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The colour evolution of the process $qq \to qqg$

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Abstract: We calculate the soft anomalous dimension matrix for a five-parton process, $qq \to qqg$. Considering different bases we unveil some interesting properties of this matrix.

Keywords: qcd, jet.

Contents

1. Introduction

The understanding of gaps-between-jets processes has been subject to great progress over the last few years. Of central importance in this context is the energy flow into the interjet region as a very useful observable in the description of gaps-between-jets processes. According to the method of Sterman et al. [1–4], the cross section for interjet energy flow can be 'refactorized' into hard and soft parts at some factorization scale μ. For all but the most trivial processes, these parts have a matrix structure in the space of colour flows of the hard process, with the hard amplitude represented by a vector $\underline{\mathcal{A}}$ in this space and the sum of all possible soft corrections to the cross section represented by a matrix S. The μ dependence of each is accounted for by an anomalous dimension Γ, also a matrix in colour space. Logarithms of the ratio of the hard and soft scales can be summed to all orders by taking μ of order the hard scale in the hard amplitude and the soft scale in the soft matrix, using the exponential of the integral of Γ to connect the two scales. The anomalous dimension matrix Γ has been calculated for all (QCD) $2 \to 2$ processes and for various definitions of the final state [5–8].

It is the purpose of this paper to calculate for the first time the anomalous dimension matrix for a $2 \to 3$ process, namely $qq \to qqg$; to distinguish it from the one for the $2 \to 2$ process $qq \to qq$, we denote it Λ. This can serve as a starting point to improve the understanding of theoretical aspects in the description of gaps-between-jets processes. The calculation of this matrix is also a first step towards energy flow analyses of 3-jet processes which are particularly interesting at the LHC.

In this paper we consider the process $qq \to qqg$ with a gap defined by a central rapidity region of length $Y < \Delta y$ where Δy is the rapidity separation of the outgoing quarks. The real gluon is restricted to the region outside the gap. Λ is then obtained by calculating the virtual corrections to this process from a softer gluon connecting the external lines in all possible ways. In common with other calculations of gaps-between-jets cross sections [5–8], we assume a perfect real–virtual cancellation outside the gap region. Thus the virtual gluon is integrated only over the rapidity interval of the gap and over all azimuthal angles.

We represent the result for Λ in three colour bases and thereby shed light on different aspects of it.

2. The anomalous dimension matrix for $qq \to qqg$

Let us start with the colour structure. For the $q_i q_j \to q_k q_l g_a$ system (where the subscripts are the colour indices) there are four independent colour states needed. We first choose the t-channel basis

$$\mathbf{C}_1 = T^a_{ki}\delta_{lj} + \delta_{ki}T^a_{lj}, \tag{2.1}$$

$$\mathbf{C}_2 = T^b_{ki}T^c_{lj}\, d^{abc}, \tag{2.2}$$

$$\mathbf{C}_3 = T^a_{ki}\delta_{lj} - \delta_{ki}T^a_{lj}, \tag{2.3}$$

$$\mathbf{C}_4 = T^b_{ki}T^c_{lj}\, if^{abc}. \tag{2.4}$$

The lowest order soft matrix (which contains the traces of the squared operators of the basis) is given in this basis by

$$\mathrm{S} = \begin{pmatrix} N_c(N_c^2-1) & 0 & 0 & 0 \\ 0 & \frac{1}{4N_c}(N_c^2-1)(N_c^2-4) & 0 & 0 \\ 0 & 0 & N_c(N_c^2-1) & 0 \\ 0 & 0 & 0 & \frac{1}{4}N_c(N_c^2-1) \end{pmatrix}. \tag{2.5}$$

The momenta of the hard process are labeled in the following way

$$q(p_1) + q(p_2) \to q(p_3) + q(p_4) + g(k). \tag{2.6}$$

We work in the frame in which 1 and 2 collide head on and the gap region is central in rapidity,

$$p_1 = E_1(1;0,0,\quad 1), \tag{2.7}$$

$$p_2 = E_2(1;0,0,-1), \tag{2.8}$$

$$p_3 = q_{\perp 3}\,(\cosh y_3;\quad 0,\quad 1, \sinh y_3)\,, \tag{2.9}$$

$$p_4 = q_{\perp 4}\,(\cosh y_4; \sin\varphi, \cos\varphi, \sinh y_4)\,, \tag{2.10}$$

$$k = k_\perp\;\;(\cosh y\;; \sin\phi, \cos\phi, \sinh y\;)\,. \tag{2.11}$$

Note that in the limit in which the emitted gluon is much softer than the quarks, $k_\perp \ll q_{\perp 3,4}$, momentum conservation implies $q_{\perp 3} = q_{\perp 4}$ and $\varphi = \pi$ and the kinematics are identical to the lowest order process $qq \to qq$. We are interested in the case that the quark jets are either side of the gap and can therefore assume $y_3 > 0$ and $y_4 < 0$.

We denote the rapidity, azimuthal angle and transverse momentum of the virtual gluon k' by y', ϕ' and $k'_\perp$, respectively. For future use, we define

$$s_y = \mathrm{sgn}(y). \tag{2.12}$$

The gap is defined by a central rapidity region of length Y. Since we are interested in the case of k outside the gap and k' within it, we have

$$|y'| < \frac{Y}{2} < |y| \tag{2.13}$$

and hence

$$\mathrm{sgn}(y - y') = \mathrm{sgn}(y) = s_y. \tag{2.14}$$

We denote the hard amplitude (2.6) evaluated at refactorization scale μ by the (four dimensional) vector $\underline{\mathcal{A}}(\mu)$. The anomalous dimension matrix Λ is then defined through the evolution of $\underline{\mathcal{A}}(\mu)$,

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \underline{\mathcal{A}} = \frac{2\alpha_s}{\pi}\, \Lambda\, \underline{\mathcal{A}}. \tag{2.15}$$

We can extract Λ from a one-loop calculation by expanding (2.15) to leading order,

$$\underline{\mathcal{A}}^{(1)} = -\frac{2\alpha_s}{\pi} \int \frac{dk'_\perp}{k'_\perp}\, \Lambda\, \underline{\mathcal{A}}^{(0)}, \tag{2.16}$$

where $\underline{\mathcal{A}}^{(0)}$ and $\underline{\mathcal{A}}^{(1)}$ are respectively the lowest order and one-loop amplitudes. The latter is calculated from the virtual corrections to the hard process from a gluon coupling two external lines in all possible ways. We work in the eikonal effective theory. The region of integration is

$$0 < \phi' < 2\pi \tag{2.17}$$

$$-Y/2 < y' < Y/2. \tag{2.18}$$

Details of the calculation of Λ can be found in the Appendix. We only state the result here written as a sum of four parts.

$$
\Lambda = \begin{pmatrix} \frac{N_c}{4}(Y-i\pi)+\frac{1}{2N_c}i\pi & (\frac{1}{4}-\frac{1}{N_c^2})i\pi & -\frac{N_c}{4}s_yY & 0 \\ i\pi & \frac{N_c}{4}(2Y-i\pi)-\frac{3}{2N_c}i\pi & 0 & 0 \\ -\frac{N_c}{4}s_yY & 0 & \frac{N_c}{4}(Y-i\pi)-\frac{1}{2N_c}i\pi & -\frac{1}{4}i\pi \\ 0 & 0 & -i\pi & \frac{N_c}{4}(2Y-i\pi)-\frac{1}{2N_c}i\pi \end{pmatrix}
$$

$$
+\begin{pmatrix} N_c & 0 & 0 & 0 \\ 0 & N_c & 0 & 0 \\ 0 & 0 & N_c & 0 \\ 0 & 0 & 0 & N_c \end{pmatrix} \frac{1}{4}\rho(2|y|)
$$

$$
+\begin{pmatrix} C_F & 0 & 0 & 0 \\ 0 & C_F & 0 & 0 \\ 0 & 0 & C_F & 0 \\ 0 & 0 & 0 & C_F \end{pmatrix} \left(\frac{1}{4}\rho(2|y_3|)+\frac{1}{4}\rho(2|y_4|)\right)
$$

$$
+\begin{pmatrix} \frac{N_c}{4}(-\frac{1}{2}\lambda) & 0 & \frac{N_c}{4}(-\frac{1}{2}s_y\lambda) & \frac{1}{4}(\frac{1}{2}s_y\lambda) \\ 0 & \frac{N_c}{4}(-\frac{1}{2}\lambda) & 0 & \frac{N_c}{4}(\frac{1}{2}s_y\lambda) \\ \frac{N_c}{4}(-\frac{1}{2}s_y\lambda) & 0 & \frac{N_c}{4}(-\frac{1}{2}\lambda) & \frac{1}{4}(-\frac{1}{2}\lambda) \\ \frac{1}{2}s_y\lambda & (\frac{N_c}{4}-\frac{1}{N_c})(\frac{1}{2}s_y\lambda) & -\frac{1}{2}\lambda & \frac{N_c}{4}(-\frac{1}{2}\lambda) \end{pmatrix}, \tag{2.19}
$$

where we have defined

$$
\rho(y) \equiv \log\frac{\sinh(y/2+Y/2)}{\sinh(y/2-Y/2)} - Y, \tag{2.20}
$$

$$
\lambda \equiv \frac{1}{2}\log\frac{\cosh(|\bar{y}|+|y|+Y)-\cos(\bar{\phi})}{\cosh(|\bar{y}|+|y|-Y)-\cos(\bar{\phi})} - Y \tag{2.21}
$$

with

$$
\bar{\phi} \equiv \begin{cases} \phi & y>0, \\ \varphi-\phi & y<0, \end{cases} \tag{2.22}
$$

and

$$
\bar{y} \equiv \begin{cases} y_3 & y>0, \\ y_4 & y<0. \end{cases} \tag{2.23}
$$

We have grouped the four terms of Λ in the following way. For fixed Y, the first line contains all the terms that remain when $y, y_3, y_4 \to \pm\infty$ (the high energy limit), the second line contains the additional terms that remain when y is finite, the third line contains the additional terms that remain when y_3 or y_4 is finite and the last line contains the additional terms that remain when y_3, y_4 and y are finite. For future reference we define these four lines to be $\Lambda_{1,2,3,4}$ respectively.

For reasons that will become apparent shortly, it will be useful to modify the matrix Λ. Adding a multiple of the identity matrix to any matrix does not change its eigenvectors and simply adds a constant to all of its eigenvalues. Moreover, adding an imaginary constant to all the eigenvalues of Λ will not change the physics, since the energy dependence comes from combinations $\lambda^{(i)^*} + \lambda^{(j)}$. Therefore we are free to add any imaginary constant times the identity matrix to Λ. From now on we shall denote Λ the matrix obtained from (2.19) by

$$\Lambda \to \Lambda + N_c/4\, i\pi. \tag{2.24}$$

The eigenvalues of this matrix Λ are:

$$\lambda^{(1)} = \frac{N_c}{2}Y + \frac{N_c-1}{2N_c}i\pi + N_c\tfrac{1}{4}\rho(2|y|) + C_F\left(\tfrac{1}{4}\rho(2|y_3|) + \tfrac{1}{4}\rho(2|y_4|)\right) + \tfrac{1}{4}\lambda, \tag{2.25}$$

$$\lambda^{(2)} = \frac{N_c}{2}Y - \frac{N_c+1}{2N_c}i\pi + N_c\tfrac{1}{4}\rho(2|y|) + C_F\left(\tfrac{1}{4}\rho(2|y_3|) + \tfrac{1}{4}\rho(2|y_4|)\right) - \tfrac{1}{4}\lambda, \tag{2.26}$$

$$\lambda^{(3)} = \frac{N_c^2Y - 2i\pi - N_c\sqrt{N_c^2Y^2 - 4Yi\pi - 4\pi^2}}{4N_c} + N_c\tfrac{1}{4}\rho(2|y|) + C_F\left(\tfrac{1}{4}\rho(2|y_3|) + \tfrac{1}{4}\rho(2|y_4|)\right) - \tfrac{N_c}{4}\lambda, \tag{2.27}$$

$$\lambda^{(4)} = \frac{N_c^2Y - 2i\pi + N_c\sqrt{N_c^2Y^2 - 4Yi\pi - 4\pi^2}}{4N_c} + N_c\tfrac{1}{4}\rho(2|y|) + C_F\left(\tfrac{1}{4}\rho(2|y_3|) + \tfrac{1}{4}\rho(2|y_4|)\right) - \tfrac{N_c}{4}\lambda \tag{2.28}$$

and it is diagonalized by

$$\mathrm{R} = \begin{pmatrix} \frac{1}{2}s_y & -\frac{1}{2}s_y & (-\frac{3}{2N_c} - i\frac{N_cY}{2\pi} + \frac{i\lambda^{(3)}}{\pi})s_y & (-\frac{1}{2N_c} - \frac{i\lambda^{(3)}}{\pi})s_y \\ \frac{N_c}{N_c+2}s_y & \frac{N_c}{N_c-2}s_y & -s_y & -s_y \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2N_c} - i\frac{N_cY}{2\pi} + \frac{i\lambda^{(3)}}{\pi} & \frac{1}{2N_c} - \frac{i\lambda^{(3)}}{\pi} \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{2.29}$$

Note that whereas Λ_1 has four different eigenvalues, two eigenvalues of Λ_4 are degenerate.

2.1 Block Diagonalization of Λ

The anomalous dimension matrix Γ for the hard process $qq \to qq$ is defined in exact analogy to Λ, (2.16). In the high energy limit ($|y_{3,4}| \to \infty$) and in the t-channel singlet–octet basis it reads:

$$\Gamma = \begin{pmatrix} 0 & \left(\frac{1}{4} - \frac{1}{4N_c^2}\right)i\pi \\ i\pi & \frac{N_c}{2}Y - \frac{1}{N_c}(i\pi) \end{pmatrix}. \tag{2.30}$$

Two of the eigenvalues of Λ_1 coincide with the eigenvalues of Γ (to enable this was the reason for the modification (2.24) of Λ). We can therefore construct a matrix

$$\hat{\mathrm{R}} = \sqrt{\frac{N_c}{2(N_c^2-1)}} \begin{pmatrix} \frac{1}{2}s_y & -\frac{1}{2}s_y & s_y & \frac{1}{2N_c}s_y \\ \frac{N_c}{N_c+2}s_y & \frac{N_c}{N_c-2}s_y & 0 & s_y \\ -\frac{1}{2} & \frac{1}{2} & 1 & -\frac{1}{2N_c} \\ 1 & 1 & 0 & -1 \end{pmatrix}. \tag{2.31}$$

which diagonalizes Λ_4 and transforms Λ_1 in the following way:

$$\hat{\mathrm{R}}^{-1}\Lambda_1\hat{\mathrm{R}} = \begin{pmatrix} \lambda_1^{(1)} & 0 & 0 & 0 \\ 0 & \lambda_1^{(2)} & 0 & 0 \\ 0 & 0 & 0 & \left(\frac{1}{4}-\frac{1}{4N_c^2}\right) i\pi \\ 0 & 0 & i\pi & \frac{N_c}{2}Y - \frac{1}{N_c}(i\pi) \end{pmatrix}. \tag{2.32}$$

where $\lambda_1^{(i)}$ are the eigenvalues of Λ_1, which can be obtained from (2.25-2.28) by setting ρ and λ to 0. Not only is this matrix block diagonal but, remarkably, the upper left block is itself diagonal, and the lower right block is identical to Γ.

Note that the soft matrix in this basis,

$$\hat{\mathrm{R}}^{\dagger}\mathrm{S}\hat{\mathrm{R}} = \begin{pmatrix} \frac{N_c^2}{2}\frac{N_c+1}{N_c+2} & 0 & 0 & 0 \\ 0 & \frac{N_c^2}{2}\frac{N_c-1}{N_c-2} & 0 & 0 \\ 0 & 0 & N_c^2 & 0 \\ 0 & 0 & 0 & \frac{1}{4}(N_c^2-1) \end{pmatrix}, \tag{2.33}$$

is still diagonal and that its lower right block is identical to the soft matrix of the $2 \to 2$ process (the latter property is the reason for our choice of normalization for $\hat{\mathrm{R}}$). It is also interesting to note that in this basis the anomalous dimension matrix Λ is s_y independent and that all of the s_y dependence is carried by the definitions of the basis states, which are different for $s_y = \pm 1$.

2.2 The s-channel basis

Reference [9] advocated using the set of s-channel projectors as the colour basis for $2 \to 2$ processes. In this section we present our results for Λ in an alternative block-diagonal form in which its lower right block is identical to Γ transformed into the s-channel basis and show that its basis states have a simple form.

For a qq state, the projectors are

$$\mathbf{P}_{\bar{3}} = \frac{1}{2}\Big(\delta_{ki}\delta_{lj} - \delta_{li}\delta_{kj}\Big), \tag{2.34}$$

$$\mathbf{P}_{6} = \frac{1}{2}\Big(\delta_{ki}\delta_{lj} + \delta_{li}\delta_{kj}\Big). \tag{2.35}$$

We can transform between the t-channel basis we have been using so far and the s-channel basis using the matrix

$$\mathrm{R}_{st} = \begin{pmatrix} \frac{N_c-1}{2N_c} & \frac{N_c+1}{2N_c} \\ -1 & +1 \end{pmatrix}. \tag{2.36}$$

That is, Γ transforms to

$$\mathrm{R}_{st}^{-1}\,\Gamma\,\mathrm{R}_{st} = \begin{pmatrix} \frac{1}{4}(N_c+1)Y - \frac{N_c+1}{2N_c}i\pi & -\frac{1}{4}(N_c+1)Y \\ -\frac{1}{4}(N_c-1)Y & \frac{1}{4}(N_c-1)Y + \frac{N_c-1}{2N_c}i\pi \end{pmatrix} \tag{2.37}$$

in the high energy limit and S to

$$\mathrm{R}_{st}^{\dagger}\,\mathrm{S}\,\mathrm{R}_{st} = \begin{pmatrix} \frac{1}{2}N_c(N_c-1) & 0 \\ 0 & \frac{1}{2}N_c(N_c+1) \end{pmatrix}. \tag{2.38}$$

Note that the imaginary terms appear only in the diagonal of Γ and that the entries in S correspond to the multiplicities of the basis states, 3 and 6 for $N_c = 3$, two of the advantages of the s-channel projector basis.

We wish to express Λ in a block diagonal form in which the bottom right block is equal to Γ in the s-channel basis, Eq. (2.37). To this end we define a matrix

$$\hat{\mathrm{R}}_{st} = \begin{pmatrix} \sqrt{\frac{N_c+2}{N_c}} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{N_c-2}{N_c}} & 0 & 0 \\ 0 & 0 & \frac{N_c-1}{2N_c} & \frac{N_c+1}{2N_c} \\ 0 & 0 & -1 & +1 \end{pmatrix}, \tag{2.39}$$

in which the bottom right block is equal to R_{st} and the diagonal entries in the top left block are arbitrary – the particular choice made here will lead to a convenient result for the soft matrix.

Transforming Λ_1 and S from the original t-channel basis, we obtain

$$\hat{\mathrm{R}}_{st}^{-1}\,\hat{\mathrm{R}}^{-1}\,\Lambda_1\,\hat{\mathrm{R}}\,\hat{\mathrm{R}}_{st} = \begin{pmatrix} \lambda_1^{(1)} & 0 & 0 & 0 \\ 0 & \lambda_1^{(2)} & 0 & 0 \\ 0 & 0 & \frac{1}{4}(N_c+1)Y - \frac{N_c+1}{2N_c}i\pi & -\frac{1}{4}(N_c+1)Y \\ 0 & 0 & -\frac{1}{4}(N_c-1)Y & \frac{1}{4}(N_c-1)Y + \frac{N_c-1}{2N_c}i\pi \end{pmatrix} \tag{2.40}$$

and

$$\hat{\mathrm{R}}_{st}^{\dagger}\,\hat{\mathrm{R}}^{\dagger}\,\mathrm{S}\,\hat{\mathrm{R}}\,\hat{\mathrm{R}}_{st} = \begin{pmatrix} \frac{1}{2}N_c(N_c+1) & 0 & 0 & 0 \\ 0 & \frac{1}{2}N_c(N_c-1) & 0 & 0 \\ 0 & 0 & \frac{1}{2}N_c(N_c-1) & 0 \\ 0 & 0 & 0 & \frac{1}{2}N_c(N_c+1) \end{pmatrix}. \tag{2.41}$$

By construction, the lower right block of Λ is equal to Γ in the high energy limit, the lower right block of S is equal to S in the $2 \to 2$ s-channel basis and the upper left entries of Λ are left unchanged. The upper left entries of S are set by our arbitrary choices in $\hat{\mathrm{R}}_{st}$ for reasons that will be seen shortly. Beyond the high energy limit, the bottom right block of Λ contains a term $\frac{1}{4}C_F(\rho(2|y_3|) + \rho(2|y_4|)) + \frac{1}{4}N_c(\rho(2|y|) - \lambda)$ times the identity matrix while Γ contains just a term $\frac{1}{4}C_F(\rho(2|y_3|) + \rho(2|y_4|))$ times the identity matrix, resulting in a small mismatch.

The actual definitions of the basis states can be read off from the columns of $\hat{\mathrm{R}}\,\hat{\mathrm{R}}_{st}$ and can be written in forms proportional to the s-channel projectors for the incoming quarks. Since the matrix $\hat{\mathrm{R}}$ depends on s_y, these states are different for

$s_y = \pm 1$ (recall that $s_y = +1$ implies that the gluon is on the same side of the rapidity gap as incoming quark i and outgoing quark k, while $s_y = -1$ implies that it is on the other side). For $s_y = +1$ we have

$$\hat{\mathbf{C}}_1 \propto \frac{1}{2}\Big(\delta_{mi}\delta_{nj} + \delta_{ni}\delta_{mj}\Big)\Big(\delta_{km}T^a_{ln} - \frac{1}{N_c+1}T^a_{km}\delta_{ln}\Big), \tag{2.42}$$

$$\hat{\mathbf{C}}_2 \propto \frac{1}{2}\Big(\delta_{mi}\delta_{nj} - \delta_{ni}\delta_{mj}\Big)\Big(\delta_{km}T^a_{ln} + \frac{1}{N_c-1}T^a_{km}\delta_{ln}\Big), \tag{2.43}$$

$$\hat{\mathbf{C}}_3 \propto \frac{1}{2}\Big(\delta_{mi}\delta_{nj} - \delta_{ni}\delta_{mj}\Big)\Big(T^a_{km}\delta_{ln}\Big), \tag{2.44}$$

$$\hat{\mathbf{C}}_4 \propto \frac{1}{2}\Big(\delta_{mi}\delta_{nj} + \delta_{ni}\delta_{mj}\Big)\Big(T^a_{km}\delta_{ln}\Big), \tag{2.45}$$

while for $s_y = -1$ we have

$$\hat{\mathbf{C}}_1 \propto \frac{1}{2}\Big(\delta_{mi}\delta_{nj} + \delta_{ni}\delta_{mj}\Big)\Big(T^a_{km}\delta_{ln} - \frac{1}{N_c+1}\delta_{km}T^a_{ln}\Big), \tag{2.46}$$

$$\hat{\mathbf{C}}_2 \propto \frac{1}{2}\Big(\delta_{mi}\delta_{nj} - \delta_{ni}\delta_{mj}\Big)\Big(T^a_{km}\delta_{ln} + \frac{1}{N_c-1}\delta_{km}T^a_{ln}\Big), \tag{2.47}$$

$$\hat{\mathbf{C}}_3 \propto \frac{1}{2}\Big(\delta_{mi}\delta_{nj} - \delta_{ni}\delta_{mj}\Big)\Big(\delta_{km}T^a_{ln}\Big), \tag{2.48}$$

$$\hat{\mathbf{C}}_4 \propto \frac{1}{2}\Big(\delta_{mi}\delta_{nj} + \delta_{ni}\delta_{mj}\Big)\Big(\delta_{km}T^a_{ln}\Big). \tag{2.49}$$

It is important to note that although we have made arbitrary choices that affect the normalizations of these states in order to get S into the form Eq. (2.41), their forms are otherwise determined entirely by the physics of Λ. We see that the two states that evolve like a $qq \to qq$ system have a colour structure given by the $qq \to qq$ projectors followed by a gluon emission from the outgoing quark it is on the same side of the gap as. The two other states are given similarly by projectors followed by an emission from the other outgoing quark, up to colour-suppressed terms coming from emission on the same side.

3. Conclusions

We have calculated the anomalous dimension matrix Λ for the five-parton process $qq \to qqg$ and presented it in several different colour bases. It seems likely that the generalization of the s-channel basis, Eq. (2.40), will be most useful both for gaining insight into the physics of Λ and for performing practical calculations. We anticipate using Λ to improve the theoretical understanding of gaps-between-jets processes and ultimately to calculate energy flow observables in 3-jet processes, which are particularly interesting at the LHC. The latter however requires the anomalous dimension matrices for all 3-jet processes to be calculated, a highly non-trivial problem: in the most complicated case of $gg \to ggg$ one expects up to 44 independent colour amplitudes and a deeper theoretical insight seems necessary to organize the calculation.

In particular it would be extremely interesting to see whether the block diagonal structure we found for $qq \to qqg$ can be generalized to arbitrary processes, with one block always equal to the anomalous dimension matrix of a lower-order process.

A. Calculation of Λ

In the basis (2.1-2.4) Λ has the following colour structure

$$\Lambda = \begin{pmatrix} \frac{1}{2N_c}(\Omega_{12}+\Omega_{34}+\Omega_{14}+\Omega_{23}) & (\frac{1}{4}-\frac{1}{N_c^2})(\Omega_{12}+\Omega_{34}+\Omega_{14}+\Omega_{23}) & 0 & \frac{1}{4}(-\Omega_{14}+\Omega_{23}) \\ \Omega_{12}+\Omega_{34}+\Omega_{14}+\Omega_{23} & \begin{matrix} -\frac{3}{2N_c}(\Omega_{12}+\Omega_{34}+\Omega_{14}+\Omega_{23}) \\ +\frac{N_c}{4}(\Omega_{14}+\Omega_{23}) \end{matrix} & 0 & \frac{N_c}{4}(-\Omega_{14}+\Omega_{23}) \\ 0 & 0 & -\frac{1}{2N_c}(\Omega_{12}+\Omega_{34}+\Omega_{14}+\Omega_{23}) & \frac{1}{4}(-\Omega_{12}+\Omega_{34}) \\ -\Omega_{14}+\Omega_{23} & (\frac{N_c}{4}-\frac{1}{N_c})(-\Omega_{14}+\Omega_{23}) & -\Omega_{12}+\Omega_{34} & \begin{matrix} -\frac{1}{2N_c}(\Omega_{12}+\Omega_{34}+\Omega_{14}+\Omega_{23}) \\ +\frac{N_c}{4}(\Omega_{14}+\Omega_{23}) \end{matrix} \end{pmatrix}$$

$$+\begin{pmatrix} \frac{N_c}{4}(\Omega_{1k}-\Omega_{2k}-\Omega_{3k}+\Omega_{4k}) & 0 & \frac{N_c}{4}(\Omega_{1k}+\Omega_{2k}-\Omega_{3k}-\Omega_{4k}) & \frac{1}{4}(\Omega_{1k}+\Omega_{2k}+\Omega_{3k}+\Omega_{4k}) \\ 0 & \frac{N_c}{4}(\Omega_{1k}-\Omega_{2k}-\Omega_{3k}+\Omega_{4k}) & 0 & \frac{N_c}{4}(\Omega_{1k}+\Omega_{2k}+\Omega_{3k}+\Omega_{4k}) \\ \frac{N_c}{4}(\Omega_{1k}+\Omega_{2k}-\Omega_{3k}-\Omega_{4k}) & 0 & \frac{N_c}{4}(\Omega_{1k}-\Omega_{2k}-\Omega_{3k}+\Omega_{4k}) & \frac{1}{4}(-\Omega_{1k}+\Omega_{2k}-\Omega_{3k}+\Omega_{4k}) \\ \Omega_{1k}+\Omega_{2k}+\Omega_{3k}+\Omega_{4k} & (\frac{N_c}{4}-\frac{1}{N_c})(\Omega_{1k}+\Omega_{2k}+\Omega_{3k}+\Omega_{4k}) & -\Omega_{1k}+\Omega_{2k}-\Omega_{3k}+\Omega_{4k} & \frac{N_c}{4}(\Omega_{1k}-\Omega_{2k}-\Omega_{3k}+\Omega_{4k}) \end{pmatrix}$$

$$+\begin{pmatrix} (\frac{N_c}{4}-\frac{1}{2N_c})(\Omega_{13}+\Omega_{24}) & 0 & \frac{N_c}{4}(-\Omega_{13}+\Omega_{24}) & 0 \\ 0 & -\frac{1}{2N_c}(\Omega_{13}+\Omega_{24}) & 0 & 0 \\ \frac{N_c}{4}(-\Omega_{13}+\Omega_{24}) & 0 & (\frac{N_c}{4}-\frac{1}{2N_c})(\Omega_{13}+\Omega_{24}) & 0 \\ 0 & 0 & 0 & -\frac{1}{2N_c}(\Omega_{13}+\Omega_{24}) \end{pmatrix}. \quad (A.1)$$

Ω_{ij} corresponds to the case in which the virtual gluon couples quarks i and j. Ω_{ik} accounts for the coupling of quark i and the gluon k. These functions are given by

$$\Omega_{ij} = \frac{1}{2}\delta_i\delta_j \left[\int_\Omega \mathrm{d}y' \frac{\mathrm{d}\phi'}{2\pi}\, \omega_{ij} - \tfrac{1}{2}(1-\delta_i\delta_j)i\pi\right] \quad (A.2)$$

and

$$\Omega_{ik} = \frac{1}{2}\delta_i\delta_k\delta_g \left[\int_\Omega \mathrm{d}y' \frac{\mathrm{d}\phi'}{2\pi}\, \omega'_{ik} - \tfrac{1}{2}(1-\delta_i\delta_k)i\pi\right] \quad (A.3)$$

where we have introduced the shorthands

$$\omega_{ij} = \frac{\frac{1}{2}k_\perp'^2\, p_i\cdot p_j}{p_i\cdot k'\, k'\cdot p_j}\,, \qquad \omega'_{ik} = \frac{\frac{1}{2}k_\perp'^2\, p_i\cdot k}{p_i\cdot k'\, k'\cdot k}. \quad (A.4)$$

We have $\delta_i\delta_j = -1$ if i and j are both incoming or both outgoing and $+1$ otherwise. δ_g depends on the topology of the triple-gluon vertex: in our convention in which the indices of if^{abc} are labeled in an anticlockwise direction around the vertex, if, with the vertex rotated so that the momentum of the eikonal gluon is flowing horizontally from left to right, the soft gluon is above it, then $\delta_g = +1$, and if below, $\delta_g = -1$.

It is worth pointing out that in the general case, the elements of Λ, Eq. (A.1) obey the following equality:

$$\Lambda_{ij}/\mathrm{S}_{jj} = \Lambda_{ji}/\mathrm{S}_{ii} \quad \text{(no sum)}, \quad (A.5)$$

implying that in an orthonormal basis in which S is equal to the identity matrix, Λ is a symmetric matrix. This property is therefore valid independently of the observable to which Λ contributes. It was pointed out in Ref. [10] that this property is true of all anomalous dimension matrices that have been calculated to date, although no explanation of this fact was offered.

Carrying out the integrations in (A.3, A.4) over the region Ω:

$$0 < \phi' < 2\pi \tag{A.6}$$

$$-Y/2 < y' < Y/2, \tag{A.7}$$

we obtain

$$\Omega_{12} = -\frac{1}{2}(Y - i\pi), \tag{A.8}$$

$$\Omega_{34} = -\frac{1}{2}(Y - i\pi) - \frac{1}{4}\rho(2|y_3|) - \frac{1}{4}\rho(2|y_4|), \tag{A.9}$$

$$\Omega_{14} = \frac{1}{2}Y + \frac{1}{4}\rho(2|y_4|), \tag{A.10}$$

$$\Omega_{23} = \frac{1}{2}Y + \frac{1}{4}\rho(2|y_3|), \tag{A.11}$$

$$\Omega_{13} = \frac{1}{4}\rho(2|y_3|), \tag{A.12}$$

$$\Omega_{24} = \frac{1}{4}\rho(2|y_4|), \tag{A.13}$$

$$\Omega_{1k} = \frac{1}{4}(1 - s_y)Y + \frac{1}{4}\rho(2|y|), \tag{A.14}$$

$$\Omega_{2k} = -\frac{1}{4}(1 + s_y)Y - \frac{1}{4}\rho(2|y|), \tag{A.15}$$

$$\Omega_{3k} = -\frac{1}{2}\left[\frac{1}{2}(1 - s_y)Y + \frac{1}{2}\rho(2|y_3|) + \frac{1}{2}\rho(2|y|) - \frac{1}{2}(1 + s_y)\lambda(|y_3| + |y|, \phi) - i\pi\right], \tag{A.16}$$

$$\Omega_{4k} = \frac{1}{2}\left[\frac{1}{2}(1 + s_y)Y + \frac{1}{2}\rho(2|y_4|) + \frac{1}{2}\rho(2|y|) - \frac{1}{2}(1 - s_y)\lambda(|y_4| + |y|, \varphi - \phi) - i\pi\right]. \tag{A.17}$$

where we have defined

$$s_y = \mathrm{sgn}(y), \tag{A.18}$$

$$\rho(y) \equiv \log\frac{\sinh(y/2 + Y/2)}{\sinh(y/2 - Y/2)} - Y, \tag{A.19}$$

$$\lambda(y, \phi) \equiv \frac{1}{2}\log\frac{\cosh(y + Y) - \cos(\phi)}{\cosh(y - Y) - \cos(\phi)} - Y. \tag{A.20}$$

It is useful to note that

$$\lambda(y, 0) = \rho(y). \tag{A.21}$$

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