Invariance group important for the interpretation of Bose-Einstein correlations

K. Zalewski

M.Smoluchowski Institute of Physics Jagellonian University, Cracow and

Institute of Nuclear Physics of the Polish Academy of Sciences

May 8, 2019

Abstract

A group of transformations changing the phases of the elements of the single-particle density matrix, but leaving unchanged the predictions for identical particles concerning the momentum distributions, momentum correlations etc., is identified. Its implications for the determinations of the interaction regions from studies of Bose-Einstein correlations are discussed.

PACS 25.75.Gz, 13.65.+i Bose-Einstein correlations.

1 Introduction

Bose-Einstein correlations are helpful when trying to derive properties of the interaction regions from the measured momentum distributions. In this report we point out an ambiguity inherent in such derivations [\[1\]](#page-6-0). There are many ways from the data to the inferred properties of the interaction regions (cf. e.g. [\[2\]](#page-6-1) and references given there). One can use density matrices, Wigner functions, emission functions, distances between the pairs of points where the identical particles are produced etc. The ambiguity seems to be common to all of them.

We will not discuss here the evolution of the density matrix during the freeze-out period. Formally, one can introduce the assumption that all the hadrons have been produced instantaneously and simultaneously at some time which we may choose as $t=0$. Then what one measures is the density matrix (in the interaction representation) at freeze-out.

As is well known, the diagonal elements of the k-particle density matrix in the momentum representation give, and are unambiguously given by, the k particle momentum distribution. On the other hand, in most models these elements can be expressed as symmetrized products of the single particle density matrix elements [\[3\]](#page-6-2):

$$
\rho(\mathbf{p}_1,\ldots,\mathbf{p}_k;\mathbf{p}_1,\ldots,\mathbf{p}_k) = \sum_{P} \prod_{j=1}^k \rho_1(\mathbf{p}_j;\mathbf{p}_{Pj}),
$$
\n(1)

where the summation is over all the k! permutations of the momenta $\mathbf{p}_1, \ldots, \mathbf{p}_k$. Thus, all the momentum distributions are unambiguously determined when the single particle density matrix $\rho_1(\mathbf{p}_1; \mathbf{p}_2)$ is known.

Our main observation [\[1\]](#page-6-0) is that the converse is not true. Given the momentum distributions for all the sets of $k = 1, 2, \ldots$ particles, it is not possible to find unambiguously the matrix ρ_1 . This trivial observation will be seen to have very non trivial consequences. Since the matrix ρ_1 is further used the derive conclusions about the interaction region, the ambiguity affects our capacity for making unambiguous statements about such regions.

2 Invariance group

Consider the transformation

$$
\rho_1(\mathbf{p}; \mathbf{p'}) \to \rho_{1\alpha}(\mathbf{p}; \mathbf{p'}) \equiv e^{i\alpha(\mathbf{p})} \rho_1(\mathbf{p}; \mathbf{p'}) e^{-i\alpha(\mathbf{p'})}, \tag{2}
$$

where α is any real-valued function of momentum. According to formula [\(1\)](#page-1-0) for $k = 1$, ρ_1 is a single particle density matrix. Therefore, it must be hermitian and have trace one. Also $\rho_{1\alpha}$, as seen from its definition, is hermitian and has trace one. Consequently, it can be substituted for ρ_1 on the right-hand side of formula [\(1\)](#page-1-0). This introduces on the right-hand side, for every p_i a term $e^{i\alpha(\mathbf{p}_i)}$ and a term $e^{-i\alpha(\mathbf{p}_i)}$ which cancels it. Thus, the diagonal matrix element on the left-hand side does not change. Experimentally, the substitution of $\rho_{1\alpha}$ for ρ_1 is invisible. The transformations from ρ_1 to $\rho_{1\alpha}$ form a local (in momentum space) $U(1)$ invariance group.

There are several quantities related to the single particle density matrix in the momentum representation and yielding information about the interaction region. In order to get the space distribution of the sources one can use

the diagonal elements of the single particle density matrix in the coordinate representation

$$
\tilde{\rho}_1(\mathbf{x}; \mathbf{x}) = \int d^3 K \frac{d^3 q}{(2\pi)^3} e^{i\mathbf{q} \cdot \mathbf{x}} \rho_1(\mathbf{p}; \mathbf{p}^*), \tag{3}
$$

where

$$
\mathbf{K} = \frac{1}{2}(\mathbf{p} + \mathbf{p}'); \qquad \mathbf{q} = \mathbf{p} - \mathbf{p}.
$$
 (4)

In order to get an approximate phase-space distribution one can use the Wigner function

$$
W_1(\mathbf{K}, \mathbf{X}) = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q} \cdot \mathbf{X}} \rho_1(\mathbf{p}; \mathbf{p}^*), \tag{5}
$$

or the emission function

$$
\rho_1(\mathbf{p}; \mathbf{p'}) = \int d^4 X S(K, X) e^{iqX}.
$$
\n(6)

In the last two formulae $X=\frac{1}{2}$ $\frac{1}{2}(x+x')$ and the approximation consists in interpreting K and X as the energy-momentum and space-time position of the particle. This approximation is not always good, but in a well-defined sense [\[4\]](#page-6-3) this is the best one can have without contradicting the principles of quantum mechanics. In the last formula the four-vector K has the same value on both sides of the equality. Since the momenta p and p' are on mass shell, for any q the value of q_0 is fixed by the condition $Kq = 0$. Thus, it is not possible to invert the Fourier transform and to express the emission function S in terms of the density matrix ρ_1 . In fact, there is an infinity of different emission functions, corresponding to various off mass shell continuations of a given on mass shell function ρ_1 . We conclude that the ambiguities when using the emission function formalism are more severe than when using Wigner functions.

The point is that the transition from ρ_1 to $\rho_{1\alpha}$, which has no effect on the momentum distributions, changes the functions $\tilde{\rho}_1$, W_1 and S and consequently changes the conclusions concerning the interaction region. In the following section we will illustrate this fact by some examples.

3 Examples

.

The class of transformations [\(2\)](#page-1-1) is very rich. We will just discuss three simple examples. Consider

$$
\alpha(\mathbf{p}) = \mathbf{b} \cdot \mathbf{p} \qquad \Rightarrow \qquad \alpha(\mathbf{p}) - \alpha(\mathbf{p'}) = \mathbf{b} \cdot \mathbf{q}, \tag{7}
$$

where **b** is any vector. This gives

$$
\tilde{\rho}_1(\mathbf{x}; \mathbf{x}) = \int dK \frac{dq}{(2\pi)^3} e^{i\mathbf{q} \cdot (\mathbf{x} + \mathbf{b})} \rho_1(\mathbf{p}; \mathbf{p}'). \tag{8}
$$

The interaction region gets shifted by b. Similarly, replacing b and p by four-vectors and using the emission function one can generate an arbitrary shift in space-time. This result is of little interest. It is obvious that the momentum distributions do not depend on where and when the experiment was done.

As our second example consider

$$
\alpha(\mathbf{p}) = \frac{1}{2}c\mathbf{p}^2 \qquad \Rightarrow \qquad \alpha(\mathbf{p}) - \alpha(\mathbf{p'}) = c\mathbf{K} \cdot \mathbf{q}, \tag{9}
$$

where c is any real number. Then

$$
W_{1\alpha}(\mathbf{K}, \mathbf{X}) = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q} \cdot (\mathbf{X} + c\mathbf{K})} \rho_1(\mathbf{p}; \mathbf{p}^{\prime}). \tag{10}
$$

This time the shift is proportional to \bf{K} with a proportionality coefficient which is unconstrained by momentum measurements. The corresponding distribution in space is

$$
\tilde{\rho}_{1\alpha}(\mathbf{x}; \mathbf{x}) = \int d^3 K W_{1\alpha}(\mathbf{K}, \mathbf{x}) \tag{11}
$$

Suppose now that for $c = 0$ there are no position-momentum correlations. Then for each \bf{K} the interaction region occupies the same portion of space and when averaged over **K** coincides with that for any given **K**. For $c \neq 0$, the interaction regions corresponding to different values of \bf{K} are shifted with respect to each other and the averaged size of the interaction region gets bigger. Its increase with respect to the situation at $c = 0$ depends on the value of $|c|$. Since this is unconstrained by the data on momentum distributions, the radius can be made as large as one wishes.

For instance, for the Gaussian

$$
\rho_1(\mathbf{p}; \mathbf{p'}) = \frac{1}{(\sqrt{2\pi\Delta^2})^3} \exp\left[-\frac{\mathbf{K}^2}{2\Delta^2} - \frac{1}{2}R^2\mathbf{q}^2\right]
$$
(12)

one obtains $\tilde{\rho}_{1\alpha}(\mathbf{x}; \mathbf{x})$ also Gaussian with root mean square width

$$
R_{\alpha}^2 = R^2 + c^2 \Delta^2. \tag{13}
$$

In order to see that this broadening is due to the averaging over \bf{K} it is enough to have a look at the corresponding Wigner function

$$
W_{1\alpha}(\mathbf{K}, \mathbf{X}) = \frac{1}{(2\pi R\Delta)^3} \exp\left[-\frac{\mathbf{K}^2}{2\Delta^2} - \frac{(\mathbf{X} + c\mathbf{K})^2}{2R^2}\right].
$$
 (14)

The conclusion from this example is that without further assumptions one can at best obtain a lower limit for the radius of the interaction region. In practice, everybody uses, more or less consciously, a model which supplies the necessary additional assumptions. The problem how to choose the best model among those which give exactly the same fit to all the data is an interesting open problem. In order to show that this problem is not purely academic let us consider the following example from the literature.

Some models [\[5\]](#page-7-0), [\[6\]](#page-7-1) assume correlations between position in space-time and energy-momentum of the type¹

$$
K^{\mu} = \lambda X^{\mu}.
$$
 (15)

A convenient notation is

$$
X_0^2 - X_{\parallel}^2 = \tau^2; \qquad K_0^2 - K_{\parallel}^2 = M_T^2 \qquad \Rightarrow \qquad \lambda = \frac{M_T}{\tau}.
$$
 (16)

Note that here M_T is a temporal-longitudinal variable. Let us assume that [\[6\]](#page-7-1)

$$
S = S_{\parallel} S_T \tag{17}
$$

where

$$
S_T = \exp\left[-\frac{\mathbf{X}_T^2}{2r_T^2} - \frac{(\mathbf{K}_T - \lambda \mathbf{X}_T)^2}{2\delta_T^2}\right].\tag{18}
$$

This is a possible quantum-mechanical rendering of the transverse part of the classical relation [\(15\)](#page-4-0). The only information about S_{\parallel} important for our

¹Sometimes called Hubble flow

purpose is that it depends neither on K_T nor on X_T . The distribution of sources in the transverse plane is obtained by integrating S_T over \mathbf{K}_T . The result is a Gaussian with constant root mean square width r_T . On the other hand S_T can be rewritten in the form

$$
S_T = \exp\left[-\frac{\phi_T^2}{2R_D^2} - \frac{(\mathbf{X}_T - \phi_T)^2}{2R_\phi^2}\right],
$$
\n(19)

where

$$
R_{\phi} = \frac{r_T}{\sqrt{1+\mu^2}}; \qquad R_D = \mu R_{\phi}; \qquad \mu = \frac{r_T}{\tau \delta_T} M_T; \qquad \phi_T = r_T \frac{\mu}{1+\mu^2} \frac{\mathbf{K}_T}{\delta_T}.
$$
\n(20)

This can be considered² as a transform with $\alpha(\mathbf{p}) = \frac{1}{2}c\mathbf{p}^2$ and

$$
c = \frac{r_T \mu}{\delta_T (1 + \mu^2)}\tag{21}
$$

of

$$
S_{\alpha T} = \exp\left[-\frac{\mathbf{K}_T^2}{2\delta_T^2} - \frac{\mathbf{X}_T^2}{2R_\phi^2}\right].\tag{22}
$$

Performing the integration of $S_{\alpha T}$ over \mathbf{K}_T one finds for the distribution of sources in the transverse plane a Gaussian with root mean square width

$$
R_{T\alpha}^2 = R_{\phi}^2 = \frac{r_T^2}{1 + \mu^2},\tag{23}
$$

which exhibits the familiar decrease of the transverse radius with the transverse mass as reported by so many experimental papers. Let us summarize the situation: if our prejudice is in favor of a Hubble flow as interpreted in [\[6\]](#page-7-1), we conclude that the transverse radius does not depend on M_T ; if our prejudice is against correlations between position in the transverse plane and transverse momentum, we conclude that the transverse radius decreases with increasing M_T ; experimental data on momentum distributions will not help us to decide which of these two prejudices is the right one.

Our last example is one dimensional. It could e.g. apply to one transverse component. We choose the Gaussian density matrix [\(12\)](#page-4-1) and

$$
\alpha(p) = \frac{4}{3a^3}p^3 \qquad \Rightarrow \qquad \alpha(p) - \alpha(p') = \frac{4}{3a^3}q\left(K^2 + \frac{q^3}{3}\right). \tag{24}
$$

²This is an approximation, because c depends on M_T , it is, however, good enough for our qualitative discussion; compare [\[6\]](#page-7-1), where the full calculation can be found

A simple calculation [\[1\]](#page-6-0) yields

$$
S_{\alpha} = \frac{a}{\sqrt{2\pi\Delta}} \exp\left[-\frac{K^2}{2\Delta^2} + B\right] Ai(A), \tag{25}
$$

where $Ai(\ldots)$ is the Airy function and

$$
A = a\tilde{X} + \frac{\omega^4}{4}; \qquad B = \frac{\omega^2}{2} \left[A - \frac{\omega^4}{12} \right]; \qquad \omega = aR; \qquad \tilde{X} = X - \frac{4}{a^3} K^2.
$$
\n(26)

For large values of a the emission function is almost Gaussian. A numerical calculation shows that $a = 2/R$ is already large enough. For smaller values of a, however, at negative \tilde{X} the emission function develops big wiggles, oscillating between positive and negative values. Its shape in the positive X region also significantly changes [\[1\]](#page-6-0). This example shows how the transformations discussed in the present report can lead to changes of the interaction region which are much more complicated than just momentum dependent shifts.

4 Conclusions

The experimental data about momentum distributions tell us little about the interaction regions, unless additional assumptions are made. The usual recommendations: reduce the experimental errors, include more particle correlations etc. are not enough. Exactly the same fit can be obtained from widely different models, differing in these additional assumptions and giving conflicting information about the interaction region. The caveat for model users is that only some of the assumptions of the model are being tested by comparison with the data, while others, which may be very important for drawing inferences about the interaction region, are unconstrained by the data.

References

- [1] A. Bialas and K. Zalewski, [hep-ph/0501017.](http://arxiv.org/abs/hep-ph/0501017)
- [2] U.A. Wiedemann and U. Heinz, Phys. Rep. 319(1999)145.
- [3] J. Karczmarczuk, Nucl. Phys. B78(1974)370.
- [4] M. Hillery et al., *Phys. Rep.* **106**(1984)121.
- [5] T. Csörgo and J Zimányi, Nucl. Phys., $A517(1990)588$.
- $[6]$ A. Bialas et al., $Phys. \ Rev.$ ${\bf D62} (2000) 114007.$