

Quark-antiquark bound state equation in the Wilson loop approach with minimal surfaces¹

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Abstract

The quark-antiquark gauge invariant Green function is studied through its dependence on Wilson loops. The latter are saturated, in the large- N_c limit and for large contours, by minimal surfaces. A covariant bound state equation is derived which in the center-of-mass frame and at equal-times takes the form of a Breit-Salpeter type equation. The large-distance interaction potentials reduce in the static case to a confining linear vector potential. In general, the interaction potentials involve contributions having the structure of flux tube like terms.

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The Wilson loop [1] is defined as the trace in color space of the path-ordered phase factor of the gluon field on a closed contour C :

$$\Phi(C) = \frac{1}{N_c} \text{tr}_c P e^{-ig \oint_C dx^\mu A_\mu(x)}. \quad (1)$$

Its vacuum expectation value, denoted $W(C)$,

$$W(C) = \langle \Phi(C) \rangle, \quad (2)$$

is a functional of the contour C . Loop equations were obtained and studied by Polyakov [2] and Makeenko and Migdal [3, 4, 5]. The Wilson loop essentially satisfies two types of equation, which are equivalent to the QCD equations of motion: The Bianchi identity and the loop equations (or Makeenko–Migdal equations). A third property, factorization, is obtained in the large- N_c limit [6] for two disjoint contours: $W(C_1, C_2) = W(C_1)W(C_2)$.

Further simplification is obtained in the large- N_c limit of the theory. In that limit, for large contours, i.e., at large distances, nonperturbative asymptotic solutions to the Wilson loops are represented by the minimal surfaces having as supports the loop contours [3, 7]. Therefore, if one is interested only in the large-distance properties of the theory, saturation of the Wilson loop averages by minimal surfaces provides a correct description of the theory in this regime. In that case, the Wilson loop average can be represented by the following functional of the contour C :

$$W(C) = e^{-i\sigma A(C)}, \quad (3)$$

where σ is the string tension and $A(C)$ the minimal area with contour C .

Minimal surfaces also appear as natural solutions to the Wilson loop averages in two-dimensional gauge theories [8].

To deal with the quarkonium bound state problem, one starts with the two-particle gauge invariant Green function for quarks q_1 and q_2 with different flavors and with masses m_1 and m_2 :

$$G(x_1, x_2; x'_1, x'_2) \equiv \langle \bar{\psi}_2(x_2) U(x_2, x_1) \psi_1(x_1) \bar{\psi}_1(x'_1) U(x'_1, x'_2) \psi_2(x'_2) \rangle_{A, q_1, q_2}. \quad (4)$$

Here, $U(x_2, x_1)$ is the path-ordered phase factor,

$$U(x_2, x_1) = P e^{-ig \int_{x_1}^{x_2} dz^\mu A_\mu(z)}, \quad (5)$$

taken along the straight-line $x_1 x_2$ (and similarly for $U(x'_1, x'_2)$). Integrating in the large- N_c limit with respect to the quark fields, one obtains:

$$G(x_1, x_2; x'_1, x'_2) = -\langle \text{tr}_c U(x_2, x_1) S_1(A; x_1, x'_1) U(x'_1, x'_2) S_2(A; x'_2, x_2) \rangle_A, \quad (6)$$

where $S_1(A)$ and $S_2(A)$ are the quark and antiquark propagators in the presence of the external gluon field and tr_c designates the trace with respect to the color group. The quark propagator $S(A)$ satisfies the equation

$$(i\gamma.\partial_x - m - g\gamma.A(x))S(A; x, x') = i\delta^4(x - x'). \quad (7)$$

At this stage, the Green function G can schematically be represented as in Fig. 1.

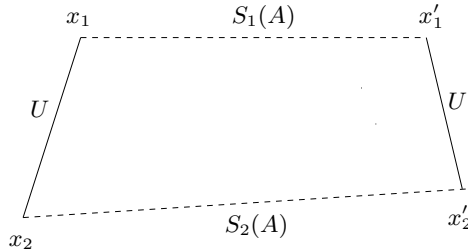


Figure 1: Schematic representation of the two-particle Green function.

In order to make the Wilson loop structure of G apparent, we adopt for the quark propagator in the external gluon field a representation based on an explicit use of the phase factor along straight lines [7]. Introducing the gauge covariant composite object $\tilde{S}(A; x, x')$, made of a free fermion propagator $S_0(x - x')$ (without color group content) multiplied by the path-ordered phase factor $U(x, x')$ [Eq. (5)] taken along the straight segment $x'x$,

$$\tilde{S}(A; x, x') \equiv S_0(x - x')U(x, x'), \quad (8)$$

one shows that the quark propagator $S(A; x, x')$ in the external gluon field satisfies the following functional integral equation in terms of \tilde{S} :

$$S(A; x, x') = \tilde{S}(A; x, x') - \int d^4x'' S(A; x, x'')\gamma^\alpha \int_0^1 d\lambda (1 - \lambda) \frac{\delta}{\delta x^\alpha(\lambda)} \tilde{S}(A; x'', x'), \quad (9)$$

where the segment $x''x'$ has been parametrized with the parameter λ as $x(\lambda) = (1 - \lambda)x'' + \lambda x'$ and where the operator $\delta/\delta x^\alpha(\lambda)$ acts on the factor U of \tilde{S} , along the internal part of the segment $x''x'$, with x' held fixed. That equation is diagrammatically represented in Fig. 2.

A similar equation in which the roles of x and x' are interchanged also holds. Those equations lead to iteration series for S in which the gauge covariance property is maintained at each order of the iteration.

Use of the above representations for the quark propagators in Eq. (6) leads for the two-particle Green function to a series expansion where each term contains a Wilson loop

$$\underline{S(A)} = \underline{\tilde{S}(A)} + \begin{array}{c} S(A) \\ \diagup \\ \text{---} \\ \diagdown \\ \tilde{S}(A) \end{array}$$

Figure 2: Diagrammatic representation of the integral equation satisfied by the quark propagator in the external gluon field. The cross represents the action of the functional derivative $\delta/\delta x(\lambda)$.

along a skew-polygon:

$$G = \sum_{i,j=1}^{\infty} G_{i,j}, \quad (10)$$

where $G_{i,j}$ represents the contribution of the term of the series having $(i - 1)$ points of integration between x_1 and x'_1 (i segments) and $(j - 1)$ points of integration between x_2 and x'_2 (j segments). We designate by $C_{i,j}$ the contour associated with the term $G_{i,j}$. A typical configuration for the contour of $G_{4,3}$ is represented in Fig. 3.

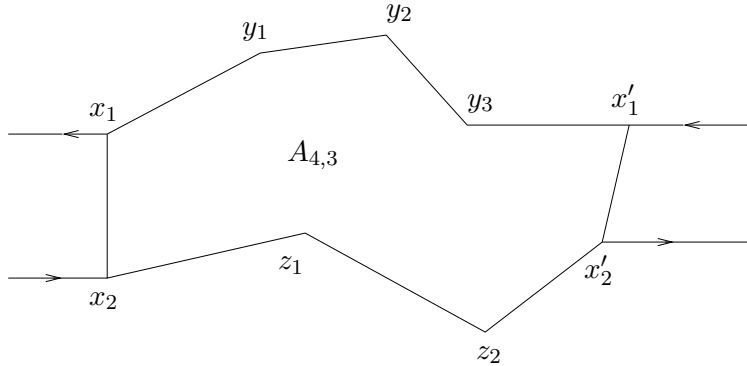


Figure 3: Contour $C_{4,3}$ associated with the term $G_{4,3}$. $A_{4,3}$ is the minimal surface with contour $C_{4,3}$.

Each segment of the quark lines supports a free quark propagator and except for the first segments (or the last ones, depending on the representation that is used) the Wilson loop is submitted to one functional derivative on each such segment. One then uses for the averages of the Wilson loops appearing in the above series the representation with minimal surfaces [Eq. (3)].

The Green function G satisfies the following equation with respect to the Dirac operator of particle 1 acting on x_1 :

$$(i\gamma \cdot \partial_{(x_1)} - m_1)G(x_1, x_2; x'_1, x'_2) = -i \langle \text{tr}_c U(x_2, x_1) \delta^4(x_1 - x'_1) U(x'_1, x'_2) S_2(x'_2, x_2) \rangle_A$$

$$-i\gamma^\alpha \langle \text{tr}_c \int_0^1 d\sigma (1-\sigma) \frac{\delta U(x_2, x_1)}{\delta x^\alpha(\sigma)} S_1(x_1, x'_1) U(x'_1, x'_2) S_2(x'_2, x_2) \rangle_A, \quad (11)$$

where the segment x_1x_2 has been parametrized with the parameter σ as $x(\sigma) = (1 - \sigma)x_1 + \sigma x_2$; furthermore, the operator $\delta/\delta x^\alpha$ does not act on the explicit boundary point x_1 of the segment, this contribution having been cancelled by the contribution of the gluon field A coming from the quark propagator S_1 . A similar equation also holds with the Dirac operator of particle 2 acting on x_2 . Representation (9) for the quark propagator can then be used in the above equation and its partner satisfied by the two-particle Green function G . One obtains two compatible equations for G where the right-hand sides involve the series of the terms $G_{i,j}$ of Eq. (10) and their functional derivative along the segment x_1x_2 . In order to obtain bound state equations, it is necessary to reconstruct in the right-hand sides the bound state poles contained in G [9]. In x -space, bound states are reached by taking the large separation time limit between the pair of points (x_1, x_2) and (x'_1, x'_2) [10]. To produce a bound state pole, it is necessary that there be a coherent sum of contributions coming from each $G_{i,j}$, since the latter, taken individually, do not have poles. Each $G_{i,j}$ involves a corresponding minimal surface $A_{i,j}$ on the contour of which act various functional derivatives. Those can be classified according to their possible irreducibility properties. Reducible contributions are those which are parts of the definition of the series of G . It does not seem possible to sum all these terms to reproduce exactly G with some kernel acting on it in the right-hand sides. However, for large separation time limits one can isolate terms that contribute to the pole terms. One notices that the derivative along the segment x_1x_2 acts on areas $A_{i,j}$ with contour $C_{i,j}$ which are different from one term of the series to the other (the number of segments being different). To have a coherent sum of those contributions it is necessary to expand each such derivative term around the derivative of the lowest-order contour $C_{1,1}$, represented in Fig. 4.

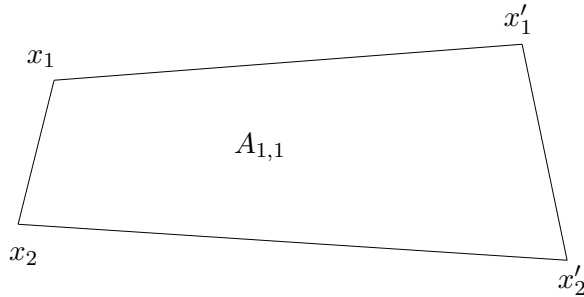


Figure 4: The lowest-order contour $C_{1,1}$ and its minimal surface $A_{1,1}$.

It is that term that can be factorized and can lead through the summation of the

factored series to the reappearance of the Green function G and to its poles. The remaining terms do not lead to pole terms. Similarly, two derivative contributions should be expanded around the lowest-order contribution coming from the contours $C_{2,1}$ or $C_{1,2}$, and so forth.

In general, the derivative of the areas along x_1x_2 depends among others on the slope of the areas in the orthogonal direction to x_1x_2 . One then associates that slope with the quark momenta. Taking then the large separation time limit and equal times in the center-of-mass frame, one ends up with a covariant three-dimensional equation, having the structure of a Breit–Salpeter type equation [11, 12] and where the interaction kernels or potentials are given by various functional derivatives involving at least one derivative along the segment x_1x_2 . Keeping for the potentials the terms containing one functional derivative of the area $A_{1,1}$ [Fig. 1], the equation takes the form [7]

$$\left[P_0 - (h_{10} + h_{20}) - \gamma_{10}\gamma_1^\mu A_{1\mu} - \gamma_{20}\gamma_2^\mu A_{2\mu} \right] \psi(\mathbf{x}) = 0, \quad (12)$$

where ψ is a 4×4 matrix wave function of the relative coordinate $x = x_2 - x_1$ considered at equal times, P_0 the center-of-mass total energy and h_{10} and h_{20} the quark and antiquark Dirac hamiltonians; the Dirac matrices of the quark (with index 1) act on ψ from the left, while the Dirac matrices of the antiquark (with index 2) act on ψ from the right. The potentials A_1 and A_2 are defined through the equations

$$A_{1\mu} = \sigma \int_0^1 d\sigma' (1 - \sigma') \frac{\delta A_{1,1}}{\delta x^\mu(\sigma')}, \quad A_{2\mu} = \sigma \int_0^1 d\sigma' \sigma' \frac{\delta A_{1,1}}{\delta x^\mu(\sigma')}, \quad (13)$$

$x(\sigma')$ belonging to the segment x_1x_2 .

The time components of A_1 and A_2 add up in the wave equation. For their sum, one has the expression (in the c.m. frame)

$$\begin{aligned} A_{10} + A_{20} = & \sigma r \frac{E_1 E_2}{E_1 + E_2} \left\{ \left(\frac{E_1}{E_1 + E_2} \epsilon(p_{10}) + \frac{E_2}{E_1 + E_2} \epsilon(p_{20}) \right) \right. \\ & \times \sqrt{\frac{r^2}{\mathbf{L}^2}} \left(\arcsin \left(\frac{1}{E_2} \sqrt{\frac{\mathbf{L}^2}{r^2}} \right) + \arcsin \left(\frac{1}{E_1} \sqrt{\frac{\mathbf{L}^2}{r^2}} \right) \right) \\ & \left. + (\epsilon(p_{10}) - \epsilon(p_{20})) \left(\frac{E_1 E_2}{E_1 + E_2} \right) \left(\frac{r^2}{\mathbf{L}^2} \right) \left(\sqrt{1 - \frac{\mathbf{L}^2}{r^2 E_2^2}} - \sqrt{1 - \frac{\mathbf{L}^2}{r^2 E_1^2}} \right) \right\}. \end{aligned} \quad (14)$$

Here, $r = \sqrt{\mathbf{x}^2}$, $E_a = \sqrt{m_a^2 + \mathbf{p}^2}$, $a = 1, 2$, with m_a the quark masses, \mathbf{p} the c.m. momentum, $\mathbf{p} = (\mathbf{p}_2 - \mathbf{p}_1)/2$, \mathbf{L} the c.m. orbital angular momentum, and $\epsilon(p_{10})$ and $\epsilon(p_{20})$ the energy sign operators of the free quark and the antiquark, respectively:

$$\epsilon(p_{a0}) = \frac{h_{a0}}{E_a}, \quad a = 1, 2. \quad (15)$$

The space components of A_1 and A_2 are orthogonal to \mathbf{x} . The expression of \mathbf{A}_1 is (in the c.m. frame):

$$\begin{aligned} \mathbf{A}_1 = & -\sigma r \frac{E_1 E_2}{E_1 + E_2} \left\{ \frac{r^2}{2\mathbf{L}^2} \frac{E_1 E_2}{E_1 + E_2} \mathbf{p}^t \right. \\ & \times \sqrt{\frac{r^2}{\mathbf{L}^2}} \left(\arcsin \left(\frac{1}{E_2} \sqrt{\frac{\mathbf{L}^2}{r^2}} \right) + \arcsin \left(\frac{1}{E_1} \sqrt{\frac{\mathbf{L}^2}{r^2}} \right) \right) \\ & + \frac{1}{E_2} \mathbf{p}^t \left(\frac{E_1 E_2}{E_1 + E_2} \right) \left(\frac{r^2}{\mathbf{L}^2} \right) \left(\sqrt{1 - \frac{\mathbf{L}^2}{r^2 E_2^2}} - \sqrt{1 - \frac{\mathbf{L}^2}{r^2 E_1^2}} \right) \\ & \left. - \frac{1}{2} \mathbf{p}^t \left(\frac{r^2}{\mathbf{L}^2} \right) \left(\frac{E_1}{E_1 + E_2} \sqrt{1 - \frac{\mathbf{L}^2}{r^2 E_2^2}} + \frac{E_2}{E_1 + E_2} \sqrt{1 - \frac{\mathbf{L}^2}{r^2 E_1^2}} \right) \right\}. \end{aligned} \quad (16)$$

Here, \mathbf{p}^t is the transverse part of \mathbf{p} with respect to \mathbf{x} :

$$\mathbf{p}^t = \mathbf{p} - \mathbf{x} \frac{1}{\mathbf{x}^2} \mathbf{x} \cdot \mathbf{p}. \quad (17)$$

The expression of \mathbf{A}_2 is obtained from that of \mathbf{A}_1 by an interchange in the latter of the indices 1 and 2 and a change of sign of \mathbf{p}^t .

For sectors of quantum numbers where $\mathbf{L}^2 = 0$, the expressions of the potentials become:

$$A_{10} + A_{20} = \frac{1}{2} (\epsilon(p_{10}) + \epsilon(p_{20})) \sigma r, \quad (18)$$

$$\mathbf{A}_1 = -\frac{1}{E_1 E_2} \left(\frac{1}{3} (E_1 + E_2) - \frac{1}{2} E_1 \right) \mathbf{p}^t \sigma r,$$

$$\mathbf{A}_2 = +\frac{1}{E_1 E_2} \left(\frac{1}{3} (E_1 + E_2) - \frac{1}{2} E_2 \right) \mathbf{p}^t \sigma r. \quad (19)$$

The potentials are generally momentum dependent operators and necessitate an appropriate ordering of terms.

From the structure of the wave equation (12) and the expressions of the potentials, one deduces that the interaction is confining and of the vector type. However, compared to the conventional timelike vector potential, it has additional pieces of terms contributing to the orbital angular momentum dependent parts. A closer analysis of those terms shows that they can be interpreted as being originated from the moments of inertia of the segment $x_1 x_2$ carrying a constant linear energy density equal to the string tension. The interaction potentials are therefore provided by the energy-momentum vector of the segment joining the quark to the antiquark, in similarity with the color flux tube picture of confinement. An analogous equation had also been proposed by Olsson *et al.* on the basis of a model where the quarks are attached at the ends of a straight string or a color flux tube [13, 14]. A similar conclusion had also been reached by Brambilla, Prospero *et al.* on the basis of

the analysis of the relativistic corrections to the nonrelativistic limit of the Wilson loop [15, 16, 17].

For heavy quarks, one can expand equation (12) around the nonrelativistic limit and obtain the hamiltonian to order $1/c^2$ [7].

The relativistic corrections to the interquark potential arising from the Wilson loop were analyzed and evaluated in the literature by Eichten and Feinberg [18], Gromes [19], Brambilla, Prosperi *et al.* [15, 16, 17], Brambilla, Pineda, Soto and Vairo [20].

The Wilson loop approach was also used for the study of quarkonium systems by Dosch, Simonov *et al.* with the use of the stochastic vacuum model [21].

In conclusion, the saturation of the Wilson loop averages in the large- N_c limit by minimal surfaces provides a systematic tool for investigating the large-distance dynamics of quark-antiquark bound state systems.

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