1)^2 + (\vec{x}-\vec{X}_2)^2\right)} \\
&= e^{-\frac{1}{4\sigma}(\vec{X}_1-\vec{X}_2)^2 - \frac{\sigma}{4}(\vec{P}_1-\vec{P}_2)^2 + \frac{i}{2}(\vec{P}_1+\vec{P}_2)(\vec{X}_1-\vec{X}_2)}.
\end{aligned} \tag{2·15}$$

Obviously matrix elements do not vanish and states are not orthogonal for different values of the momenta and coordinates. Despite of the nonorthogonality they satisfy

$$\int \frac{d\vec{P}_0 d\vec{X}_0}{(2\pi)^3} | \vec{P}_0, \vec{X}_0 \rangle \langle \vec{P}_0, \vec{X}_0 | \vec{P}, \vec{X} \rangle = | \vec{P}, \vec{X} \rangle. \tag{2·16}$$

Hence $\langle \vec{P}_1, \vec{X}_1 | \vec{P}_2, \vec{X}_2 \rangle$ plays a role of the Dirac delta function.

2.1.2. Wave packets defined at arbitrary time

Wave packets defined at a certain time T_0 is constructed when one particle energy is known. Let $E(\vec{p})$ stands one particle energy of the momentum $\vec{p}$,

$$E(\vec{p}) = (\vec{p}^2 + m^2)^{1/2} \tag{2·17}$$

in the unit with $c = 1$, then the wave packet defined at T_0 is,

$$\langle \vec{p} | \vec{P}_0, \vec{X}_0, T_0 \rangle = \langle \vec{p} | \vec{P}_0, \vec{X}_0 \rangle e^{\frac{-E(\vec{p})}{i} T_0}. \tag{2·18}$$

This set of functions for a given time T_0 satisfy the completeness condition,

$$\int \frac{d\vec{P}_0 d\vec{X}_0}{(2\pi)^3} \langle \vec{p}_1 | \vec{P}_0, \vec{X}_0, T_0 \rangle \langle \vec{P}_0, \vec{X}_0, T_0 | \vec{p}_2 \rangle = \langle \vec{p}_1 | \vec{p}_2 \rangle. \tag{2·19}$$

The representation of these wave packets in the coordinates space is obtained by the Fourier transformation,

$$\langle \vec{x}|\vec{P}_0,\vec{X}_0,T_0\rangle = \int d^3p \langle \vec{x}|\vec{p}\rangle\langle \vec{p}|\vec{P}_0,\vec{X}_0\rangle e^{\frac{-E(\vec{p})}{i}T_0}. \quad (2\cdot20)$$

It is easy to verify the completeness condition by combining Eq.(2·14) and (2·18) with Eq.(2·20),

$$\int \frac{d\vec{P}_0 d\vec{X}_0}{(2\pi)^3}\langle \vec{x}_1|\vec{P}_0,\vec{X}_0,T_0\rangle\langle \vec{P}_0,\vec{X}_0,T_0|\vec{x}_2\rangle = \langle \vec{x}_1|\vec{x}_2\rangle. \quad (2\cdot21)$$

The completeness condition is simply written as

$$\int \frac{d\vec{P}_0 d\vec{X}_0}{(2\pi)^3}|\vec{P}_0,\vec{X}_0,T_0\rangle\langle \vec{P}_0,\vec{X}_0,T_0| = 1. \quad (2\cdot22)$$

The matrix elements of the wave packets at different times are computed as,

$$\langle \vec{P}_1,\vec{X}_1,T_1|\vec{P}_2,\vec{X}_2,T_2\rangle = \int d^3p \langle \vec{P}_1,\vec{X}_1,T_1|\vec{p}\rangle\langle \vec{p}|\vec{P}_2,\vec{X}_2,T_2\rangle \quad (2\cdot23)$$

and satisfy

$$\int \frac{d\vec{P}_0 d\vec{X}_0}{(2\pi)^3}|\vec{P}_0,\vec{X}_0,T_0\rangle\langle \vec{P}_0,\vec{X}_0,T_0|\vec{P}_1,\vec{X}_1,T_1\rangle = |\vec{P}_1,\vec{X}_1,T_1\rangle, \quad (2\cdot24)$$

$$\langle \vec{P}_1,\vec{X}_1,T_1|\vec{P}_2,\vec{X}_2,T_1\rangle = \langle \vec{P}_1,\vec{X}_1|\vec{P}_2,\vec{X}_2\rangle. \quad (2\cdot25)$$

Explicit forms of these matrix elements are given later. We use these wave functions for expanding the field operator.

2.2. *Matrix elements*

2.2.1. Time dependent transformation function

We calculate the matrix elements of the mixed states appeared in the previous section. It is convenient to define time dependent transformation functions,

$$\begin{aligned}
&\langle t,\vec{p}|\vec{P}_0,\vec{X}_0,T_0\rangle \\
&= e^{\frac{E(\vec{p})}{i}(t-T_0)}\langle \vec{p}|\vec{P}_0,\vec{X}_0\rangle \\
&= N_3\sigma^{3/2}e^{\frac{E(\vec{p})}{i}(t-T_0)}e^{-i\vec{p}\cdot\vec{X}_0-\frac{\sigma}{2}(\vec{p}-\vec{P}_0)^2}
\end{aligned} \quad (2\cdot26)$$

and

$$\begin{aligned}
&\langle t,\vec{x}|\vec{P}_0,\vec{X}_0,T_0\rangle \\
&= \int d\vec{p}\langle \vec{x}|\vec{p}\rangle\langle t,\vec{p}|\vec{P}_0,\vec{X}_0,T_0\rangle \\
&= N_3(\frac{\sigma}{2\pi})^{3/2}\int d\vec{p}e^{\frac{E(\vec{p})}{i}(t-T_0)+i\vec{p}\cdot\vec{x}-i\vec{p}\cdot\vec{X}_0-\frac{\sigma}{2}(\vec{p}-\vec{P}_0)^2} \\
&= N_3(\frac{\sigma}{2\pi})^{3/2}\int d\vec{p}e^{-iE(\vec{p})(t-T_0)+i\vec{p}\cdot(\vec{x}-\vec{X}_0)-\frac{\sigma}{2}(\vec{p}-\vec{P}_0)^2}.
\end{aligned} \quad (2\cdot27)$$

The absolute value of the integrand becomes maximum at $\vec{P}_0$ but the phase becomes large in large $t-T_0$ region or large $\vec{x}-\vec{X}_0$ region. So we integrate on $\vec{p}$ in two regions separately. In the small $t-T_0$ region we use the approximation of the integrand around the $\vec{P}_0$ and in the large $t-T_0$ region we use the approximation of the integrand around the stationary momentum.

(A) Small $T-T_0$ case:translational motion.

In the small $t-T_0$ region, the integral is written and computed around $\vec{P}_0$ as

$$\begin{aligned}
&\langle t,\vec{x}|\vec{P}_0,\vec{X}_0,T_0\rangle \\
&= N_3(\frac{\sigma}{2\pi})^{3/2}\int d\vec{p}e^{-i(E(\vec{p}_0)+(\vec{p}-\vec{p}_0)\cdot\vec{v}_0)(t-T_0)+i(\vec{p}_0+(\vec{p}-\vec{p}_0))\cdot(\vec{x}-\vec{X}_0)-\frac{\sigma}{2}(\vec{p}-\vec{P}_0)^2} \\
&= Ne^{i\phi}, && (2\cdot28) \\
&N = N_3e^{-\frac{1}{2\sigma}(\vec{x}-\vec{X}_0-\vec{v}_0(t-T_0))^2}, && (2\cdot29) \\
&e^{i\phi} = e^{-iE(\vec{P}_0)(t-T_0)+i\vec{P}_0\cdot(\vec{x}-\vec{X}_0)}, && (2\cdot30) \\
&\vec{v}_0 = \frac{\partial}{\partial p_i}E(\vec{p})|_{\vec{p}=\vec{p}_0}. && (2\cdot31)
\end{aligned}$$

The wave packet keeps its shape and moves with a constant velocity $\vec{v}_0$. The center of wave packet is $\vec{X}_0$ at $t=T_0$ and is $\vec{X}_0+\vec{v}_0(t-T_0)$ at a time t.

(B) Large $t-T_0$: expanding wave packet.

In the large $t-T_0$ region the momentum integration applied in the previous method is not a good approximation any more because the phase oscillates rapidly in this region. The phase of the integrand becomes stationary at $\vec{P}_X$ which satisfies

$$\frac{\partial}{\partial p_i}\left(-iE(\vec{p})(t-T_0)+i\vec{p}\cdot(\vec{x}-\vec{X}_0)-\frac{\sigma}{2}(\vec{p}-\vec{P}_0)^2\right)=0. \quad (2\cdot32)$$

The solution is obtained by expanding the momentum in $\frac{1}{t-T_0}$ and is given by,

$$\begin{aligned}
&\vec{P}_X = \vec{P}_X^{(0)}+\vec{P}_X^{(1)}+\vec{P}_X^{(2)}, && (2\cdot33) \\
&\vec{P}_X^{(0)} = m\frac{1}{\sqrt{(t-T_0)^2-(\vec{x}-\vec{X}_0)^2}}(\vec{x}-\vec{X}_0), && (2\cdot34) \\
&\vec{P}_X^{(1)} = i\frac{1}{t-T_0}\sigma E(\vec{P}_X)(\vec{P}_X-\vec{P}_0), && (2\cdot35) \\
&\vec{P}_X^{(2)} = o((t-T_0)^{-2}). && (2\cdot36)
\end{aligned}$$

So the exponent of the integrand is expanded around $\vec{P}_X$ and we have

$$\begin{aligned}
&\langle t,\vec{x}|\vec{P}_0,\vec{X}_0,T_0\rangle && (2\cdot37) \\
&= Ne^{-iE(\vec{p})(t-T_0)+i\vec{p}\cdot(\vec{x}-\vec{X}_0)}, \\
&N = N_3(\frac{1}{2i\frac{\gamma_L}{\sigma}+1})^{1/2}(\frac{1}{2i\frac{\gamma_T}{\sigma}+1})e^{-\frac{1}{2}\sigma(\vec{P}_X^{(0)}-\vec{P}_0)^2+\delta}, && (2\cdot38) \\
&\delta = \frac{1}{2}\sigma(\vec{P}_X-\vec{P}_0)^2(2i-\xi)\xi, && (2\cdot39)
\end{aligned}$$

$$\xi = \frac{\sigma E(\vec{P}_X)}{(t-T_0)}, \tag{2·40}$$

in a large $t - T_0$ where the wave packet parameters are given by

$$\gamma_L = \frac{1}{2}\frac{m^2|t-T_0|}{E(\vec{P}_X)^3}, \tag{2·41}$$

$$\gamma_T = \frac{1}{2}\frac{|t-T_|}{E(\vec{P}_X)}. \tag{2·42}$$

The longitudinal component and transverse component of a momentum $\vec{q}$ are defined as

$$\vec{q}_T = \vec{q} - \vec{P}_X\frac{(\vec{P}_X,\vec{q})}{(\vec{P}_X,\vec{P}_X)} \tag{2·43}$$

$$\vec{q}_L = \vec{P}_X\frac{(\vec{P}_X,\vec{q})}{(\vec{P}_X,\vec{P}_X)}. \tag{2·44}$$

The phase factor in the Eq.(2·37) is written in leading order of $|t - T_0|$ as

$$\begin{aligned} & e^{-i(t-T_0)E(\vec{P}_X)+i\vec{P}_X\cdot(\vec{x}-\vec{X}_0))} \\ &= e^{-i\frac{m}{}\sqrt{(t-T_0)^2-(\vec{x}-\vec{X}_0)^2}} \\ &= e^{-i\frac{m^2}{E(\vec{P}_X)}(t-T_0)}. \end{aligned} \tag{2·45}$$

This phase factor becomes very small if the mass is very small and vanishes in the massless case. The fact that the phase becomes small in the high energy region or in the small mass region is a characteristic property of relativistic invariant theory.[12)]

The absolute magnitude of N becomes maximum when the momentum $\vec{P}_X$ agrees to $\vec{P}_0$. This is realized at a particular position of $\vec{x}$, $\vec{x}_0$, which satisfies

$$\vec{P}_0 = m\frac{1}{\sqrt{(t-T_0)^2-(\vec{x}_0-\vec{X}_0)^2}}(\vec{x}_0-\vec{X}_0). \tag{2·46}$$

The solution is

$$\vec{x}_0 = \vec{X}_0 + (t-T_0)\frac{\vec{P}_0}{E(\vec{P}_0)}, \tag{2·47}$$

where $\vec{x}_0$ is the center of wave packet at a time t.

We write the difference of the momenta, $\vec{P}_X - \vec{P}_0$, in the Gaussian exponent in Eq.(2·37) using $\vec{x}$ and $\vec{x}_0$ and a unit vector $\vec{n}_1$ in the $\vec{p}_1$ direction as,

$$\vec{P}_X - \vec{P}_0 = \frac{E(\vec{P}_0)}{t-T_0}\{(\vec{x}-\vec{x}_0)_T + \frac{E(\vec{P}_0)^2}{m^2}|(\vec{x}-\vec{x}_0)_L|\vec{n}_1\}. \tag{2·48}$$

We substitute this expression into Eq.(2·37) and we have

$$\begin{aligned} & \exp(-\frac{1}{2}\sigma(\vec{P}_X-\vec{P}_0)^2) \\ &= \exp(-\frac{1}{2}\sigma\frac{(E(\vec{P}_0))^2}{(t-T_0)^2}{(\vec{x}-\vec{x}_0)_T}^2 - \frac{1}{2}\sigma\frac{(E(\vec{P}_0))^6}{m^4(t-T_0)^2}{(\vec{x}-\vec{x}_0)_L}^2). \end{aligned} \tag{2·49}$$

The normalization factor in the small $|t - T_0|$ region has also

$$\exp(-\frac{1}{2\sigma}(\vec{x} - \vec{x}_0)^2). \quad (2\cdot50)$$

Hence the size of the wave packet in the longitudinal direction is given as

$$\delta x_L = \sqrt{\frac{2}{\sigma}\frac{m^2|t - T_0|}{E(\vec{P}_0)^3}} + \sqrt{2\sigma} \quad (2\cdot51)$$

and in the transverse direction is given as

$$\delta x_T = \sqrt{\frac{2}{\sigma}\frac{|t - T_0|}{E(\vec{P}_0)}} + \sqrt{2\sigma}. \quad (2\cdot52)$$

A wave packet expansion is characterized by its velocity. The velocity of expansion in the transverse direction, v_T, is determined by the momentum variance as

$$v_T = \sqrt{\frac{2}{\sigma}}\frac{1}{E(\vec{P}_0)}. \quad (2\cdot53)$$

and that in the longitudinal direction,v_L, is determined as

$$v_L = \sqrt{\frac{2}{\sigma}}\frac{m^2}{(E(\vec{P}_0))^3}. \quad (2\cdot54)$$

The velocity of expansion satisfy uncertainty relations

$$v_T\delta x(t = 0)E(\vec{P}_0) = 1 \quad (2\cdot55)$$

$$v_L\delta x(t = 0)E(\vec{P}_0) = (\frac{m}{E(\vec{P}_0)})^2, \quad (2\cdot56)$$

where $\delta x(t = 0)$ is the spatial extension of wave packet at $t = 0$. The v_L is given by multiplying $\frac{m^2}{E(\vec{P}_0)^2}$ to the v_T. This ratio becomes one in the non-relativistic energy region where $E(\vec{p}_0)$ is nearly equal to m. Consequently wave packets expand symmetrically in the non-relativistic region. In the relativistic region where $E(\vec{p}_0)$ is much larger than m, the ratio between both values becomes very small and wave packets expand an-symmetrically. The shape becomes a circular thin disk after certain time. The wave packet size is given as a function of the propagation time, t, for various values of the initial wave packet size in Fig(1). The δx_T becomes huge size in 500 sec.,the period between the sun and the earth.

Due to the expansion of wave packet, the normalization of the wave function is inversely proportional to $\gamma_T\sqrt{\gamma_L}$ in the asymptotic region.

2.2.2. Two body matrix elements

Various matrix elements of wave packets are studied in this section.

1.Matrix elements of wave packets defined at equal time

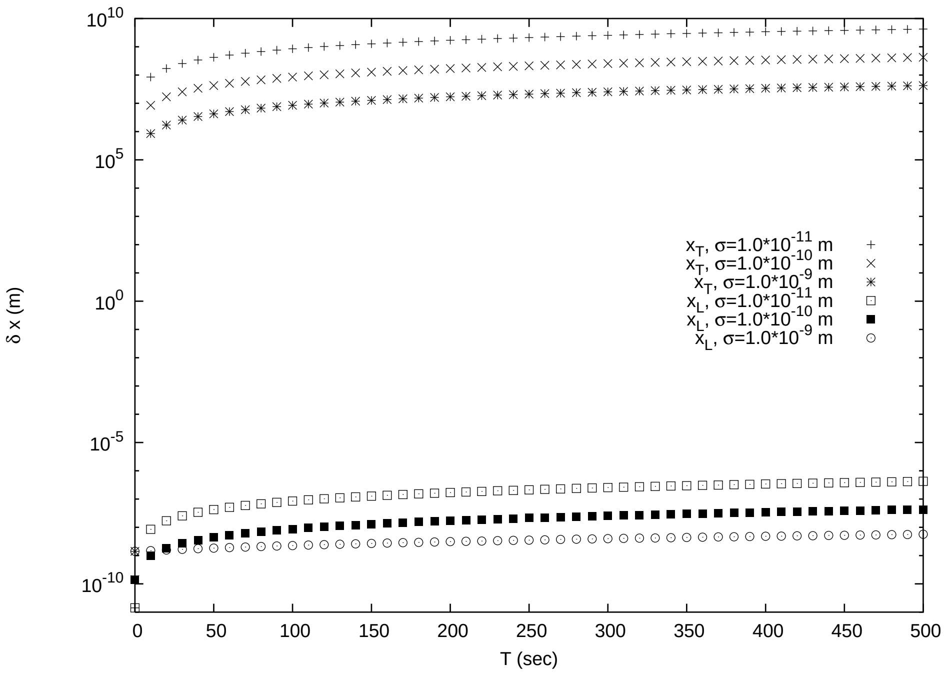

Fig. 1. Time dependence of wave packet size in the physical unit is given. δ_T is the size in the transverse direction given in Eq.(2·52) and δ_L is the size in the longitudinal direction given in Eq.(2·51) . $mc^2 = 10^{-2}eV$ is assumed.

The overlap between two wave packets is given as,

$$\langle \vec{P}_1, \vec{X}_1 | \vec{P}_2, \vec{X}_2 \rangle = e^{-\frac{1}{4\sigma}(\vec{X}_1-\vec{X}_2)^2-\frac{\sigma}{4}(\vec{P}_1-\vec{P}_2)^2} e^{\frac{i}{2}(\vec{P}_1+\vec{P}_2)(\vec{X}_1-\vec{X}_2)}. \tag{2·57}$$

Thus the overlap decreases fast as the distance between two positions in the real space and in the momentum become large. Matrix elements for the same coordinates is

$$\langle \vec{P}_1, \vec{X} | \vec{P}_2, \vec{X} \rangle = e^{-\frac{\sigma}{4}(\vec{P}_1-\vec{P}_2)^2} \tag{2·58}$$

and for the same momenta is

$$\langle \vec{P}, \vec{X}_1 | \vec{P}, \vec{X}_2 \rangle = e^{-\frac{1}{4\sigma}(\vec{X}_1-\vec{X}_2)^2+i\vec{P}_1\cdot(\vec{X}_1-\vec{X}_2)}. \tag{2·59}$$

From these matrix elements if a particle is prepared at $(\vec{P}', \vec{X}')$ probabilities of finding a particle in a region of the momentum and coordinate between $(\vec{P}, \vec{X})$ and $(\vec{P}+d\vec{P}, \vec{X}+d\vec{X})$ is given by

$$P = \frac{d\vec{P}d\vec{X}}{(2\pi)^3} e^{-\frac{1}{2\sigma}(\vec{X}-\vec{X}')^2-\frac{\sigma}{2}(\vec{P}-\vec{P}_2')^2} \tag{2·60}$$

Heisenberg uncertainty relation between the variance in the coordinates and the variance in the momenta is satisfied. It is important to notice that the momentum and the coordinate is measured simultaneously.

2. Time dependent matrix elements

The time dependent matrix element is computed by inserting the complete set of momentum eigenstates,

$$\begin{aligned}&\langle \vec{P}_1, \vec{X}_1, T_1|\vec{P}_2, \vec{X}_2, T_2\rangle \\ &= \int d^3p \langle \vec{P}_1, \vec{X}_1|\vec{p}\rangle e^{\frac{E(\vec{p})(T_1-T_2)}{i}} \langle \vec{p}|\vec{P}_2, \vec{X}_2\rangle \\ &= {N_3}^2\sigma^3 e^{-\frac{\sigma}{4}(\vec{P}_1-\vec{P}_2)^2} \int d^3p e^{\frac{E(\vec{p})(T_1-T_2)}{i}+i\vec{p}\cdot(\vec{X}_1-\vec{X}_2)-\sigma(\vec{p}-\vec{P}_0)^2}, \\ &\vec{P}_0 = \frac{1}{2}(\vec{P}_1+\vec{P}_2).\end{aligned} \tag{2·61}$$

In the small $|T_1 - T_2|$ region this matrix element is given as,

$$\begin{aligned}&\langle \vec{P}_1, \vec{X}_1, T_1|\vec{P}_2, \vec{X}_2, T_2\rangle && (2·62)\\ &= N e^{-iE(\vec{P}_0)(T_1-T_2)+i\vec{P}_0\cdot(\vec{X}_1-\vec{X}_2)}, \\ &N = e^{-\frac{1}{4\sigma}(\vec{X}_1-\vec{X}_2-\vec{v}(T_1-T_2))^2-\frac{\sigma}{4}(\vec{P}_1-\vec{P}_2)^2} && (2·63)\end{aligned}$$

where the velocity is given by

$$\vec{v}_i = \frac{\partial E(\vec{p})}{\partial p_i}|_{\vec{p}} = \frac{\vec{P}_0}{E(\vec{P})}. \tag{2·64}$$

This matrix element shows that the shape of the wave packet is preserved but the center position is moving with the velocity $\vec{v}$ in the small $|T_1 - T_2|$ region.

The matrix element in a large $|T_1 - T_2|$ region is obtained by the stationary phase approximation in $\vec{p}$ integration. The stationary point $\vec{P}_X$ is obtained from the stationarity condition,

$$\frac{\partial}{\partial p_i}(i\vec{p}\cdot(\vec{X}_1-\vec{X}_2) - iE(\vec{p})(T_1-T_2) - \sigma(\vec{p}-\vec{P}_0)^2) = 0, \tag{2·65}$$

and is given as,

$$\vec{P}_X = (\vec{X}_1-\vec{X}_2)\frac{m}{((T_1-T_2)^2-(\vec{X}_1-\vec{X}_2)^2)^{1/2}} + i\frac{1}{T_1-T_2}\delta\vec{P}_x, \tag{2·66}$$

$$\delta\vec{P}_X = 2\sigma E(\vec{P}_X)(\vec{P}_X-\vec{P}_0). \tag{2·67}$$

Using this momentum, we have

$$\begin{aligned}&\langle \vec{P}_1, \vec{X}_1, T_1|\vec{P}_2, \vec{X}_2, T_2\rangle \\ &= \tilde{N}\exp(i\phi), && (2·68)\\ &\tilde{N} = (1+\frac{i(T_1-T_2)}{2\sigma E(\vec{P}_X)})^{-1}(1+\frac{i(T_1-T_2)m^2}{2\sigma E(\vec{P}_X)^3})^{-1/2} && (2·69)\\ &\times e^{-\frac{\sigma}{4}\{(\vec{P}_1-\vec{P}_2)^2+4(\vec{P}_X-\vec{P}_0)^2}, \\ &\exp(i\phi) = \exp(-iE(\vec{P}_X)(T_1-T_2)+i\vec{P}_X(\vec{X}_1-\vec{X}_2)) && (2·70)\end{aligned}$$

$$= \exp(im\sqrt{(T_1 - T_2)^2 - (\vec{X}_1 - \vec{X}_2)^2})$$
$$= exp(im^2 \frac{|T_1 - T_2|}{E(\vec{P}_X)})$$

In the above equations, a transverse component and a longitudinal component of the momentum are defined as Eq.(2·43). The variances in the transverse directions and the longitudinal direction, σ_T and σ_L, are given as

$$\sigma_T = \frac{\sigma}{1 + (\frac{T_1 - T_2}{2\sigma E(\vec{P}_X)})^2} \tag{2·71}$$

$$\sigma_L = \frac{\sigma}{1 + (\frac{(T_1 - T_2)m^2}{2\sigma E(\vec{P}_X)^3})^2}. \tag{2·72}$$

Thus the variances σ_T and σ_L decrease with time. The spatial extension of the wave packet in the transverse direction is inversely proportional to ${\sigma_T}^{1/2}$ and the extension of the wave packet in the longitudinal direction is inversely proportional to ${\sigma_L}^{1/2}$. Hence the wave packet expands differently in two directions and the expansion in the longitudinal direction is determined by the absolute value of mass. For the massless case or very small mass case, a wave packet does not expand in the longitudinal direction, since the velocity is the constant in this case and σ_L is equal to σ.

The stationary momentum $\vec{P}_X$ is proportional to the direction $\vec{X}_1 - \vec{X}_2$. After the wave packet expands with time, the wave becomes almost spherical wave which is a linear combination of many plane waves. When the measurement is made at the position $\vec{X}_2$, the corresponding wave has a wave vector which is proportional to $\vec{X}_1 - \vec{X}_2$ and the phase which is proportional to the proper time in the asymptotic region.

In the asymptotic region,$|T_1 - T_2| \to \infty$, the matrix element behaves as,

$$\int d^3p \langle \vec{P}_1, \vec{X}_1 | \vec{p} \rangle e^{\frac{E(\vec{p})(T_1 - T_2)}{i}} \langle \vec{p} | \vec{P}_2, \vec{X}_2 \rangle = \tilde{N} \exp(i\phi), \tag{2·73}$$

$$\tilde{N} = (\frac{i(T_1 - T_2)}{2\sigma E(\vec{P}_X)})^{-1} (\frac{i(T_1 - T_2)m^2}{2\sigma E(\vec{P}_X)^3})^{-1/2} \exp(-\frac{\sigma}{2}\{(\vec{P}_X - \vec{P}_1)^2 + (\vec{P}_X - \vec{P}_2)^2\}), \tag{2·74}$$

$$\exp(i\phi) = \exp(im\sqrt{(T_1 - T_2)^2 - (\vec{X}_1 - \vec{X}_2)^2}). \tag{2·75}$$

The probability to find a particle with a momentum $\vec{P}_2$ at a position $\vec{X}_2$ is given as

$$P = d\vec{P}_2 d\vec{X}_2 \frac{1}{(2\pi)^3} |\tilde{N}|^2 \exp(-\sigma\{(\vec{P}_X - \vec{P}_1)^2 + (\vec{P}_X - \vec{P}_2)^2\}). \tag{2·76}$$

For a massless case or an extremely small mass case, in large time of $|T_1 - T_2|$, the matrix element Eq.(2·73) is replaced with,

$$\int d^3p \langle \vec{P}_1, \vec{X}_1 | \vec{p} \rangle e^{\frac{E(\vec{p})(T_1 - T_2)}{i}} \langle \vec{p} | \vec{P}_2, \vec{X}_2 \rangle = \tilde{N} \exp(i\phi), \tag{2·77}$$

$$\tilde{N} = (\frac{i(T_1 - T_2)}{2\sigma E(\vec{P}_X)})^{-1} \exp(-\frac{\sigma}{2}\{(\vec{P}_X - \vec{P}_1)^2 + (\vec{P}_X - \vec{P}_2)^2\}),$$

§3. Generalized scattering matrix

We study many particle systems where one particle wave functions are described by wave packets. A field operator is expanded by a complete set of wave packets and coefficients become operators.

3.1. *Expansion of field*

A field operator $\phi(x)$ is expanded by a set of momentum eigenstates as

$$\phi(\vec{x},t) = \int d\vec{p}(\langle\vec{p}|\vec{x}\rangle \frac{1}{\sqrt{2\omega(\vec{p})}} a(\vec{p},t) + \langle\vec{x}|\vec{p}\rangle \frac{1}{\sqrt{2\omega(\vec{p})}} a^\dagger(\vec{p},t)). \tag{3·1}$$

Operators $a(\vec{p},t)$ and its conjugate $a(\vec{p},t)^\dagger$ are creation and annihilation operator of the momentum $\vec{p}$.

The operator $A(\vec{P}_0,\vec{X}_0,T_0,t)$ and its conjugate are defined as linear combinations of the operators $a(\vec{p},t)$ and its conjugate,

$$A(\vec{P}_0,\vec{X}_0,T_0,t) = \int d\vec{p} a(\vec{p},t)\langle\vec{p}|\vec{P}_0,\vec{X}_0,T_0\rangle \tag{3·2}$$

$$A^\dagger(\vec{P}_0,\vec{X}_0,T_0,t) = \int d\vec{p}\langle\vec{P}_0,\vec{X}_0,T_0|\vec{p}\rangle a^\dagger(\vec{p},t). \tag{3·3}$$

Conversely $a(\vec{p},t)$ and its conjugate are solved from the above equations by using Eq.(2·22) as

$$a(\vec{p},t) = \int \frac{d\vec{P}_0 d\vec{X}_0}{(2\pi)^3} A(\vec{P}_0,\vec{X}_0,T_0,t)\langle\vec{P}_0,\vec{X}_0,T_0|\vec{p}\rangle \tag{3·4}$$

$$a^\dagger(\vec{p},t) = \int \frac{d\vec{P}_0 d\vec{X}_0}{(2\pi)^3} \langle\vec{p}|\vec{P}_0,\vec{X}_0,T_0\rangle A^\dagger(\vec{P}_0,\vec{X}_0,T_0,t) \tag{3·5}$$

The wave packet size σ in in-state is determined from the beam and the σ in out-state is determined from the detector and they are different generally. For simplicity, we use the same value and omit to write in most parts of the present paper. In several places we write σ explicitly.

By substituting the above expansions to Eq.(3·2), we have

$$A(\vec{P}_0,\vec{X}_0,T_0,t) = \int \frac{d\vec{P}'_0 d\vec{X}'_0}{(2\pi)^3} A(\vec{P}'_0,\vec{X}'_0,T'_0,t)\langle\vec{P}'_0,\vec{X}'_0,T'_0|\vec{P}_0,\vec{X}_0,T_0\rangle \tag{3·6}$$

$$A^\dagger(\vec{P}_0,\vec{X}_0,T_0,t) = \int \frac{d\vec{P}'_0 d\vec{X}'_0}{(2\pi)^3} \langle\vec{P}_0,\vec{X}_0,T_0|\vec{P}'_0,\vec{X}'_0,T'_0\rangle A^\dagger(\vec{P}'_0,\vec{X}'_0,T'_0,t). \tag{3·7}$$

These equations show that in the space of operator $A(\vec{P}_0,\vec{X}_0,T_0,t)$ the transformation function $\langle\vec{P}'_0,\vec{X}'_0,T'_0|\vec{P}_0,\vec{X}_0,T_0\rangle$ plays a role of Dirac's delta function.

By substituting the above expansion to Eq.(3·1), the same field operator is expanded by a set of minimum wave packets of the center momentum, $\vec{P}_0$, the center coordinates, $\vec{X}_0$, and the time, T_0 as,

$$\phi(\vec{x},t) = \int \frac{d\vec{P}_0 d\vec{X}_0}{(2\pi)^3}(C(\vec{P}_0,\vec{X}_0,T_0;\vec{x})A(\vec{P}_0,\vec{X}_0,T_0,t) \tag{3·8}$$
$$+C(\vec{P}_0,\vec{X}_0,T_0;\vec{x})^* A^\dagger(\vec{P}_0,\vec{X}_0,T_0,t)),$$
$$C(\vec{P}_0,\vec{X}_0,T_0;\vec{x}) = \int \frac{d\vec{p}}{\sqrt{2\omega(\vec{p})}}\langle\vec{P}_0,\vec{X}_0,T_0|\vec{p}\rangle\langle\vec{p}|\vec{x}\rangle \tag{3·9}$$

is obtained. It is convenient to define transformation matrices,

$$\tilde{C}(\vec{P}_0,\vec{X}_0,T_0;\vec{x}) = \int d\vec{p}\sqrt{2\omega(\vec{p})}\langle\vec{P}_0,\vec{X}_0,T_0|\vec{p}\rangle\langle\vec{p}|\vec{x}\rangle. \tag{3·10}$$

These matrices satisfy

$$C(\vec{P}_0,\vec{X}_0,T_0;\vec{x}) = \int \frac{d\vec{P}_1 d\vec{X}_1}{(2\pi)^3}\langle\vec{P}_0,\vec{X}_0,T_0|\vec{P}_1,\vec{X}_1,T_1\rangle C(\vec{P}_1,\vec{X}_1,T_1;\vec{x}) \tag{3·11}$$

$$\tilde{C}(\vec{P}_0,\vec{X}_0,T_0;\vec{x}) = \int \frac{d\vec{P}_1 d\vec{X}_1}{(2\pi)^3}\langle\vec{P}_0,\vec{X}_0,T_0|\vec{P}_1,\vec{X}_1,T_1\rangle \tilde{C}(\vec{P}_1,\vec{X}_1,T_1;\vec{x}) \tag{3·12}$$

$$\int d\vec{x} C(\vec{P}_0,\vec{X}_0,T_0;\vec{x})(\tilde{C}(\vec{P}_0,\vec{X}_0,T_0;\vec{x}))^* = \langle\vec{P}_0,\vec{X}_0,T_0|\vec{P}_1,\vec{X}_1,T_1\rangle. \tag{3·13}$$

3.2. *Complete set of many body states*

Many body state is constructed by the second quantized operators and the vacuum. In the momentum representation the operators $a(\vec{p})$ and $a^\dagger(\vec{p})$ satisfy

$$[a(\vec{p},t),a^\dagger(\vec{p}',t')]\delta(t-t') = \delta(\vec{p}-\vec{p}')\delta(t-t') \tag{3·14}$$
$$a(\vec{p},t)|0\rangle = 0. \tag{3·15}$$

A complete set of many body states are constructed from

$$|0\rangle, |\vec{p}_1\rangle, |\vec{p}_1,\vec{p}_2\rangle, |\vec{p}_1,\vec{p}_2,\vec{p}_3\rangle, -, |\vec{p}_1,\vec{p}_2,-,-\vec{p}_N\rangle, \tag{3·16}$$

where

$$|\vec{p}_1\rangle = a^\dagger(\vec{p}_1,t|0\rangle, \tag{3·17}$$
$$|\vec{p}_1,\vec{p}_2\rangle = a^\dagger(\vec{p}_1,t)a^\dagger(\vec{p}_2,t)|0\rangle, \tag{3·18}$$
$$|\vec{p}_1,\vec{P}_2,-,-\vec{p}_N\rangle = \Pi_l a^\dagger(\vec{p}_l,t)|0\rangle. \tag{3·19}$$

These particle states satisfy orthonormality conditions with Dirac delta function,

$$\langle\vec{p}_1|\vec{p}_1'\rangle = \delta(\vec{p}_1-\vec{p}_1'), \tag{3·20}$$
$$\langle\vec{p}_1,\vec{p}_2|\vec{p}_1',\vec{p}_2'\rangle = \delta(\vec{p}_1-\vec{p}_1')\delta(\vec{p}_2-\vec{p}_2') + permutation. \tag{3·21}$$

Let define a projection operator I as

$$I = |0\rangle\langle 0| + \int d\vec{p}a^\dagger(\vec{p})|0\rangle\langle 0|a(\vec{p}) + \int \frac{d\vec{p}_1 d\vec{p}_2}{2!} a^\dagger(\vec{p}_1)a^\dagger(\vec{p}_2)|0\rangle\langle 0|a(\vec{p}_1)a(\vec{p}_2) + \int \frac{1}{l!}\Pi_l d\vec{p}_l a^\dagger(\vec{p}_l)|0\rangle\langle 0|\Pi_l a(\vec{p}_l) + \cdots , \qquad (3\cdot22)$$

and multiply I to a state

$$|\Psi\rangle = \int f(\vec{q}_i)\Pi_l a^\dagger(\vec{p}_l)|0\rangle. \qquad (3\cdot23)$$

Then from Eq.(3·14) and (3·15), we have

$$\begin{aligned} I|\Psi\rangle &= \int \Pi_l d^3q_l f(\vec{q}_i) \int \frac{1}{l!}\Pi_l d\vec{p}_l a^\dagger(\vec{p}_l)|0\rangle\delta(\vec{q}_l - \vec{p}_j) \\ &= \int f(\vec{q}_i)\Pi_l a^\dagger(\vec{p}_l)|0\rangle \\ &= |\Psi\rangle \end{aligned} \qquad (3\cdot24)$$

Hence the completeness condition,

$$I = 1 \qquad (3\cdot25)$$

is satisfied. The time t is arbitrary and is omitted in the above equations.

Similarly the operators in the mixed representation satisfy

$$[A(\vec{P}_0,\vec{X}_0,T_0,t), A^\dagger(\vec{P}'_0,\vec{X}'_0,T'_0,t')]\delta(t-t') = \langle \vec{P}_0,\vec{X}_0,T_0|\vec{P}'_0,\vec{X}'_0,T'_0\rangle\delta(t-t') \qquad (3\cdot26)$$

$$A(\vec{P}_0,\vec{X}_0,T_0,t)|0\rangle = 0. \qquad (3\cdot27)$$

The particle states defined by these operators are normalized and are not orthogonal even though momenta and positions are different. However the same complete set is constructed by the vacuum and creation operators in the mixed representation. Let define an projection operator I',

$$\begin{aligned} I' = {} & |0\rangle\langle 0| + \int \frac{d\vec{P}d\vec{X}}{(2\pi)^3} A^\dagger(\vec{P},\vec{X},T)|0\rangle\langle 0|A(\vec{P},\vec{X},T) \\ &+ \int \frac{1}{2}\frac{d\vec{P}_1 d\vec{X}_1}{(2\pi)^3}\frac{d\vec{P}_2 d\vec{X}_2}{(2\pi)^3} A^\dagger(\vec{P}_1,\vec{X}_1,T_1)A^\dagger(\vec{P}_2,\vec{X}_2,T_2)|0\rangle\langle 0|A(\vec{P}_2,\vec{X}_2,T_2)A(\vec{P}_1,\vec{X}_1,T_1) \\ &+ \int \frac{1}{l!}\Pi_l \frac{d\vec{P}_l d\vec{X}_l}{(2\pi)^3}\Pi_l A^\dagger(\vec{P}_l,\vec{X}_l,T_l)|0\rangle\langle 0|\Pi_l A(\vec{P}_l,\vec{X}_l,T_l) + \cdots \end{aligned} \qquad (3\cdot28)$$

The time t in the operators is arbitrary and is omitted in the above equations. Let multiply the operator I' to an state,

$$|\Psi'\rangle = \int \Pi_l \frac{d\vec{Q}_l d\vec{Y}_l}{(2\pi)^3} F(\vec{Q}_i,\vec{Y}_i,S_i)\Pi_l A^\dagger(\vec{Q}_l,\vec{Y}_l,S_i)|0\rangle. \qquad (3\cdot29)$$

Then from Eq.(3·26) and (3·27), we have

$$
\begin{aligned}
&I'|\Psi'\rangle \\
&= \int \Pi_l \frac{d\vec{Q}_l d\vec{Y}_l}{(2\pi)^3} F(\vec{Q}_i, \vec{Y}_i, S_i) \int \frac{1}{L!} \Pi_m \frac{d\vec{P}_m d\vec{X}_m}{(2\pi)^3} A^\dagger(\vec{P}_m, \vec{X}_m, T_m)|0\rangle \\
&\quad \langle 0|A(\vec{P}_m, \vec{X}_m, T_m)\Pi_l A^\dagger(\vec{Q}_l, \vec{Y}_l, S_i)|0\rangle \\
&= \int \Pi_l \frac{d\vec{Q}_l d\vec{Y}_l}{(2\pi)^3} F(\vec{Q}_i, \vec{Y}_i, S_i) \Pi_m \frac{d\vec{P}_m d\vec{X}_m}{(2\pi)^3} A^\dagger(\vec{P}_m, \vec{X}_m, T_m)\langle \vec{P}_m, \vec{X}_m, T_m|\vec{Q}_l, \vec{Y}_l, S_l\rangle|0\rangle \\
&= \int \Pi_l \frac{d\vec{Q}_l d\vec{Y}_l}{(2\pi)^3} F(\vec{Q}_i, \vec{Y}_i, S_i) A^\dagger(\vec{Q}_l, \vec{Y}_l, T_i)|0\rangle \qquad (3\cdot30) \\
&= |\Psi'\rangle.
\end{aligned}
$$

Hence the completeness condition,

$$I' = 1 \tag{3·31}$$

is satisfied. The time t is arbitrary and is omitted in the above equations.

3.3. *Time evolution*

A unitary operator which translates a time of field operators by a finite value, t, is given by a Hamiltonian H as,

$$U(t) = e^{\frac{Ht}{i}}. \tag{3·32}$$

In a free scalar theory the Hamiltonian is given as,

$$H = \int d^3x (\frac{1}{2}(\pi(x)^2 + \frac{1}{2}(\vec{\nabla}\phi(x))^2 + \frac{1}{2}m^2\phi(x)^2). \tag{3·33}$$

A commutation relation between the field operator and its conjugate,

$$[\phi(x), \pi(y)]\delta(x_0 - y_0) = i\delta^{(4)}(x - y) \tag{3·34}$$

leads the operators $a(\vec{p})$ and $a^\dagger(\vec{p})$ in Eq.(3·1) satisfy the equal time commutation relation Eq.(3·14). The Hamiltonian is expressed as

$$
\begin{aligned}
H &= \int d^3p E(\vec{p})(a^\dagger(\vec{p})a(\vec{p}) + \frac{1}{2}) \qquad (3\cdot35) \\
E(\vec{p}) &= \sqrt{\vec{p}^2 + m^2}
\end{aligned}
$$

and satisfy

$$[H, a^\dagger(\vec{p})] = E(\vec{p})a^\dagger(\vec{p}) \tag{3·36}$$

$$[H, a(\vec{p})] = -E(\vec{p})a^\dagger(\vec{p}) \tag{3·37}$$

From the above commutation relations, we have the time dependence of the creation and annihilation operators in the momentum space,

$$a^\dagger(\vec{p}, t)$$

$$= U(t)a^\dagger(\vec{p},0)U^\dagger(t) = e^{\frac{E(\vec{p})t}{i}}a^\dagger(\vec{p},0) \tag{3·38}$$

$$a(\vec{p},t)$$
$$= U(t)a(\vec{p},0)U^\dagger(t) = e^{-\frac{E(\vec{p})t}{i}}a(\vec{p},0). \tag{3·39}$$

The creation and annihilation operators in the momentum space change only the c-number phase with time. The states created by the creation operators in the momentum space stay in the same state,

$$U(t)|\Psi\rangle = e^{\frac{\sum_j E(\vec{p}_j)}{i}t}|\Psi\rangle \tag{3·40}$$
$$|\Psi\rangle = a^\dagger(\vec{p}_1,0)a^\dagger(\vec{p}_2,0) - - - -a^\dagger(\vec{p}_l,0)|0\rangle.$$

The creation operators and annihilation operators in the mixed space satisfy a commutation relation Eq.(3·26). The overlap function in the right hand side is computed in the next subsection. This function does not vanish even though the momentum or the coordinates are different. Hence the operator of one set of center coordinate and center momentum do not commute with the operator of a different coordinate and a different momentum, but a set of operators satisfy completeness condition from Eq.(3·31). The operators in the mixed space evolve with time as,

$$A^\dagger(\vec{P}_0,\vec{X}_0,T_0,t) \tag{3·41}$$
$$= U(t)A^\dagger(\vec{P}_0,\vec{X}_0,T_0,o)U^\dagger(t) = \int d^3p e^{\frac{E(\vec{p})t}{i}}\langle\vec{P}_0,\vec{X}_0,T_0|\vec{p}\rangle a^\dagger(\vec{p},0)$$
$$= \int \frac{d\vec{P}_0' d\vec{X}_0'}{(2\pi)^3}\int d^3p e^{\frac{E(\vec{p})t}{i}}\langle\vec{P}_0,\vec{X}_0,T_0|\vec{p}\rangle\langle\vec{p}|\vec{P}_0',\vec{X}_0',T_0\rangle A^\dagger(\vec{P}_0',\vec{X}_0',T_0,0)$$

$$A(\vec{P}_0,\vec{X}_0,T_0,t) \tag{3·42}$$
$$= U(t)A(\vec{P}_0,\vec{X}_0,T_0,0)U^\dagger(t) = \int d^3p e^{-\frac{E(\vec{p})t}{i}}\langle\vec{p}|\vec{P}_0,\vec{X}_0,T_0\rangle a(\vec{p},t)$$
$$= \int \frac{d\vec{P}_0' d\vec{X}_0'}{(2\pi)^3}\int d^3p e^{-\frac{E(\vec{p})t}{i}}\langle\vec{P}_0',\vec{X}_0',T_0|\vec{p}\rangle\langle\vec{p}|\vec{P}_0,\vec{X}_0,T_0\rangle A(\vec{P}_0',\vec{X}_0',T_0,0).$$

The time dependent phase factor is not factored out and the states created by the operators in the mixes space, $A^\dagger(\vec{P}_0,\vec{X}_0)$ change with time,

$$U(t)|\Psi'\rangle \neq e^{\frac{\sum_j E(\vec{p}_j)}{i}t}|\Psi'\rangle \tag{3·43}$$
$$|\Psi'\rangle = A^\dagger(\vec{P}_1,\vec{X}_1)A^\dagger(\vec{P}_2,\vec{X}_2) - - - -A^\dagger(\vec{P}_l,\vec{X}_l)|0\rangle.$$

The matrix elements in the above equations are obtained in the next subsection and we will see that these states are approximate eigenstates.

3.4. *Generalized scattering amplitude and transition probability*

A scattering amplitude where particles in the initial state of momentum $\vec{P}_l^i$ are prepared at positions $\vec{X}_l^i$ and times T_l^i and particles in the final state of momentum $\vec{P}_m^o$ are measured at positions $\vec{X}_m^o$ and times T_m^o are described as,

$$S_{out,in} \tag{3·44}$$

$$= \langle \vec{P}_1^o \vec{X}_1^o {T^o}_1; -; \vec{P}_L^o \vec{X}_L^o {T^o}_L; out | in; \vec{P}_1^i \vec{X}_1^i {T^i}_1; -; \vec{P}_M^i \vec{X}_M^i {T^i}_M \rangle,$$

$$|in; \vec{P}_1^i \vec{X}_1^i {T^i}_1; -; \vec{P}_M^i \vec{X}_M^i {T^i}_M \rangle \tag{3·45}$$

$$= A^\dagger(\vec{P}_1^i, \vec{X}_1^i, T_1^i) A^\dagger(\vec{P}_2^i, \vec{X}_2^i, T_2^i) - -A^\dagger(\vec{P}_l^i, \vec{X}_l^i, T_l^i)|0\rangle$$

$$|out; \vec{P}_1^o \vec{X}_1^o {T^o}_1; -; \vec{P}_M^o \vec{X}_M^o {T^o}_M \rangle \tag{3·46}$$

$$= A^\dagger(\vec{P}_1^o, \vec{X}_1^o, T_1^o) A^\dagger(\vec{P}_2^o, \vec{X}_2^o, T_2^o) - -A^\dagger(\vec{P}_l^o, \vec{X}_l^o, T_l^o)|0\rangle,$$

$$T_l^i \ll T_l^o \tag{3·47}$$

The differential transition probability from an initial state to a final state is given as,

$$dP = \Pi_{l=1}^{l=L} \frac{d\vec{P}_l d\vec{X}_l}{(2\pi)^3} |S_{out,in}|^2, \tag{3·48}$$

and the total transition probability is obtained by integrating positions and momenta as,

$$P = \int \frac{1}{L!} \Pi_{l=1}^{l=L} \frac{d\vec{P}_l d\vec{X}_l}{(2\pi)^3} |S_{out,in}|^2 \tag{3·49}$$

In the interaction picture, the S-matrix element is computed by

$$\langle 0 | \Pi_{l=1}^L A(\vec{P}_l, \vec{X}_l, T_l) T \exp \int dt' \frac{H_{int}(t')}{i} \Pi_{m=1}^M A^\dagger(\vec{P}_m, \vec{X}_m, T_m)|0\rangle. \tag{3·50}$$

As will be seen in examples of the next section, the differential probability depends upon the sizes of wave packets in the initial states and final states. But total probabilities become universal values that are independent from the sizes of wave packets owing to the completeness of the many body states.

Firstly, the total probability from one initial state to a final state of fixed number of particles become independent from the wave packet sizes of final states. To see this, let define a generalized S-matrix of wave packet size σ_o of the final state and σ_i of the initial state,

$$S_{out,in}(\sigma_o, \sigma_i) \tag{3·51}$$

$$= \langle \vec{P}_1^o \vec{X}_1^o {T^o}_1, \sigma_o; -; \vec{P}_L^o \vec{X}_L^o {T^o}_L \sigma_o; out | in; \vec{P}_1^i \vec{X}_1^i {T^i}_1 \sigma_i; -; \vec{P}_M^i \vec{X}_M^i {T^i}_M \sigma_i \rangle.$$

Using complexness relation of wave packets for an arbitrary wave packet size , the total probability from one initial state to a L-particle state of one value of σ_o satisfy

$$P_M(\sigma_0, \sigma_i) \tag{3·52}$$

$$= \frac{1}{L!} \Pi_{l=0}^{l=L} \int \frac{d\vec{P}_l d\vec{X}_l}{(2\pi)^3} \langle in; \sigma_i | S | L, \vec{P}_l, \vec{X}_l, out; \sigma_o \rangle \langle L, \vec{P}_l, \vec{X}_l out; \sigma_o | S^\dagger | in; \sigma_i \rangle$$

$$= \frac{1}{L!} \Pi_{l=0}^{l=L} \int \frac{d\vec{P}_l d\vec{X}_l}{(2\pi)^3} \langle in; \sigma_i | S | L, \vec{P}_l, \vec{X}_l, out; \sigma_o' \rangle \langle L, \vec{P}_l, \vec{X}_l, out; \sigma_o' | S^\dagger | in; \sigma_i \rangle$$

$$= P_M(\sigma_0', \sigma_i).$$

Thus the total probability is independent from the size σ_o. The above total probability also agrees with the total probability from the initial state of wave packets to the final states of momentum eigenstates,

$$\begin{aligned}
&P_M(\sigma_0,\sigma_i) \\
&= \frac{1}{L!}\Pi_{l=0}^{l=L}\int \frac{d\vec{p}_l}{(2\pi)^{(3/2)}}\langle in;\sigma_i|S|L,p_l,out;\rangle\langle L,p_l,out;|S^\dagger|in;\sigma_i\rangle \\
&= P_M(momentumstate,\sigma_i).
\end{aligned} \tag{3·53}$$

In the above derivations, we have used the fact that the set of N-particle states is complete regardless of the wave packet sizes. Thus the scattering amplitude for any complete set of functions gives the same total probability. Conversely we can compute the total probability by using the momentum eigenstates for the unobserved particles as far as the boundary conditions are satisfied.

Secondly, the total probability from one state to any possible final states,

$$\begin{aligned}
&\sum_L P(in\to L) \\
&= \sum_L \frac{1}{L!}\Pi_{l=0}^{l=L}\int \frac{d\vec{P}_l d\vec{X}_l}{(2\pi)^3}\langle in|S|L,\vec{P}_l,\vec{X}_l,\sigma_o;out\rangle\langle L,\vec{P}_l,\vec{X}_l,\sigma_o;out|S^\dagger|in\rangle \\
&= \langle in|SS^\dagger|in\rangle \\
&= 1
\end{aligned} \tag{3·54}$$

becomes unity. The facts that the particle states are normalized and S is unitary are used in the above derivation. Thus the standard probability interpretation for the square of the absolute value of the amplitudes is applicable with the phase space defined in Eq.(3·55) despite of the non-orthogonality of the states.

Finally an inclusive probability where a partial set of kinematical variables are measured and other variables are unmeasured satisfies also a similar universal relation as the total probability. Namely this probability depends on the size of wave packet of measured particles but does not depend upon the sizes of wave packets of unmeasured particles, if the different values of wave packet sizes are used.

It is summarized as follows: The probability depends upon the sizes of wave packets of measured particles and the probability does not depend on the variables of unmeasured particles, such as the momenta, positions and wave packet sizes.

§4. Few body scattering

Few body scattering are studied as examples. We study a scattering process where a particle of a momentum $\vec{P}_1$ which is prepared at a space and time coordinate $\vec{X}_1, T_1$ and another particle of a momentum $\vec{P}_2$ which is prepared at $\vec{X}_2, T_2$ collide and a particle of a momentum $\vec{P}_3$ at $\vec{X}_3, T_3$ and another particle of a momentum $\vec{P}_4$ at $\vec{X}_4, T_4$ are measured first.

A system with an interaction Hamiltonian,

$$H_{int} = \int d\vec{x}\frac{\lambda}{4}\phi(x)^4 \tag{4·1}$$

is studied. This interaction is of short range, and the generalized amplitude shows characteristic dependences on the momentum as well as coordinate.

4.1. *effective sizes of the interaction region*

Let substitute the expansion of the field Eq.(3·8) and the interaction Hamiltonian Eq.(4·1) into Eq.(3·50). Then we have the scattering matrix in the first order of H_{int},

$$\begin{aligned}&\langle 0|\Pi_{l=1}^2 A(\vec{P}_l,\vec{X}_l,T_l)\int dt'\frac{H_{int}(t')}{i}\Pi_{m=1}^2 A^\dagger(\vec{P}_m,\vec{X}_m,T_m)|0\rangle\\ &=\lambda\int dtd\vec{x}\Pi_l C(\vec{P}_l,\vec{X}_l,T_l,\vec{x},t)\Pi_m(C(\vec{P}_m,\vec{X}_m,T_m,\vec{x},t))^*.\end{aligned} \tag{4·2}$$

We have used the two point function of the field of mixed representation and the field of coordinate representation,

$$\begin{aligned}&\langle 0|\phi(\vec{x},t)A^\dagger(\vec{P},\vec{X},T)|0\rangle\\ &=\int\frac{d\vec{P}_0 d\vec{X}_0}{(2\pi)^3}C(\vec{P}_0,\vec{X}_0,\vec{x},t)\langle 0|A(\vec{P}_0,\vec{X}_0,t)A^\dagger(\vec{P},\vec{X},T)|0\rangle.\end{aligned} \tag{4·3}$$

This is written further by combining Eq.(3·8) and Eq.(3·11) as,

$$\langle 0|\phi(\vec{x},t)A^\dagger(\vec{P},\vec{X},T)|0\rangle = C(\vec{P},\vec{X},T,\vec{x},t). \tag{4·4}$$

In the parameter regions we are interested in this paper, this function and its partner is approximated well with a very good accuracy as,

$$C(\vec{P},\vec{X},T,\vec{x},t) = \frac{1}{\sqrt{2E(\vec{P})}}\langle\vec{P},\vec{X},T|\vec{x},t\rangle \tag{4·5}$$

$$\tilde{C}(\vec{P},\vec{X},T,\vec{x},t) = \sqrt{2E(\vec{P})}\langle\vec{P},\vec{X},T|\vec{x},t\rangle \tag{4·6}$$

In the following calculations we use these formula.

The coefficients $C(\vec{P}_l,\vec{X}_l,T_l,\vec{x},t)$ and their complex conjugate give the values of wave functions at $(\vec{x},t)$. In the region where times T_l are close to the t, the product of the functions is a Gaussian function around the peak in the variables $\vec{x}$ and t and is expressed as

$$\begin{aligned}&\Pi_l C(\vec{P}_l,\vec{X}_l,T_l,\vec{x},t)\Pi_m(C(\vec{P}_m,\vec{X}_m,T_m,\vec{x},t))^* \\ &=(N_3)^L\Pi\exp(-\frac{1}{2\sigma}(\vec{x}-\vec{X}_j-\vec{v}_j(t-T_j))^2)\times\\ &\quad\exp(-iE(\vec{P}_l)(t-T_l)+i\vec{P}_l\cdot(\vec{x}-\vec{X}_l)+iE(\vec{P}_m)(t-T_m)-i\vec{P}_m\cdot(\vec{x}-\vec{X}_m))\\ &=N\exp(-\frac{1}{2\sigma_S}(\vec{x}-\vec{x}_0)^2-\frac{1}{2\sigma_T}(t-t_0)^2)e^{i\phi}.\end{aligned} \tag{4·7}$$

The peak position $\vec{x}_0$ and t_0 are determined as,

$$\vec{x}_0 = \sigma_S \sum \frac{1}{\sigma_l} \vec{x}_l(t), \tag{4·8}$$

$$t_0 = \frac{B}{\sigma_T}, \tag{4·9}$$

$$B = \sum_l \frac{1}{\sigma_l} (\vec{X}_l - \vec{v}_l T_l) \cdot \vec{v}_l - \sigma_S (\sum_l \frac{1}{\sigma_l} \vec{v}_l) \cdot (\sum_l \frac{1}{\sigma_l} (\vec{X}_l - \vec{v}_l T)), \tag{4·10}$$

$$\vec{x}_l(t) = \vec{X}_l + \vec{v}_l (t - T_l) \tag{4·11}$$

and the variances σ_S and σ_T are determined as

$$\sigma_S = \frac{\sigma}{4} \tag{4·12}$$

$$\sigma_T = \sigma \left(\sum_l \vec{v}_l^2 - \frac{1}{4} (\sum_l \vec{v}_l)^2 \right)^{-1}. \tag{4·13}$$

The region around the peak within the spatial distance $\sqrt{\sigma_S}$ and the time distance $\sqrt{\sigma_T}$ gives a dominant contribution to the integral. Both distances are proportional to the σ. So if the σ_S is small, the σ_T is also small. The normalization N and the phase ϕ are complicated functions of the energies, momenta, spatial positions, and temporal positions. Explicit formulas are given in Appendix.

4.1.1. Complete measurements

So far, all particles have the same wave packet size and all particles are measured. When this is not hold and different particles have different wave packet sizes and some particles are not measured, the effective sizes σ_S and σ_T become different from the above values. We study the behavior of these variances in general cases where each wave packet has its own size and all momenta and positions are measured here. Let specify the wave packet size of the l-th particle as σ_l, and its velocity as $\vec{v}_l$, then the variances σ_S and σ_T are given by

$$\sigma_S = \left(\sum_l \frac{1}{\sigma_l} \right)^{-1} \tag{4·14}$$

$$\sigma_T = \left(\sum_l \frac{1}{\sigma_l} \vec{v}_l^2 - \sigma_S (\sum_l \frac{1}{\sigma_l} \vec{v}_l)^2 \right)^{-1} \tag{4·15}$$

The σ_S is determined mainly by the small σ_l of the measured particles but the σ_T is determined by the σ_l and $\vec{v}_l$ of the measured particles. The small σ_l does not contribute if the corresponding $\vec{v}_l$ vanishes. So the time variance σ_T could become large even though the space variance σ_S is small. The large σ_t is compatible with the small σ_S in this case.

The $\vec{v}_l$ depends on the momentum $\vec{P}_l$. So σ_T is not a constant generally but varies in the kinematical region of the final state. The exception is the case when σ_l is infinity, i.e., plane wave.

4.1.2. Partial measurements

When some portion of particles are measured and others are unmeasured, the probability depends on the wave packet sizes of the measured particles. When one particle, $l=1$, is measured and other particles are unmeasured,it depends on the wave packet σ_1. Hence the effective sizes of the vetcex area for computing the total probability are obtained by letting $\sigma_l=\infty$ for $l\neq 1$

$$\sigma_S=\sigma_1 \tag{4·16}$$

$$\sigma_T=(\frac{1}{\sigma_1}\vec{v}_1^2-\frac{1}{\sigma_1}\vec{v}_1^2)^{-1}=\infty \tag{4·17}$$

The spatial size σ_S is determined from the σ_1 of the observed particle but the temporal size σ_T diverges.

Next, the probability when two particles are measured and other particles are unmeasured depends on the wave packet σ_1 and σ_2 and velocities $\vec{v}_1$ and $\vec{v}_2$ of the measured particles. The effective sizes are obtained by letting $\sigma_l=\infty$ for $l\neq 1,2$

$$\sigma_S=\left(\sum_{l=1}^{l=2}\frac{1}{\sigma_l}\right)^{-1}=\frac{\sigma_1\sigma_2}{\sigma_1+\sigma_2} \tag{4·18}$$

$$\sigma_T=\left(\sum_{l=1}^{l=2}\frac{1}{\sigma_l}\vec{v}_l^2-\sigma_S(\sum_{l=1}^{l=2}\frac{1}{\sigma_l}\vec{v}_l)^2\right)^{-1}=\frac{\sigma_1+\sigma_2}{(\vec{v}_1-\vec{v}_2)^2} \tag{4·19}$$

σ_T diverges if $\vec{v}_1$ is equal to $\vec{v}_2$.

4.2. *Short distance scattering*

When two particles are in the initial states and two particles are in the final states and the distances $|\vec{X}_i-\vec{X}_j|$ and $|T_i-T_j|$ are small, the formula Eq.(2·28) for the small time differences is used. We expect that the amplitude in this region shows features of the translational motion of wave packets and other features of the generalized scattering amplitude. For simplicity, we present the results when all the particles have the same wave packet size. The general case is given in the Appendix.

The transition amplitude in the lowest order of H_{int} is given by

$$\lambda\int dtd\vec{x}\Pi_l C(\vec{P}_l,\vec{X}_l,T_l,\vec{x},t)\Pi_m(C(\vec{x},t,\vec{P}_m,\vec{X}_m,T_m,\vec{x},t))^* \tag{4·20}$$

$$=\lambda N_3^4\Pi_l(2E(\vec{P}_l))^{-1/2}\int dtd\vec{x}\exp\sum\{-is_i\{(t-T_l)E(\vec{P}_l)-\vec{P}_l\cdot(\vec{x}-\vec{X}_l)\}$$

$$\exp\sum\{-\frac{1}{2\sigma}\{\vec{x}-\vec{X}_l-\vec{v}_l(t-T_l)\}^2\},$$

$$=\tilde{N}\exp(i\phi-R),$$

$$\tilde{N}=(\frac{4}{\sum_{i,j}(\vec{v}_i-\vec{v}_j)^2})^{3/2}\Pi_l(2E(\vec{P}_l))^{-1/2}, \tag{4·21}$$

$$\phi=\sum_i sgn_i(T_iE(\vec{P}_i)-\vec{P}_i\cdot\vec{X}_i)+\tilde{\phi}, \tag{4·22}$$

$$R = \frac{\sigma}{8}(\sum_i \vec{P}_i)^2 + \frac{1}{8\sigma}\sum_{i,j}(\vec{X}_i - \vec{v}_i T_i - \vec{X}_j + \vec{v}_j T_j)^2$$
$$+\frac{2\sigma}{\sum_{i,j}(\vec{v}_i - \vec{v}_j)^2}(\sum E(\vec{P}_i)s_i)^2 + \tilde{R}, \tag{4·23}$$

where sgn_i is a signature

$$sgn_i = +1(in - state), -1(out - state) \tag{4·24}$$

and $\tilde{\phi}$ and $\tilde{R}$ are small quantities and are given in the appendix for most general case. In the above equations, the dominant part in the phase, ϕ, is the standard phase of the plane wave of the momentum $\vec{p}_0$ and energy E_0. From the first factor and third factor of normalization R, the momentum conservation is approximately satisfied with the variance, $(\frac{\sigma}{8})^{-1/2}$ and the energy conservation is also approximately satisfied with the variance $(\frac{2\sigma}{\sum_{i,j}(\vec{v}_i - \vec{v}_j)^2})^{-1/2}$. The second factor of the R shows that the particle trajectories coincide and coordinates $\vec{X}_i - \vec{v}_i T_i$ are the same within the distance $(8\sigma)^{1/2}$. So particles follow classical trajectories.

From the amplitudes, we define new transition probabilities that depend upon the coordinates in addition to the momenta and argue on an asymptotic condition of the standard scattering amplitude which depends upon the momenta.

The transition probability is a square of the absolute value of the amplitude and is expressed as

$$P(\vec{P}_3, \vec{X}_3, T_3; \vec{P}_4, \vec{X}_4, T_4) \tag{4·25}$$
$$= \frac{1}{2!}\frac{1}{(2\pi)^3}(\tilde{N})^2 \exp(-\frac{\sigma}{4}(\sum_i \vec{P}_i)^2)$$
$$\exp(-\frac{1}{4\sigma}\sum_{i,j}(\vec{X}_i - \vec{v}_i T_i - \vec{X}_j + \vec{v}_j T_j)^2 - \frac{4\sigma}{\sum_{i,j}(\vec{v}_i - \vec{v}_j)^2}(\sum E(\vec{P}_i)s_i)^2).$$

This has a peak at the positions where the conditions

$$\vec{X}_i - \vec{v}_i T_i - \vec{X}_j + \vec{v}_j T_j = 0 \tag{4·26}$$

are satisfied. Thus the peak is along a line

$$\vec{X}_i = \vec{v}_i T_i + \vec{C}, \tag{4·27}$$

where $\vec{C}$ is a constant vector. These positions depend upon the times and the time T_i are arbitrary in the present formalism, hence it is possible to choose the times in such manner that these positions are inside of detectors if the detector is located in the direction of the momentum. Then total probability integrated on this direction is measured. To see this probability, let us decompose the position vector of the i-th particle into the longitudinal component and the transverse components with respects to the velocity $\vec{v}^i$,

$$\vec{X}^i = \vec{v}^i S^i + \vec{n}_T^i {X^i}_T, \tag{4·28}$$
$$\vec{v}^i \cdot \vec{n}_T^i = 0, \tag{4·29}$$
$$\vec{n}_T^i \cdot \vec{n}_T^i = \delta_{ij}. \tag{4·30}$$

The volume element is written as

$$d\vec{X}^j = dS^j d{X^i}_T |v^i|. \tag{4·31}$$

Using these variables, the Gaussian factor of the differential probability is written as

$$\sum_{ij} (\vec{X}^i - \vec{v}^i T^i - \vec{X}^j + \vec{v}^j T^j)^2 \tag{4·32}$$

$$= \sum_{ij} (\vec{v}^i (S^i - T^i) + \vec{n}^i_T X^i_T - \vec{v}^j (S^j - T^j) - \vec{n}^j_T X^j_T)^2$$

$$= \sum_{ij} (\vec{v}^i \tilde{S}^i + \vec{n}^i_T X^i_T - \vec{v}^j \tilde{S}^j - \vec{n}^j_T X^j_T)^2,$$

$$\tilde{S}^i = S^i - T^i. \tag{4·33}$$

Thus the longitudinal variable S_i is combined with time T^i and it is possible to replace the longitudinal coordinate with the time variable in Eq.(4·34).

The transition probability is given by an integration of a differential probability over the momenta and coordinates as,

$$P = \int \Pi_{m=3}^{m=4} \frac{d\vec{P}_m d\vec{X}_m}{(2\pi)^3} P(\vec{P}_3, \vec{X}_3, T_3; \vec{P}_4, \vec{X}_4, T_4). \tag{4·34}$$

In the ordinary detectors neither the the precise value of the time T^i nor the longitudinal coordinates ${X^i}_L$ are measurable but the total probability $P(\vec{P}^3, \vec{X}^3_T, ; \vec{P}^4, \vec{X}^4_T)$ integrated on these variables is measured.

To obtain this probability,the variable S_i is integrated. Then the probability $P(\vec{P}^3, \vec{X}^3_T, T^3; \vec{P}^4, \vec{X}^4_T, T^4)$ is found and is written as

$$P(\vec{P}^3, \vec{X}^3_T, T^3; \vec{P}^4, \vec{X}^4_T, T^4) \tag{4·35}$$

$$= \int \Pi_{m=3}^{m=4} d\vec{X}^m_L \frac{1}{(2\pi)^{1/2}} P(\vec{P}^3, \vec{X}^3, T^3; \vec{P}^4, \vec{X}^4, T^4)$$

$$= P(\vec{P}^3, \vec{X}^3_T, ; \vec{P}^4, \vec{X}^4_T).$$

The time dependence disappears in $P(\vec{P}^3, \vec{X}^3_T, T^3; \vec{P}^4, \vec{X}^4_T, T^4)$.

Next, we make a connection of the present result with a standard scattering matrix where an asymptotic condition is satisfied and only the momenta are observed. In the ordinary scattering processes the initial time T_i is $-\infty$ and the final time is $+\infty$. The distance $|\vec{X}_i - \vec{X}_j|$ in the initial state is proportional to $|\vec{v}_i - \vec{v}_j| T_i$ and the distance $|\vec{X}_i - \vec{X}_j|$ in the final state is proportional to $|\vec{v}_i - \vec{v}_j| T_i$. They become large except $|\vec{v}_i - \vec{v}_j| = 0$. When we define this case from a limit $|\vec{v}_i - \vec{v}_j| \to 0$ where a large T limit is taken first, a distance between two wave packets becomes infinity and the wave packets at $T \to \pm\infty$ do not overlap each others. The theory thus defined satisfies asymptotic condition.

By integrating the coordinates, we have the momentum dependent differential probability,

$$P(\vec{P}_3,;\vec{P}_4) = N\exp(-\frac{\sigma}{4}(\sum_i \vec{P}_i)^2 - \frac{4\sigma}{\sum_{i,j}(\vec{v}_i - \vec{v}_j)^2}(\sum E_i s_i)^2) \qquad (4{\cdot}36)$$

$$N = \frac{1}{2}(4\sigma\pi)^3 2^{-3/2}\exp(-\frac{3}{8\sigma}(\vec{X}_1 - \vec{v}_1 T_1 - \vec{X}_2 + \vec{v}_2 T_2)^2). \qquad (4{\cdot}37)$$

In Eq.(4·36), the normalization factor N is a constant which does not depend on the final state and the momentum dependent probability is almost the same as the probability of the plane waves. By integrating momenta of the final states,we have the total probability.

If the initial state and the final state have different values of the σ, we use the formula given in the appendix. Let σ_o be the size for all final particles and the σ_i be the size for all initial particles, the probability is given as,

$$P(\vec{P}_3,;\vec{P}_4) = N\exp(-\sigma_s(\sum_i \vec{P}_i)^2 - \sigma_t(\sum E_i s_i)^2) \qquad (4{\cdot}38)$$

$$N = \frac{1}{2}(4\pi)^3(\frac{\sigma_i\sigma_o}{2})^{3/2}\exp(-\frac{3}{8\sigma_i}(\vec{X}_1 - \vec{v}_1 T_1 - \vec{X}_2 + \vec{v}_2 T_2)^2), \qquad (4{\cdot}39)$$

$$\sigma_s = \frac{1}{2}(\frac{1}{\sigma_o} + \frac{1}{\sigma_i})^{-1}, \qquad (4{\cdot}40)$$

$$\sigma_t = (\Sigma_j \frac{\vec{v}_j^2}{\sigma_j} - \frac{\vec{v}_0^2}{\sigma_s})^{-1}. \qquad (4{\cdot}41)$$

4.3. *Long distance scatterings:first order*

When one of the times, T_1, is in a position far away from other times $T_l(l \neq 1)$, and the classical trajectories meet at around a time near $T_l(l \neq 1)$ and coordinate $\vec{X}_l(l \neq 1)$, the dominant contribution in the integration comes from the region near $\vec{X}_l(l \neq 1)$ and the position $\vec{X}_l - \vec{v}_l T_l$. One of the time difference $t - T_l$ becomes large and asymptotic expansion for the corresponding $C(\vec{P}_1, \vec{X}_1, T_1|t, \vec{x})$ is used. The transition matrix element becomes,then,

$$\begin{aligned}
&\langle 0|\Pi_{l=1}^2 A(\vec{P}_l, \vec{X}_l, T_l)\int dt' \frac{H_{int}(t')}{i}\Pi_{m=1}^2 A^\dagger(\vec{P}_m, \vec{X}_m, T_m)|0\rangle \\
&= \lambda\int dt d\vec{x}\,\Pi_l C(\vec{P}_l, \vec{X}_l, T_l|\vec{x}, t)\Pi_m C(\vec{P}_m, \vec{X}_m, T_m|\vec{x}, t)^* \\
&= \lambda(\frac{1}{2E(\vec{P}_1)2E(\vec{P}_2)2E(\vec{P}_3)2E(\vec{P}_4)})^{1/2}\int dt d\vec{x} N_{asym}^* e^{iE(\vec{P}_{(\vec{X}_1-\vec{x})})(t-T_1) - i\vec{P}_1(\vec{x}-\vec{X}_1)} \\
&N_3{}^* e^{-\frac{1}{2\sigma}(\vec{x}-\vec{X}_2-\vec{v}_2(t-T_2))^2 + iE(\vec{P}_2)(t-T_2) - i\vec{P}_2(\vec{x}-\vec{X}_2)} \\
&\times\Pi_{j=3,4} N_3 e^{-\frac{1}{2\sigma}(\vec{x}-\vec{X}_j-\vec{v}_j(t-T_j))^2 - iE(\vec{P}_j)(t-T_j) + \vec{P}_j(\vec{x}-\vec{X}_j)}
\end{aligned} \qquad (4{\cdot}42)$$

where N_{asym} and the stationary momentum $\vec{P}_{\vec{X}_1-\vec{x}}$ are

$$N_{asym} = N_3(\frac{1}{\frac{2i\gamma_L}{\sigma}+1})^{1/2}(\frac{1}{\frac{2i\gamma_T}{\sigma}+1})e^{-\frac{1}{2}\sigma(\vec{P}_X - \vec{P}_1)^2} \qquad (4{\cdot}43)$$

$$\vec{P}_{\vec{X}_1-\vec{x}} = (\vec{X}_1-\vec{x})\frac{m}{((T_1-t)^2-(\vec{X}_1-\vec{x})^2)^{1/2}}$$

Substituting these expressions, we have

$$\lambda\int dtd\vec{x}\Pi_l C(\vec{P}_l,\vec{X}_l,T_l|\vec{x},t)\Pi_m C(\vec{P}_m,\vec{X}_m,T_m|\vec{x},t)^*$$
$$= \lambda(\frac{1}{2E(\vec{P}_1)2E(\vec{P}_2)2E(\vec{P}_3)2E(\vec{P}_4)})^{1/2}(|N_3|^2)^2(\frac{1}{\frac{2i\gamma_L}{\sigma}+1})^{1/2}(\frac{1}{\frac{2i\gamma_T}{\sigma}+1})$$
$$\int dtd\vec{x}e^{im\sqrt{(t-T_1)^2-(\vec{x}-\vec{X}_1)^2}}\exp(-\frac{1}{2}\sigma\frac{(E(\vec{P}_0))^2}{(t-T_1)^2}(\vec{x}-\vec{x}_0)_T{}^2-\frac{1}{2}\sigma\frac{(E(\vec{P}_1))^6}{m^4(t-T_1)^2}(\vec{x}-\vec{x}_0)_L{}^2)$$
$$\tilde{N}e^{i\tilde{\phi}}\exp(-\frac{1}{2\sigma_s}(\vec{x}-\vec{x}_0')^2-\frac{1}{2\sigma_t}(t-t_0')^2), \quad (4\cdot44)$$

where the center position $\vec{x}_0$ is given by

$$\vec{x}_0 = \vec{X}_1+(t-T_1)\frac{\vec{P}_1}{E(\vec{P}_1)}, \quad (4\cdot45)$$

and the normalization , the phase, the variances, and center positions are defined by diagonalizing the products of wave packets,

$$\tilde{N}e^{\tilde{\phi}}\exp(-\frac{1}{2\sigma_s}(\vec{x}-\vec{x}_0')^2-\frac{1}{2\sigma_t}(t-t_0')^2) = N_3{}^*e^{-\frac{1}{2\sigma}(\vec{x}-\vec{X}_2-\vec{v}_2(t-T_2))^2+iE(\vec{P}_2)(t-T_2)-i\vec{P}_2(\vec{x}-\vec{X}_2)}$$
$$\times\Pi_{j=3,4}N_3e^{-\frac{1}{2\sigma}(\vec{x}-\vec{X}_j-\vec{v}_j(t-T_j))^2-iE(\vec{P}_j)(t-T_j)+\vec{P}_j(\vec{x}-\vec{X}_j)}. \quad (4\cdot46)$$

If the variances σ_s and σ_t are small values, the amplitude is further written as,

$$\lambda(\frac{1}{2E(\vec{P}_1)2E(\vec{P}_2)2E(\vec{P}_3)2E(\vec{P}_4)})^{1/2}(|N_3|^2)^2(\frac{1}{\frac{2i\gamma_L}{\sigma}+1})^{1/2}(\frac{1}{\frac{2i\gamma_T}{\sigma}+1})$$
$$e^{im\sqrt{(t_0'-T_1)^2-(\vec{x}_0'-\vec{X}_1)^2}}\exp(-\frac{1}{2}\sigma\frac{(E(\vec{P}_0))^2}{(t_0'-T_1)^2}(\vec{x}_0'-\vec{x}_0)_T{}^2-\frac{1}{2}\sigma\frac{(E(\vec{P}_1))^6}{m^4(t_0'-T_1)^2}(\vec{x}_0'-\vec{x}_0)_L{}^2)$$
$$\tilde{N}e^{i\tilde{\phi}}(2\sigma_s\pi)^{\frac{3}{2}}(2\sigma_t)^{\frac{1}{2}}. \quad (4\cdot47)$$

This expression of the amplitude shows that the wave packet expands and has the phase factor which is proportional to the square root of the proper time.

When we integrate on the variables $\vec{x},t$ first in Eq.(4·42), we have,

$$\lambda\int dtd\vec{x}\Pi_l C(\vec{P}_l,\vec{X}_l,T_l|\vec{x},t)\Pi_m C(\vec{P}_m,\vec{X}_m,T_m|\vec{x},t)^* \quad (4\cdot48)$$
$$= (|N_3|^2)^2(\frac{2\sigma\pi}{3})^{4/2}(\frac{1}{<(\vec{v})^2>-<\vec{v}>^2})^{1/2}\tilde{N}$$
$$\int d\vec{p}e^{-iE(\vec{p})(T_1-\delta T_1)+i\vec{p}(\vec{X}_1+\delta\vec{X}_1)-\frac{\sigma'}{2}(\vec{p}-\vec{P}_1-\delta\vec{P}_1)^2},$$
$$<\vec{v}> = \frac{1}{3}\sum_{j=1,3}\vec{v}_j, \quad (4\cdot49)$$
$$<\vec{v}^2> = \frac{1}{3}\sum_{j=1,3}(\vec{v}_j)^2, \quad (4\cdot50)$$

where δT_1, δX_1, and $\delta \vec{P}_1$ are of the order $O(1)$. $\tilde{N}$ is a normalization factor which depends upon kinematical variables. The integration on the variable $\vec{p}$ is carried with a use of the stationary phase approximation as in the previous cases.

Finally we have the amplitude,

$$\lambda \int dt d\vec{x} \Pi_l C(\vec{P}_l, \vec{X}_l, T_l|\vec{x}, t) \Pi_m C(\vec{P}_m, \vec{X}_m, T_m|\vec{x}, t)^* \\ = (|N_3|^2)^2 (\frac{2\sigma\pi}{3})^{4/2} (\frac{1}{< (\vec{v})^2 > - < \vec{v} >^2})^{1/2} \tilde{N} \\ e^{-im\sqrt{(T_1-\delta T_1)^2-(\vec{X}_1+\delta\vec{X}_1)^2} - \frac{1}{2}\sigma \frac{(E(\vec{P}_1))^2}{(T_1-\delta T_1)^2}(\vec{X}_1-\delta\vec{X}_1)_T{}^2 - \frac{1}{2}\sigma \frac{(E(\vec{P}_1))^6}{m^4(T_1-\delta T_1)^2}(\vec{X}_1-\delta\vec{X}_1)_L{}^2}. \tag{4·51}$$

Thus the amplitude depends on the large variables $\vec{X}_1$ and T_1 in a simple form. The normalization factor is inversely proportional to T_1 and the phase factor is proportional to the mass and the proper time $m\sqrt{c^2(T_1)^2 - (\vec{X}_1)^2}$.

4.4. *Long distance scattering :second order*

Next we study the few body scattering amplitude in the second order of interaction where there is one propagator $D(t, \vec{x})$. The propagator connects two interaction points,$(\vec{x}_1, t_1)$ and $(\vec{x}_2, t_2)$,

$$\langle 0|\Pi_{l=1}^3 A(\vec{P}_l, \vec{X}_l, T_l) \int dt_1 dt_2 \frac{T(H_{int}(t_1)H_{int}(t_2))}{i^2} \Pi_{m=1}^3 A^\dagger(\vec{P}_m, \vec{X}_m, T_m)|0\rangle \\ = \lambda^2 \int dt_1 d\vec{x}_1 \int dt_2 d\vec{x}_2 V_g(t_1, \vec{x}_1, \vec{P}_l, \cdots) D(t_1 - t_2, \vec{x}_1 - \vec{x}_2) V_g(t_2, \vec{x}_2, \vec{P}_2, \cdots)^*, \\ V_g(t_1, \vec{x}_1, \vec{P}_l, \cdots) = \Pi_l C(\vec{P}_l, \vec{X}_l, T_l|\vec{x}_1, t_1), \\ V_g(t_2, \vec{x}_2, \vec{P}_2, \cdots)^* = \Pi_m C(\vec{P}_m, \vec{X}_m, T_m|\vec{x}_2, t_2)^*. \tag{4·52}$$

The propagator $D(t_1 - t_2, \vec{x}_1 - \vec{x}_2)$ is given by,

$$D(t_1 - t_2, \vec{x}_1 - \vec{x}_2) = i \int \frac{d^3p}{(2\pi)^3 2E(\vec{p})} e^{ip(x_1-x_2)}|_{E(\vec{p})=\sqrt{\vec{p}^2+m^2}}. \tag{4·53}$$

We study the configuration when times $T_l(l = 1, 3)$ are close each others and times $T_m(m = 4, 6)$ are close each others but the first group of times is separated from the second group of times with a large distance. Regions when the time variable t_1 is near $T_l(l = 1, 3)$ and the other time variable t_2 is near $T_m(m = 4, 6)$ or the opposite give the dominant contribution to the amplitude in the time integration. Since the distance $|t_1 - t_2|$ is large, on mass shell kinematical region where $p^2 = m^2$ is satisfied is dominant in the momentum integration. Using the stationary phase approximation in the momentum integration, we replace the propagator with the asymptotic form that is obtained at the stationary momentum $\vec{p}_x$

$$D_{asym}(t_1 - t_2, \vec{x}_1 - \vec{x}_2) = iN_x \frac{1}{(2\pi)^3 2E(\vec{p}_x)} e^{ip_x(x_1-x_2)}|_{E(\vec{p}_x)} \tag{4·54}$$

$$N_x = (\frac{1}{i\gamma_L})^{1/2} (\frac{1}{i\gamma_T}), \tag{4·55}$$

$$\gamma_T = \frac{1}{2}\frac{(t_1-t_2)}{E(\vec{p}_x)}, \gamma_L = \gamma_T \frac{m^2}{E(\vec{p}_x)^2}, \tag{4·56}$$

where the momentum $\vec{p}_x$ is given as,

$$\vec{p}_{\vec{x}} = (\vec{x}_1 - \vec{x}_2)\frac{m}{((t_1-t_2)^2 - (\vec{x}_1-\vec{x}_2)^2)^{1/2}}. \tag{4·57}$$

We substitute these expressions into the amplitude and we have,

$$\begin{aligned} &\lambda^2 \int dt_1 d\vec{x}_1 dt_2 d\vec{x}_2 V_g(t_1,\vec{x}_1,\vec{P}_l,)D(t_1-t_2,\vec{x}_1-\vec{x}_2)V_g(t_2,\vec{x}_2,\vec{P}_m,\vec{X}_m,T_m)^* \\ &= \lambda \int dt_1 d\vec{x}_1 V_g(t_1,\vec{x}_1,\vec{P}_l,\vec{X}_l,T_l)e^{ip_x x_1}|_{E(\vec{p}_x)} iN_x(\frac{1}{(2\pi)^3 2E(\vec{p}_x)})^{1/2} \\ &\lambda \int dt_2 d\vec{x}_2 V_g(t_2,\vec{x}_2,\vec{P}_m,\vec{X}_m,T_m)^* e^{-ip_x x_2}|_{E(\vec{p}_x)}(\frac{1}{(2\pi)^3 2E(\vec{p}_x)})^{1/2}. \end{aligned} \tag{4·58}$$

In integrating $(t_i,\vec{x}_i), i=1,2$ the momentum $\vec{p}_x$ of Eq.(4·57) becomes a constant vector, if these variables are in the narrow regions. Then the total amplitude is proportional to the product of the two amplitudes and is inversely proportional to the spreading of the wave packet γ_T. The total probability is proportional to the product of probabilities of two processes and is inversely proportional to ${\gamma_T}^2$. Interference effect of propagating wave is negligible in this case. Thus in the present regime the intermediate state is treated as an observed particle. Hence the single particle treatment of the intermediate state is applicable. Using the argument of Ref.[7] and,[6] the effect of the wave packets are described by the ensemble of the energy eigenstates in this regime.

In an opposite situation where these integration variables cover wide regions and $\vec{p}_x$ is not a constant vector but varies with these variables $(t_1,\vec{x}_1)$ or $(t_2,\vec{x}_2)$, the naive single particle treatment is not justified. The total amplitude becomes linear combinations of the amplitudes of the various values of the momentum, $\vec{p}_x$, and become different from the product of two amplitudes. We split the integration regions into small regions V_l and obtain the momentum $\vec{p}_x^{ll'}$ defined from a pair of these regions. Using them we have the amplitude,

$$\begin{aligned} &\lambda^2 \int dt_1 d\vec{x}_1 dt_2 d\vec{x}_2 V_g(t_1,\vec{x}_1,\vec{P}_l,)D(t_1-t_2,\vec{x}_1-\vec{x}_2)V_g(t_2,\vec{x}_2,\vec{P}_m,\vec{X}_m,T_m)^* \\ &= \sum_{ll'} \lambda \int_{V_l} dt_1 d\vec{x}_1 V_g(t_1,\vec{x}_1,\vec{P}_l,\vec{X}_l,T_l)e^{ip_x^{ll'} x_1}|_{E(\vec{p}_x^{ll'})} iN_x(\frac{1}{(2\pi)^3 2E(\vec{p}_x^{ll'})})^{1/2} \\ &\lambda \int_{V_{l'}} dt_2 d\vec{x}_2 V_g(t_2,\vec{x}_2,\vec{P}_m,\vec{X}_m,T_m)^* e^{-ip_x^{ll'} x_2}|_{E(\vec{p}_x^{ll'})}(\frac{1}{(2\pi)^3 2E(\vec{p}_x^{ll'})})^{1/2}. \end{aligned} \tag{4·59}$$

The total amplitude is a linear combination of the amplitudes of different momentum $\vec{p}_x^{ll'}$. Probability may show the interference of the different intermediate momentum.

In this situation we are able to write the amplitude in a different manner. Let write the propagator as,

$$D(t_1-t_2,\vec{x}_1-\vec{x}_2) = -2i\int d^3x D(t_1-t,\vec{x}_1-\vec{x})\dot{D}(t-t_2,\vec{x}-\vec{x}_2). \tag{4·60}$$

We substitute this expression and we have the amplitude,

$$\lambda^2 \int dt_1 d\vec{x}_1 dt_2 d\vec{x}_2 V_g(t_1,\vec{x}_1,\vec{P}_l,)D(t_1-t_2,\vec{x}_1-\vec{x}_2)V_g(t_2,\vec{x}_2,\vec{P}_m,\vec{X}_m,T_m)^*$$
$$= -2i\lambda^2 \int d\vec{x} \int dt_1 d\vec{x}_1 V_g(t_1,\vec{x}_1,\vec{P}_l,\vec{X}_l,T_l)D(t_1-t,\vec{x}_1-\vec{x}) \quad (4\cdot61)$$
$$\times \int dt_2 d\vec{x}_2 \dot{D}(t-t_2,\vec{x}-\vec{x}_2)V_g(t_2,\vec{x}_2,\vec{P}_m,\vec{X}_m,T_m)^*.$$

Using the stationary phase approximation in the momentum integration, we have the propagators,

$$D_{asym}(t_1-t,\vec{x}_1-\vec{x}) = iN_x^{(1)}\frac{1}{(2\pi)^3 2E(\vec{p}_x^{(1)})}e^{ip_x^{(1)}(x_1-x)}|_{E(\vec{p}_x^{(1)})} \quad (4\cdot62)$$
$$N_x^{(1)} = (\frac{1}{i\gamma_L^{(1)}})^{1/2}(\frac{1}{i\gamma_T^{(1)}}), \quad (4\cdot63)$$
$$\gamma_T^{(1)} = \frac{1}{2}\frac{(t_1-t)}{E(\vec{p}_x)}, \gamma_L^{(1)} = \gamma_T^{(1)}\frac{m^2}{E(\vec{P}_x^{(1)})^2} \quad (4\cdot64)$$
$$\vec{P}_{\vec{x}}^{(1)} = (\vec{x}_1-\vec{x})\frac{m}{((t_1-t)^2-(\vec{x}_1-\vec{x})^2)^{1/2}} \quad (4\cdot65)$$

and

$$\dot{D}_{asym}(t-t_2,\vec{x}-\vec{x}_2) = i^2 N_x^{(2)}\frac{1}{(2\pi)^3 2}e^{ip_x^{(2)}(x-x_2)}|_{E(\vec{p}_x^{(2)})} \quad (4\cdot66)$$
$$N_x^{(2)} = (\frac{1}{i\gamma_L^{(2)}})^{1/2}(\frac{1}{i\gamma_T^{(2)}}), \quad (4\cdot67)$$
$$\gamma_T^{(2)} = \frac{1}{2}\frac{(t-t_2)}{E(\vec{p}_X^{(2)})}, \gamma_L^{(2)} = \gamma_T^{(2)}\frac{m^2}{E(\vec{P}_X)^2} \quad (4\cdot68)$$
$$\vec{P}_{\vec{x}}^{(2)} = (\vec{x}-\vec{x}_2)\frac{m}{((t-t_2)^2-(\vec{x}-\vec{x}_2)^2)^{1/2}}. \quad (4\cdot69)$$

The amplitude becomes,

$$-2i\lambda^2 \int d\vec{x} T_1(\vec{X}_l,\vec{P}_l,\cdots,;\vec{x},t)N_x^{(1)}N_x^{(2)}T_2(\vec{X}_m,\vec{P}_m,\cdots,;\vec{x},t), \quad (4\cdot70)$$
$$T_1(\vec{X}_l,\vec{P}_l,\cdots,;\vec{x},t) = \frac{1}{2(2\pi)^3} \quad (4\cdot71)$$
$$\times \int dt_1 d\vec{x}_1 V_g(t_1,\vec{x}_1,\vec{P}_l,\vec{X}_l,T_l)\frac{1}{E(\vec{p}_x^{(1)})}e^{ip_x^{(1)}(x_1-x)}|_{E(\vec{p}_x^{(1)})},$$
$$T_2(\vec{X}_m,\vec{P}_m,\cdots,;\vec{x},t)^* = \frac{1}{2(2\pi)^3} \quad (4\cdot72)$$
$$\times \int dt_2 d\vec{x}_2 e^{ip_x^{(2)}(x-x_2)}|_{E(\vec{p}_x^{(2)})}V_g(t_2,\vec{x}_2,\vec{P}_m,\vec{X}_m,T_m)^*.$$

This amplitude agrees with the previous form if the momentum $\vec{p}_x^{(1)}$ and $\vec{p}_x^{(2)}$ are regarded as constant vectors. If these momenta are not constant vectors, the $\vec{x}$ dependence and $\vec{x}_j, j = 1,2$ dependence of $\vec{p}_x^{(j)}, j = 1,2$ are taken explicitly in the time and coordinate integrations.

4.5. *Factorization*

Factorization is a general feature of the amplitudes obtained in the previous sections. Namely the amplitudes are factorized into amplitudes of sub-processes that depend on close space and time coordinates $\vec{X}_{i_1}, T_{i_1}$ where $|T_{i_1} - T_{j_1}| \approx 0$ and $\vec{X}_{i_2}, T_{i_2}$ where $|T_{i_2} - T_{j_2}| \approx 0$ and $|T_{i_1} - T_{i_2}| \approx \infty$. In these short distance amplitudes, Gaussian momentum integration around minimal are applied and amplitudes become almost equivalent to ordinary scattering amplitudes. On the other hand, the long distance parts have particular forms that are proportional to the inverses of time difference $|T_{i_1} - T_{i_2}|$ in the small mass case and to the phases $\exp(im\sqrt{(T_{i_1} - T_{i_2})^2 - (\vec{X}_{i_1} - \vec{X}_{i_2})^2})$, where T_{i_1} and $\vec{X}_{i_1}$ are average values of times and positions in a group 1 and T_{i_2} and $\vec{X}_{i_2}$ are average values of times and positions in a group 2. The former behavior is due to the expansion of the wave packets. It is possible to decompose amplitudes in general many body amplitudes in which space time coordinates are separated into many groups. Each amplitude for the process of close coordinates is almost equivalent to ordinary scattering amplitude and the amplitudes for the long distance parts have the particular normalization and the phase factor of the above forms.

§5. Summary

We have defined the generalized scattering amplitudes which have dependence upon particle's positions in addition to the particle's momenta. Idealistic cases where the positions and the momenta satisfy minimum uncertainty relations are studied by the use of minimum wave packets, coherent states.

Since wave packets are linear combinations of eigenfunctions of free Hamiltonian, wave packets change with time. Wave packets move with a constant group velocity and expand. These behaviors occur since each wave of definite momentum has a different velocity. They reveal a particle's nature and a wave nature of wave packets. Expansion is slow and has been irrelevant to any observations in high energy experiments till recently. They are relevant in some long distance experiments and its effects are analyzed in the present work. We found also that the expansion speeds satisfy new uncertainty relations expressed in Eq.(2·55) and Eq.(2·56).

Several relations which must be satisfied for the transition amplitudes and probabilities are proven. Completeness of the mixed representation is proven and is used for defining the weight of phase space integral for both variables of momenta and coordinates. The particle states which are specified by momenta and positions are normalized to unity and Dirac delta function is unnecessary for the normalization of states in mixed representation since the states are normalized but are not extended in space. The whole transition probability from one state to states of a fixed particle number becomes an intrinsic value which is independent from the wave packet sizes and the total transition probability from one state to all possible states becomes unity under the use of the present measure of phase space even though states of different momenta and positions are nonorthogonal. So probability interpretation holds.

Several examples in few body scatterings are analyzed and amplitudes are explicitly computed in the lowest order and the second order of the interaction Hamiltonian. Translational motions and expansions of wave packets are taken into account explicitly and their effects are seen in the manifest manner. It is shown that scattering amplitudes which have long distance part in addition to short distance part are factorized. The asymptotic condition for the ordinary scattering, which is satisfied by the addition of $i\epsilon$ in propagators in the standard s-matrix, is realized automatically in the present formalism by taking a suitable limit of the present amplitude.

Applications to neutrino long distance experiments,[10)13)] and others will be given in separate works.

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Appendix A
—— *Generalized vertices of arbitrary wave packets* ——

The product of the wave functions at $(t,\vec{x})$ are the Gaussian function of the space time coordinates $(t,\vec{x})$,

$$\begin{aligned}&\Pi_j N_j^* e^{-\frac{1}{2\sigma_j}(\vec{x}-\vec{X}_j-\vec{v}_j(t-T_j))^2+iE(\vec{p}_j)(t-T_j)-i\vec{p}_j(\vec{x}-\vec{X}_j)}\\ &\times \Pi_l N_l e^{-\frac{1}{2\sigma_l}(\vec{x}-\vec{X}_l-\vec{v}_l(t-T_l))^2-iE(\vec{p}_l)(t-T_l)+i\vec{p}_l(\vec{x}-\vec{X}_l)}\\ &= \Pi_j N_j^* \Pi_l N_l e^{-\frac{1}{2\sigma_s}(\vec{x}-\vec{x}_0(t))^2-\frac{1}{2\sigma_t}(t-t_0)^2} e^{R+i\phi}.\end{aligned} \tag{A·1}$$

Wave packet parameters in the spatial directions and the temporal direction are

$$\frac{1}{\sigma_s} = \Sigma_j \frac{1}{\sigma_j} \tag{A·2}$$

$$\frac{1}{\sigma_t} = \Sigma_j \frac{1}{\sigma_j}\vec{v}_j^2 - \frac{1}{\sigma_s}\vec{v}_0^2 \tag{A·3}$$

and the central values of the space-time coordinates are

$$\vec{x}_0(t) = \vec{v}_0 t + \vec{x}_0(0), \tag{A·4}$$

$$\vec{v}_0 = \sigma_s \Sigma_j \frac{1}{\sigma_j}\vec{v}_j, \tag{A·5}$$

$$\vec{x}_0(0) = \sigma_s(\Sigma_j \frac{1}{\sigma_j}\tilde{\vec{X}}_j - i(\Sigma_j(\pm)\vec{p}_j)) \tag{A·6}$$

$$t_0 = \sigma_t(\frac{1}{\sigma_s}\vec{v}_0 \cdot \vec{x}_0 - \Sigma_j \frac{1}{\sigma_j}\vec{v}_j \cdot \tilde{\vec{X}}_j + i\Sigma_j(\pm)E(\vec{p}_j)) \tag{A·7}$$

$$\tilde{\vec{X}}_j = \vec{X}_j - \vec{v}_j T_j. \tag{A·8}$$

The real part determines the magnitude of the amplitude and is composed of the trajectory terms and the energy-momentum terms. The former give constraints on the particle trajectories and the latter give constraints the total energy and total momentum and are determined as,

$$R = R_{trajectory} + R_{momentum}, \tag{A·9}$$

$$R_{trajectory} = -\Sigma_j \frac{1}{2\sigma_j}\tilde{\vec{X}}_j^{\,2} + 2\sigma_s(\Sigma_j \frac{1}{2\sigma_j}\tilde{\vec{X}}_j)^2 + 2\sigma_t(\Sigma_j(\vec{v}_0 - \vec{v}_j)\tilde{\vec{X}}_j)^2, \tag{A·10}$$

$$R_{momentum} = -\frac{\sigma_t}{2}(\Sigma_j(\pm)(E(\vec{p}_j) - \vec{v}_0\vec{p}_j))^2 - \frac{\sigma_s}{2}(\Sigma_j(\pm)\vec{p}_j)^2. \tag{A·11}$$

The phase factor is composed of the primary term which expresses the energy momentum dependent phase and the secondary terms which are due to finite sizes of wave packets and are determined as,

$$\phi = \phi_0 + \phi_1, \tag{A·12}$$

$$\phi_0 = \Sigma_j(\pm)(\vec{p}_j\vec{X}_j - E(\vec{P}_j)T_j), \tag{A·13}$$

$$\phi_1 = -2\sigma_t(\Sigma_j \frac{1}{2\sigma_j}(\vec{v}_0 - \vec{v}_j)\tilde{\vec{X}}_j)(\Sigma(\pm)\vec{v}_0(\vec{P}_j - E(\vec{p}_j)))$$
$$-2\sigma_s(\Sigma_j(\pm)\vec{p}_j)(\Sigma_j \frac{1}{2\sigma_j}\tilde{\vec{X}}_j). \tag{A·14}$$

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11) It should be noted that the following completeness is satisfied for general wave functions which are localized and have the same phase factors. So, generalized S-matrix can be formulated in wide systems where uncertainty relations between variances of momenta and coordinates are different from the minimum uncertainty.
12) In non-relativistic theory, one particle energy is given as $E(\vec{p}) = \frac{\vec{p}^2}{2m}$ and the stationary phase condition $\frac{\partial}{\partial p_i}(E(\vec{p})t - \vec{p} \cdot \vec{x}) = 0$ is solved easily. The momentum and the phase at the stationary momentum are given as $m\frac{x_i}{t}, -\frac{m}{2}(\frac{\vec{x}}{t})^2$. The phase is not canceled.
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