Lorentz-violating brane worlds and cosmological perturbations

M.V. Libanov and V.A. Rubakov[1]

[1] *Institute for Nuclear Research of the Russian Academy of Science,*
60th October Anniversary prospect 7a, Moscow 117312, Russia

Abstract

We consider an inflating brane-world setup in which 4-dimensional Lorentz-invariance is violated at high 3-momentum scale $P_{LV} \gg H$, where H is the inflationary Hubble parameter. We study massless scalar field in this background as a model for cosmological perturbations. Towards the end of inflation, the spectrum has both the standard, 4-dimensional part due to a brane-localized mode, and exotic, bulk induced contribution. The suppression of the latter is power-law only, $(H/P_{LV})^{\alpha}$, provided that there exist bulk modes with energies $\omega \ll H$. Contrary to general expectations, the exponent α may be smaller than 2, and even smaller than 1, depending on details of the bulk geometry. Furthermore, the overall amplitude of the bulk-induced perturbations is enhanced as compared to the standard part, so the effects due to Lorentz-violation may dominate over the standard mechanism even for $P_{LV} \gg H$.

I. INTRODUCTION AND SUMMARY

In brane-world scenarios, it is not inconceivable that four-dimensional Lorentz-invariance is violated in the bulk [1, 2, 3, 4]. Models of this sort provide an interesting framework [5] for addressing the cosmological "trans-Planckian" problem [6, 7, 8] of whether or not possible Lorentz-violation at high 3-momenta may affect the predictions for cosmological perturbations generated at inflation. A conservative possibility is that 3-momentum scale of Lorentz-violation, P_{LV}, is much higher than the Hubble parameter towards the end of inflation, H. It has been argued [8] that for $P_{LV} \gg H$, the standard inflationary predictions should remain almost intact, although there are fairly exotic four-dimensional counterexamples [6, 9] based, e.g., on Corley–Jacobson dispersion relation [10]. Barring exotica, there is still some debate [8, 11] on whether the effects of Lorentz-violation, and "heavy physics" in general, are at best of order $(H/P_{LV})^2$, or weaker suppression is possible.

In our previous discussion [5] of the effects due to Lorentz-violation on cosmological perturbations in the brane-world framework, we introduced an inflating version of the setup of Ref. [3] with Lorentz-violating bulk. A drawback of this approach is that the bulk geometry is completely *ad hoc*; neither the source of Lorentz-violation nor mechanism of inflation are specified. An advantage, however, is that the behavior of quantum fields, in particular, their initial state, are well under control, so the calculation of the spectrum is unambiguous. The particular setup used in Ref. [5] was not interesting at $P_{LV} \gg H$ — the corrections to the standard predictions were suppressed as $\exp\left(-\text{const} \cdot \frac{P_{LV}}{H}\right)$ — so we concentrated there on strong effects occurring in a less conservative case, $P_{LV} \lesssim H$. In this paper we modify our setup in such a way that the effects due to Lorentz-violating bulk are only power-law suppressed at $P_{LV} \gg H$. Needless to say, the above remarks concerning the entire approach apply to this work as well.

Our main findings are somewhat unexpected. First, the suppression of the bulk-induced contribution, which is generated on top of the standard spectrum, strongly depends on the bulk geometry, and may be weaker than $(H/P_{LV})^2$ and even H/P_{LV}, in obvious contradiction to the claim of Ref. [8]. Second, this contribution to the spectrum is enhanced by ϵ^{-3} where ϵ is another small free parameter inherent in our model. Thus, for small enough ϵ the effects due to Lorentz-violating bulk may compete with, and even dominate over the standard four-dimensional mechanism in spite of the hierarchy $P_{LV} \gg H$. There is a simple reason for

the enhancement: Lorentz-violating bulk modes exit the cosmological horizon earlier, and hence get frozen at higher amplitudes, as compared to brane-localized ones. Relatively large perturbations in the bulk are then partially transferred to the brane due to subsequent (but still occurring at inflationary stage) mixing between bulk and brane modes.

In our model, the spectrum of additional, bulk-induced perturbations is flat for $H = \text{const}$ (in accord with the scaling argument of Ref. [5]) and almost flat for inflationary Hubble parameter slowly varying in time. However, the amplitude and tilt are determined by the expansion rate at earlier stages of inflation, as compared to the standard theory. In the slow roll scenario this would mean that this part of perturbations has less tilted spectrum. A potentially observable property that the primordial perturbations are a sum of two Gaussian fields with different tilts (and amplitudes) appears to be an interesting feature of our model, and likely a whole class of models with Lorentz-violating bulk.

Our overall conclusion is that in Lorentz-violating brane-world models, the properties of cosmological perturbations generated at inflation may strongly depend on dynamics in the bulk, even if 3-momentum scale of Lorentz-violation on the brane largely exceeds the inflationary Hubble parameter. Needless to say, this dynamics may naturally be entirely different for, e.g., inflaton and graviton modes, so that the standard relations between the scalar and tensor perturbations may be completely (or partially) destroyed.

II. GENERALITIES

A. Background geometry

Let us consider $(4+1)$-dimensional model with the coordinates (t, x^i, y), $i = 1, 2, 3$. We choose the background five-dimensional metric as follows,

$$ds^2 = [\alpha^2(y)dt^2 - \beta^2(y)a^2(t)d\mathbf{x}^2] - \alpha^2(y)dy^2 \tag{1}$$

where the coordinate choice for y is made for convenience. There is a single brane at

$$y = y_B$$

The warp factors $\alpha(y)$ and $\beta(y)$ are continuous across the brane, while their derivatives $\partial_y \alpha \equiv \alpha'$ and $\partial_y \beta \equiv \beta'$ are not. The static background, $a(t) = \text{const}$, is not four-dimensinally

Lorentz-invariant; it is this case that has been considered in Refs. [2, 3], where is has been shown that, with appropriate choice of the warp factors, four-dimensional Lorenz-invariance still holds for brane modes. An equivalent form of the metric is

$$ds^2 = a^2(\eta)[\alpha^2(y)d\eta^2 - \beta^2(y)d\mathbf{x}^2] - \alpha^2(y)dy^2 \tag{2}$$

where the conformal time η is related to time t in the usual way,

$$dt = a(\eta)d\eta$$

We will use both forms of the metric in what follows.

We choose the warp factors in such a way that both $\alpha(y)$ and $\beta(y)$ are Z_2-symmetric across the brane, monotonically decrease towards large y with

$$\beta'' > 0 \tag{3}$$

and decay away from the brane,

$$\alpha(y), \beta(y) \to 0 , \quad \text{as } y \to \infty$$

so that the integrals in (4) are finite. For what follows it is convenient to rescale the coordinates x^i and t in such a way that

$$\int_{y_B}^{\infty} dy \; \alpha^2\beta = \int_{y_B}^{\infty} dy \; \beta^3 \tag{4}$$

Furthermore, we assume that the ratio of warp factors tends to a small constant,

$$\frac{\alpha(y)}{\beta(y)} \to \epsilon , \quad \text{as } y \to \infty \tag{5}$$

$$\epsilon \ll 1$$

The parameter ϵ is a free small parameter of our model. The static case with $\epsilon = 0$ was discussed in Ref. [3]. Here we will be interested in inflating background,

$$a(t) = \exp\left(\int H(t)dt\right) \tag{6}$$

where $H(t)$ is a slowly varying function of time.

Finally, we assume that $\beta(y)$ decays sufficiently slowly as $y \to \infty$, so that

$$\frac{\beta'(y)}{\beta(y)}, \frac{\beta''(y)}{\beta(y)} \to 0 , \quad \text{as } y \to \infty \tag{7}$$

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The latter assumption is the key property that differs our background from that considered in Ref. [5]; the peculiar features of bulk perturbations that we alluded to in Introduction can be traced back precisely to this property. In fact, the main results of this paper hold for milder requirements on the behavior of $\beta(y)$ as $y \to \infty$: it is sufficient to assume that $\left(\frac{\beta'(y)}{\beta(y)}\right)^2$ and $\frac{\beta''(y)}{\beta(y)}$ tend to constant values which are sufficiently small compared to H^2. In what follows we concentrate on the case (7) for definiteness.

To illustrate the dependence on the bulk geometry, we will take as a concrete example

$$\beta(y) = \frac{1}{y^\kappa}$$
$$\alpha^2(y) = \beta^2(y) \left(\frac{3\kappa + 1}{3\kappa - 1} \cdot \frac{y_B^2}{y^2} + \epsilon^2\right) \tag{8}$$

where

$$\kappa > 1/3 \tag{9}$$

The numerical factor in (8) is chosen in such a way that Eq. (4) is satisfied modulo unimportant ϵ^2-corrections. We will see that another small parameter of our model, y_B, determines the 3-momentum P_{LV} at which the effects of Lorentz-violating bulk are substantial for brane modes.

It is worth stressing that our setup is completely *ad hoc*; we are not aware of any realistic brane construction that would produce this geometry. As mentioned in Introduction, this is the major disadvantage of the entire approach.

B. Brane and bulk modes

In this paper we consider a real massless scalar field Φ in the background (1), meant to model perturbations of gravitational and/or inflaton field. The action is

$$S_\Phi = \frac{1}{2} \int d^5X \ \sqrt{g} g^{AB} \partial_A \Phi \partial_B \Phi$$

It is consistent with Z_2 symmetry across the brane to impose the Neumann boundary condition,

$$\Phi'(y = y_B) = 0$$

Indeed, this boundary condition applies to appropriately defined gravitational perturbations [12], and also to free scalar field [13] in the Randall–Sundrum background. It is

convenient to introduce the fields

$$\phi(t, x^i, y) = \beta^{3/2}(y) \cdot \Phi(X^A)$$

and

$$\chi(\eta, x^i, y) = a(\eta)\beta^{3/2}(y) \cdot \Phi(X^A)$$

which obey the field equations, in terms of three-dimensional Fourier harmonics,

$$\frac{\partial^2 \phi}{\partial t^2} + 3H\frac{\partial \phi}{\partial t} + U(y)P^2(t)\phi + L_y\phi = 0 \tag{10}$$

and

$$\ddot{\chi} - \frac{\ddot{a}}{a}\chi + U(y)k^2\chi + a^2(\eta)L_y\chi = 0 \tag{11}$$

Hereafter dot denotes $d/d\eta$, k is time-independent conformal 3-momentum, while

$$P(t) = \frac{k}{a(t)}$$

is physical 3-momentum,

$$U(y) = \frac{\alpha^2(y)}{\beta^2(y)} \tag{12}$$

and

$$L_y = -\frac{\partial^2}{\partial y^2} + V(y)$$

with

$$V(y) = \frac{3}{2}\frac{\beta''}{\beta} + \frac{3}{4}\frac{\beta'^2}{\beta^2}$$

The boundary conditions for ϕ and χ are

$$\left(\phi' - \frac{3}{2}\frac{\beta'}{\beta}\phi\right)_{y=y_B} = \left(\chi' - \frac{3}{2}\frac{\beta'}{\beta}\chi\right)_{y=y_B} = 0 \tag{13}$$

The field χ is canonically normalized: its action is

$$S_\chi = \int d\eta \, d^3x \, dy \left(\frac{1}{2}\dot{\chi}^2 + \dots\right)$$

where dots denote terms independent of $\dot{\chi}$.

Let us first discuss bulk modes, assuming for the time being that the background is static,

$$H = 0$$

According to our assumptions (3) and (7), the potential $V(y)$ is positive and vanishes as $y \to \infty$, so the spectrum of the operator L_y is continuous and starts from zero[1]. Recalling Eq. (5), we find that the dispersion relation for these modes is

$$\omega^2 = \epsilon^2 P^2 + \lambda^2 \tag{14}$$

where λ^2 are eigenvalues of L_y, which can be arbitrarily small.

Let us now consider brane modes, still in static background. At $P^2 = 0$ there is a zero mode,

$$\phi_0(y) = \beta^{3/2}$$

which is normalizable, since the integrals in (4) are assumed to be finite. It is worth noting that in terms of the original filed Φ this mode is constant along extra dimension. Now, at finite but small P, this mode gets lifted; the third term in Eq. (10) can be treated as perturbation, and for the (real part of) energy one finds

$$\omega^2 = P^2 \cdot \frac{\int_{y_B}^{\infty} dy \ U(y)|\phi_0|^2}{\int_{y_B}^{\infty} dy \ |\phi_0|^2}$$

Making use of Eqs. (4) and (12), one obtains the Lorentz-invariant dispersion relation,

$$\omega^2 = P^2$$

Thus, the theory on the brane is (almost) Lorentz-invariant at small 3-momenta.

This is not the whole story, however. At small but finite P, the would be zero mode is embedded in the continuum of bulk modes: its energy is larger than the lowest energy $\omega = \epsilon P$ of continuum modes, see Eq. (14). Therefore, the brane mode is *quasi*-localized even at low 3-momenta, i.e., it has finite width against escape into extra dimension. This effect was found in Ref. [3]. The quasi-localization is due to mixing with bulk modes, which comes from the third term in Eq. (10). Introducing the overlap between the (normalized) zero mode ϕ_0 and continuum modes ϕ_λ,

$$I_\lambda = \int \ dy U(y)\phi_0^*(y)\phi_\lambda(y) \tag{15}$$

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[1] In the model of Ref. [5], the spectrum of the operator analogous to L_y started from a large positive value (in the case of large Lorentz-violation scale). This difference is behind different results obtained in the present paper and in Ref. [5].

we estimate the width of the quasilocalized state as

$$\Gamma(P) \sim P^2 \cdot |I_{\lambda=P}|^2 \tag{16}$$

The whole picture of weak Lorentz-violation on the brane at low 3-momenta and strong Lorentz-violation at high 3-momenta works provided that the warp factors are chosen in such a way that

$$\lambda|I_\lambda|^2 \to 0 \,, \quad \text{as} \quad \frac{\lambda}{P_{LV}} \to 0 \tag{17}$$

and

$$\lambda|I_\lambda|^2 \sim 1 \,, \quad \text{at} \ \lambda \sim P_{LV}$$

It is this case that we consider in what follows. Then, the width of the brane mode is small compared to its energy at low P, but it becomes comparable to energy $\omega = P$ at $P \gtrsim P_{LV}$. At $P > P_{LV}$ even quasi-localized brane mode ceases to exist, so four-dimensional Lorentz-invariance is indeed completely destroyed.

Let us illustrate these features by making use of the concrete form (8) of the warp factors. In this case one has[2]

$$\begin{aligned}
V(y) &= \frac{1}{y^2} \cdot \left(\frac{9}{4}\kappa^2 + \frac{3}{2}\kappa \right) \\
U(y) &= \frac{3\kappa+1}{3\kappa-1} \cdot \frac{y_B^2}{y^2} + \epsilon^2
\end{aligned} \tag{18}$$

The appropriately normalized eigenfunctions of L_y are the zero mode

$$\phi_0(y) = C(\kappa) \cdot \frac{y_B^{\frac{3\kappa-1}{2}}}{y^{\frac{3\kappa}{2}}} \tag{19}$$

and bulk modes

$$\phi_\lambda(y) = \frac{1}{\sqrt{2}} \cdot \sqrt{\lambda y} \cdot \frac{J_{\nu-1}(\lambda y_B)Y_\nu(\lambda y) - Y_{\nu-1}(\lambda y_B)J_\nu(\lambda y)}{\sqrt{J_{\nu-1}(\lambda y_B)^2 + Y_{\nu-1}(\lambda y_B)^2}} \tag{20}$$

where J_ν and Y_ν are the Bessel functions and

$$\nu = \frac{3\kappa+1}{2}$$

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[2] The field equation in this case coincides, up to notations and field redefinition, with the equation considered in Ref. [3]. The results in the end of this section agree with Ref. [3] where they overlap.

Note that because of (9), one has

$$\nu > 1$$

In Eq. (19) and below $C(\kappa)$ denotes unimportant constants of order 1, which may be different in different formulas, but never are equal to zero or infinity, provided that κ obeys (9).

At $\lambda \ll y_B^{-1}$ the overlap integral (15) is equal to

$$I_\lambda = C(\kappa) y_B^{\nu-1} \cdot \lambda^{\nu - \frac{3}{2}}$$

so that

$$\lambda I_\lambda^2 \sim (y_B \lambda)^{2\nu - 2} \tag{21}$$

and the estimate (16) for the width of the quasi-localized mode reads

$$\Gamma = C(\kappa) P \cdot (P y_B)^{2\nu - 2}$$

The direct calculation of the width by techniques well known from quantum mechanics [14] confirms this estimate. At low 3-momenta this width is small compared to the real part of energy, $\omega = P$, although the suppression is mild at ν close to 1. On the other hand, the width becomes large and the quasi-localized mode disappears at

$$P \gtrsim P_{LV} = \frac{1}{y_B} \tag{22}$$

We see that in this particular setup, the position of the brane determines the length scale of Lorentz-violation for brane-based observer.

III. GENERATION OF BRANE-LOCALIZED PERTURBATIONS

A. Evolution of modes: zeroth order

Let us now consider the scalar field in inflating setup, Eq. (6). It is convenient to work with the field $\chi(\eta, y)$ and decompose it in eigenfunctions of the operator L_y,

$$\chi(\eta, y) = \psi_0(\eta) \phi_0(y) + \int_0^\infty d\lambda \psi_\lambda(\eta) \phi_\lambda(y)$$

From Eq. (11) one obtains the system of equations

$$\ddot{\psi}_0 - \frac{\ddot{a}}{a} \psi_0 + k^2 \psi_0 = -k^2 \int d\lambda \, \psi_\lambda I_\lambda \tag{23}$$

$$\ddot{\psi}_\lambda - \frac{\ddot{a}}{a} \psi_\lambda + \epsilon^2 k^2 \psi_\lambda + a^2 \lambda^2 \psi_\lambda = -k^2 I_\lambda^* \psi_0 - k^2 \int d\lambda' \, \psi_{\lambda'} I_{\lambda, \lambda'} \tag{24}$$

where

$$I_{\lambda\lambda'} = \int dy \, \phi_\lambda^*[U(y) - \epsilon^2]\phi_{\lambda'} \tag{25}$$

Our approach is as follows. We will be interested in the amplitude of the brane mode, ψ_0, towards the end of inflation. To this end, we will treat the right hand sides of Eqs. (23) and (24) as perturbations. This approximation is certainly *not* valid at very early times, when $P(\eta) \equiv k/a(\eta) \gtrsim P_{LV}$: at those times the very notion of the brane mode does not make sense. However, for $P_{LV} \gg H$ the relevant modes are in the adiabatic regime at those times, mixing between positive- and negative-frequency components of the field is negligible, and the field remains in its adiabatic vacuum state. This point (mode generation in the adiabatic regime) has been discussed in more detail in Ref. [5]. At later times, perturbation theory *is* justified by the fact that $U(y)$ peaks near the brane, where $\phi_\lambda(y)$ are suppressed; hence the overlap integrals (15) and (25) are small. In the zeroth approximation, with overlaps neglected, equations for the brane mode and bulk modes decouple and can be straighforwardly solved. At this level, equation for the brane mode is

$$\ddot{\psi}_0 - \frac{\ddot{a}}{a}\psi_0 + k^2\psi_0 = 0$$

It exactly coincides with the corresponding equation in four-dimensional theory, so the zeroth order result for the spectrum is exactly the same as in four dimensions. Namely, towards the end of inflation, $\psi_0^{(0)}(\mathbf{k}, \eta)$ (the superscript here refers to the zeroth order approximation) is a Gaussian field with the correlation function

$$\langle \psi_0^{(0)}(\mathbf{k}), \psi_0^{(0)}(\mathbf{k}') \rangle = a^2(\eta) \frac{2\pi^2}{k^3} \mathcal{P}^{(0)}(k)\delta^3(\mathbf{k} - \mathbf{k}')$$

with

$$\mathcal{P}^{(0)} = \frac{H_k^2}{4\pi^2}$$

and

$$H_k = H(\eta_k)$$

where η_k is the time at which the mode of momentum k crosses out the horizon,

$$H(\eta_k) = \frac{k}{a(\eta_k)}$$

Thus, four-dimensional behavior of the field in the brane mode is trivially obtained in the zeroth approximation.

The first non-trivial contribution to the field in the brane mode occurs at the first order of perturbation theory, and is due to the overlap with bulk modes in the right hand side of Eq. (23). To evaluate this contribution, we first have to solve Eq. (24) at the zeroth order.

At the zeroth order, Eq. (24) becomes

$$\ddot{\psi}_\lambda - \frac{\ddot{a}}{a}\psi_\lambda + \epsilon^2 k^2 \psi_\lambda + a^2\lambda^2\psi_\lambda = 0$$

In the asymptotic past, the second and fourth terms in this equation are negligible, the field is in the adiabatic regime, and we immediately write its decomposition in creation and annihilation operators,

$$\psi_\lambda^{(0)} = \frac{1}{\sqrt{2\epsilon k}}(\psi_\lambda^+(\eta)A_{\lambda,k}^+ + \text{h.c.}) \tag{26}$$

where in the asymptotic past

$$\psi_\lambda^+ = e^{i\epsilon k\eta}, \quad \eta \to -\infty$$

As we will see in the next subsection, the main effect on the brane mode comes from the bulk modes with

$$\lambda \ll H \tag{27}$$

where H is, roughly speaking, the inflationary Hubble parameter. These modes get out from the adiabatic regime at the time $\eta_{\epsilon k}$ such that

$$H(\eta_{\epsilon k}) \equiv H_{\epsilon k} \sim \frac{\epsilon k}{a(\eta_{\epsilon k})}$$

Let us call this moment of time "ϵ-horizon crossing". One of our key observations is that for small ϵ, this moment occurs much earlier than the horizon crossing by the brane mode. In terms of the original field Φ, the bulk modes thus freeze out at much larger amplitudes than the brane mode, and their effect on brane mode is enhanced.

Immediately after ϵ-horizon crossing, the bulk mode behaves as

$$\psi_\lambda^+(\eta) = a(\eta)\frac{H_{\epsilon k}}{\epsilon k}, \quad \eta \gtrsim \eta_{\epsilon k} \tag{28}$$

We will need this mode at the moment η_k at which the brane mode crosses out the horizon. For small ϵ this moment occurs much later than ϵ-horizon crossing, so some care must be taken at this point. The easiest way to proceed is to make use of the original equation (10) for the field ϕ. After ϵ-horizon crossing, the field is in the slow roll regime and the physical

momentum is small, so that the first and third terms in Eq. (10) are negligible. The field $\phi(t)$ thus evolves as follows,

$$\phi(t) \propto \exp\left(-\int_{t_{\epsilon k}}^{t} dt' \frac{\lambda^2}{3H(t')}\right)$$

Making use of the initial condition (28), one obtains in terms of ψ_λ

$$\psi_\lambda^+(\eta) = a(\eta)\frac{H_{\epsilon k}}{\epsilon k}\exp\left(-\int_{\eta_{\epsilon k}}^{\eta} d\eta' \frac{\lambda^2}{3H(\eta')}a(\eta')\right) , \quad \eta \gg \eta_{\epsilon k}$$

Let us introduce the following "mean value" $\hat{H}$ of the inflationary Hubble parameter in the interval $(\eta_{\epsilon k}, \eta_k)$

$$\begin{aligned}
\frac{1}{\hat{H}^2} &= \frac{1}{|\log \epsilon|}\int_{\eta_{\epsilon k}}^{\eta_k} d\eta' \frac{1}{H(\eta')}a(\eta') \\
&= \frac{1}{|\log \epsilon|}\int_{a(\eta_{\epsilon k})}^{a(\eta_k)} \frac{da}{a}\frac{1}{H^2(a)}
\end{aligned} \tag{29}$$

For time-independent H one has $\hat{H} = H$, while in general $\hat{H}$ is a non-trivial function of ϵ and k. In terms of this parameter, the bulk mode behaves near η_k as follows,

$$\psi_\lambda^+(\eta) = a(\eta)\frac{H_{\epsilon k}}{\epsilon k}\exp\left(-\frac{\lambda^2}{3\hat{H}^2}|\log \epsilon|\right) , \quad \eta \sim \eta_k \tag{30}$$

This completes the discussion of the zeroth approximation.

B. Bulk contribution into brane mode

We now wish to calculate the effect of the bulk modes on the brane mode, in the lowest non-trivial order of perturbation theory. To this end, we insert the zeroth order expression for the bulk modes, Eq. (26) into Eq. (23) and obtain the equation for the first correction to ψ_0,

$$\ddot{\psi}_0^{(1)} - \frac{\ddot{a}}{a}\psi_0^{(1)} + k^2\psi_0^{(1)} = -k^2\int d\lambda\, \psi_\lambda^{(0)}I_\lambda \tag{31}$$

We are interested in the solution to this equation with zero initial condition at infinite past: all modes oscillate as $\eta \to -\infty$, and mixing between the modes is negligible at that time. This solution is

$$\psi_0^{(1)}(\eta) = -k^2\int d\eta' G(\eta, \eta')\int d\lambda\, \psi_\lambda^{(0)}(\eta')I_\lambda \tag{32}$$

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where G is the retarded Green's function of the operator in the left hand side of Eq. (31). We will see that the effect of mixing is most relevant at $\eta \sim \eta_k$, so we approximate

$$a(\eta) = -\frac{1}{H_k \eta} \tag{33}$$

and obtain

$$G(\eta, \eta') = \theta(\eta - \eta')F(\eta, \eta')$$

where

$$F(\eta, \eta') = \frac{1}{2ik}e^{ik(\eta - \eta')}\left(1 + \frac{i}{k\eta}\right)\left(1 - \frac{i}{k\eta'}\right) + \text{c.c.}$$

Our purpose is to calculate the amplitude of the brane mode after the horizon crossing, $\eta \gg \eta_k$. We find in this region

$$F(\eta, \eta') = -a(\eta)\frac{H_k}{k^2}\left(\cos(k\eta') - \frac{\sin(k\eta')}{k\eta'}\right), \quad \eta \gg \eta_k$$

The integrand in Eq. (32) rapidly oscillates at $\eta' \ll \eta_k$ and decays at $\eta' \gg \eta_k$. Thus, the integral is saturated at $\eta' \sim \eta_k$. The bulk modes at that time have the form (30), where one can again use the relation (33). Combining all factors and performing the integration over η' we obtain at $\eta \gg \eta_k$

$$\psi_0^{(1)}(\eta) = -a(\eta)\frac{1}{\sqrt{2}}\frac{H_{\epsilon k}}{\epsilon^{\frac{3}{2}}k^{\frac{3}{2}}}\int d\lambda\, I_\lambda e^{-\frac{\lambda^2}{3H^2}|\log \epsilon|}A_\lambda^+ + \text{h.c.} \tag{34}$$

This is again a Gaussian field whose spectrum is

$$\mathcal{P}^{(1)} = \frac{H_{\epsilon k}^2}{4\pi^2\epsilon^3}\int d\lambda\, |I_\lambda|^2 e^{-\frac{2\lambda^2}{3H^2}|\log \epsilon|}$$

Because of the relation (17), the integral here is convergent at small λ, and therefore it is saturated at

$$\lambda \sim \lambda_c = \frac{\hat{H}}{|\log \epsilon|^{1/2}}$$

This justifies our assumption (27). We obtain finally

$$\mathcal{P}^{(1)} = \text{const} \cdot \frac{H_{\epsilon k}^2}{\epsilon^3}\lambda_c|I_{\lambda_c}|^2 \tag{35}$$

where the constant is of order 1. This is our final result.

Several remarks are in order. First, the Gaussian field (34) is independent of the field $\psi_0^{(0)}$ considered in the previous subsection, as the latter contains creation and annihilation operators in the incoming brane mode. Thus, towards the end of inflation, perturbations

on the brane are the sum of two independent Gaussian fields with spectra $\mathcal{P}^{(0)}$ and $\mathcal{P}^{(1)}$. Second, because of the relation (17), the contribution (35) indeed tends to zero as $P_{LV} \to \infty$. However, the suppression at small H/P_{LV} may be quite weak, depending on the form of the warp factors. As an example, for the choice (8) the suppression factor is (see Eqs. (21) and (22))

$$\lambda_c I_{\lambda_c}^2 \sim \left(\frac{\hat{H}}{P_{LV}} \right)^{2\nu-2}$$

(up to $\log \epsilon$), while the only restriction on the parameter ν is $\nu > 1$. Third, the bulk-induced contribution to the spectrum is enhanced by ϵ^{-3}; we have already discussed the origin of this enhancement in previous subsection. Fourth, the overall magnitude of $\mathcal{P}^{(1)}$ is determined by $H_{\epsilon k}$, the value of the Hubble parameter at ϵ-horizon crossing, which occurs earlier than the usual horizon crossing by the brane mode, and also by $\hat{H}$, which is a certain average of the Hubble parameter over rather long time. Thus, the two contributions to the spectrum, $\mathcal{P}^{(0)}$ and $\mathcal{P}^{(1)}$ generically have different tilts. Finally, it is straightforward to check that both the back reaction of the brane mode on bulk modes (first integral in the right hand side of Eq. (24)) and mixing between bulk modes (second integral) indeed give small effects, even if $\mathcal{P}^{(1)} > \mathcal{P}^{(0)}$, so our perturbative treatment is justified.

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[1] M. Visser, Phys. Lett. B **159**, 22 (1985) [arXiv:hep-th/9910093]; D. J. H. Chung and K. Freese, Phys. Rev. D **61**, 023511 (2000) [arXiv:hep-ph/9906542]; D. J. H. Chung and K. Freese, Phys. Rev. D **62**, 063513 (2000) [arXiv:hep-ph/9910235]; D. J. H. Chung, E. W. Kolb and A. Riotto, Phys. Rev. D **65**, 083516 (2002) [arXiv:hep-ph/0008126].

[2] C. Csaki, J. Erlich and C. Grojean, Nucl. Phys. B **604**, 312 (2001) [arXiv:hep-th/0012143].

[3] S. L. Dubovsky, JHEP **0201**, 012 (2002) [arXiv:hep-th/0103205].

[4] D. S. Gorbunov and S. M. Sibiryakov, "Ultra-large distance modification of gravity from Lorentz symmetry breaking at the Planck scale," arXiv:hep-th/0506067.

[5] M. V. Libanov and V. A. Rubakov, JCAP **0509**, 005 (2005) [arXiv:astro-ph/0504249].

[6] J. Martin and R. H. Brandenberger, Phys. Rev. D **63**, 123501 (2001) [arXiv:hep-th/0005209]; R. H. Brandenberger and J. Martin, Mod. Phys. Lett. A **16**, 999 (2001) [arXiv:astro-ph/0005432].

[7] J. C. Niemeyer, Phys. Rev. D **63**, 123502 (2001) [arXiv:astro-ph/0005533]; T. Tanaka, "A comment on trans-Planckian physics in inflationary universe," arXiv:astro-ph/0012431; A. A. Starobinsky, Pisma Zh. Eksp. Teor. Fiz. **73**, 415 (2001) [JETP Lett. **73**, 371 (2001)] [arXiv:astro-ph/0104043]; M. Lemoine, M. Lubo, J. Martin and J. P. Uzan, Phys. Rev. D **65**, 023510 (2002) [arXiv:hep-th/0109128]; R. H. Brandenberger, S. E. Joras and J. Martin, Phys. Rev. D **66**, 083514 (2002) [arXiv:hep-th/0112122]; U. H. Danielsson, Phys. Rev. D **66**, 023511 (2002) [arXiv:hep-th/0203198]; U. H. Danielsson, JHEP **0207**, 040 (2002) [arXiv:hep-th/0205227]; V. Bozza, M. Giovannini and G. Veneziano, JCAP **0305**, 001 (2003) [arXiv:hep-th/0302184]; A. Kempf, Phys. Rev. D **63**, 083514 (2001) [arXiv:astro-ph/0009209]; A. Kempf and J. C. Niemeyer, Phys. Rev. D **64**, 103501 (2001) [arXiv:astro-ph/0103225]; R. Easther, B. R. Greene, W. H. Kinney and G. Shiu, Phys. Rev. D **64**, 103502 (2001) [arXiv:hep-th/0104102]; F. Lizzi, G. Mangano, G. Miele and M. Peloso, JHEP **0206**, 049 (2002) [arXiv:hep-th/0203099]; M. Porrati, Phys. Lett. B **596**, 306 (2004); R. H. Brandenberger and J. Martin, Phys. Rev. D **71**, 023504 (2005) [arXiv:hep-th/0410223].

[8] N. Kaloper, M. Kleban, A. E. Lawrence and S. Shenker, Phys. Rev. D **66**, 123510 (2002) [arXiv:hep-th/0201158]; K. Schalm, G. Shiu and J. P. van der Schaar, JHEP **0404**, 076 (2004) [arXiv:hep-th/0401164].

[9] J. Martin and R. H. Brandenberger, Phys. Rev. D **65**, 103514 (2002) [arXiv:hep-th/0201189].

[10] S. Corley and T. Jacobson, Phys. Rev. D **54**, 1568 (1996) [arXiv:hep-th/9601073].

[11] C. P. Burgess, J. M. Cline, F. Lemieux and R. Holman, JHEP **0302** (2003) 048 [arXiv:hep-th/0210233]; C. P. Burgess, J. M. Cline and R. Holman, JCAP **0310** (2003) 004.

[12] L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999) 4690 [arXiv:hep-th/9906064].

[13] B. Bajc and G. Gabadadze, Phys. Lett. B **474** (2000) 282 [arXiv:hep-th/9912232].

[14] L.D. Landau and E.M. Lifshits, "Quantum Mechanics. Non-Relativistic Theory", Moscow, Nauka, 1989, Section 134.