

Rephasing Invariance and Hierarchy of the CKM Matrix

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Abstract

We identified a set of four rephasing invariant parameters of the CKM matrix. They are found to exhibit hierarchies in powers of λ^2 , from λ^2 to λ^8 . It is shown that, at the present level of accuracy, only the first three parameters are needed to fit all available data on flavor physics.

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In the standard model, the essential input in understanding flavor physics and CP violation is given by the Cabibbo-Kobayashi-Maskawa (CKM) matrix, V_{CKM} , or rather, by the rephasing invariant portions thereof, since one can always change the phases (rephasing) of quark fields without affecting the physics. What is remarkable is that, to this date, all observations, both CP-conserving and CP-violating, while greatly over-constraining the parameters in V_{CKM} , are nevertheless completely in accord with this picture. Another remarkable feature of V_{CKM} is its hierarchical structure, characterized by the Wolfenstein parameter [1], $\lambda = |V_{us}| \cong 0.22$. This implies that flavor processes can be naturally classified by their strengths in powers of λ (e.g. CP-violating phenomena only appear at $O(\lambda^6)$ or higher). In this paper we make use of a manifestly rephasing invariant parametrization introduced in a previous work [2]. It consists of six parameters, (x_i, y_j) , $i, j = 1, 2, 3$, which satisfy two constraints, so that any four of them can be used as a complete set. Using known measurements, we find that three of those (y_2, y_1, x_2) are of order $O(\lambda^2)$, $O(\lambda^4)$ and $O(\lambda^6)$, respectively. There is also considerable evidence that a fourth, y_3 , is of order $O(\lambda^8)$, and is consistent with zero, at the present level of accuracy in the determination of V_{CKM} . It is then shown that, for V_{CKM} , the set of three parameters (y_2, y_1, x_2) , with $y_3 = 0$, suffices to describe all extant data on flavor physics. An accurate assessment of the value of y_3 can only be done after more precise measurements become available.

We begin by summarizing the main properties of the (x, y) parametrization and detailing its relations with other parametrizations which are in common use.

We assume, without loss of generality, that

$$\det V_{CKM} = +1. \tag{1}$$

There are then six rephasing invariants defined by

$$\Gamma_{ijk} = V_{1i}V_{2j}V_{3k}, \quad (2)$$

where (i, j, k) is a permutation of $(1, 2, 3)$. It was proved that all six Γ_{ijk} 's have the same imaginary part, $-iJ$, where J is the invariant CP measure [3], so that

$$\Gamma_{ijk} = \text{Re}\Gamma_{ijk} - iJ. \quad (3)$$

It is useful to separate the even and odd permutation Γ 's and define

$$\text{Re}(\Gamma_{123}, \Gamma_{231}, \Gamma_{312}) = (x_1, x_2, x_3); \quad (4)$$

$$\text{Re}(\Gamma_{132}, \Gamma_{213}, \Gamma_{321}) = (y_1, y_2, y_3). \quad (5)$$

They are found to satisfy two constraints,

$$(x_1 + x_2 + x_3) - (y_1 + y_2 + y_3) = 1, \quad (6)$$

$$x_1x_2 + x_2x_3 + x_3x_1 = y_1y_2 + y_2y_3 + y_3y_1. \quad (7)$$

Thus, any four of the set (x_i, y_j) may be used as a complete set of parameters of V_{CKM} . In addition, it was also established that

$$J^2 = x_1x_2x_3 - y_1y_2y_3. \quad (8)$$

All of the parameters (x_i, y_j) take values between ± 1 , $-1 \leq (x_i, y_j) \leq +1$, with $y_j \leq x_i$, for any (i, j) .

We now turn to the relations between the set (x, y) and other familiar parametrizations. The simplest is that with $|V_{ij}|^2$ [4], the absolute square of the elements of V_{CKM} . It is given by

$$W = \begin{pmatrix} |V_{11}|^2 & |V_{12}|^2 & |V_{13}|^2 \\ |V_{21}|^2 & |V_{22}|^2 & |V_{23}|^2 \\ |V_{31}|^2 & |V_{32}|^2 & |V_{33}|^2 \end{pmatrix} \quad (9)$$

$$= \begin{pmatrix} x_1 - y_1 & x_2 - y_2 & x_3 - y_3 \\ x_3 - y_2 & x_1 - y_3 & x_2 - y_1 \\ x_2 - y_3 & x_3 - y_1 & x_1 - y_2 \end{pmatrix} \quad (10)$$

It is also straight-forward to obtain the relations between (x, y) and the “standard” parametrization of the Particle Data Group [5]

$$V^{(s)} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}. \quad (11)$$

Note that $\det V^{(s)} = +1$. However, $V^{(s)}$ is not invariant under rephasing. We will not write down the explicit relations between (x, y) and the angles in $V^{(s)}$. These relations are exact but they tend to be rather cumbersome.

Another well-established parametrization, due to Wolfenstein [1], is often used. It is an expansion of V_{CKM} in the parameter $\lambda = |V_{us}| \cong 0.22$, whose validity is related to the fact that the three angles in $V^{(s)}$ are small and hierarchical. Wolfenstein’s representation of V_{CKM} reads:

$$V^{(W)} = \begin{pmatrix} 1 - \lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \lambda^2/2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4). \quad (12)$$

Note that $\det V^{(W)} = 1 + O(\lambda^2)$, and, like $V^{(s)}$, $V^{(W)}$ is not rephasing invariant. A systematic expansion into higher orders of λ is also available [6], and is often used in the literature. In the spirit of an expansion in λ , we may compute (x, y) by using $\Gamma_{ijk}^{(W)} = V_{1i}^{(W)}V_{2j}^{(W)}V_{3k}^{(W)}$, which form rephasing invariant combinations. We find, to leading order in λ^2 ,

$$\Gamma_{123}^{(W)} = 1 - \lambda^2 = x_1, \quad (13)$$

$$\Gamma_{213}^{(W)} = -\lambda^2 = y_2, \quad (14)$$

$$\Gamma_{132}^{(W)} = -A^2\lambda^4 = y_1. \quad (15)$$

None of these Γ 's contains the imaginary part, $-iJ$, which is $O(\lambda^6)$, although they do give correct values for x_1, y_2 and y_1 , to leading order. On the other hand,

$$\Gamma_{231}^{(W)} = A^2\lambda^6[(1 - \rho) - i\eta] = x_2 - iJ, \quad (16)$$

$$\Gamma_{321}^{(W)} = A^2\lambda^6[(\rho - \rho^2 - \eta^2) - i\eta] = y_3 - iJ, \quad (17)$$

$$\Gamma_{312}^{(W)} = A^2\lambda^6(\rho - i\eta) = x_3 - iJ. \quad (18)$$

All of these quantities are $O(\lambda^6)$, and they have the same imaginary part, with

$$J = A^2\lambda^6\eta, \quad (19)$$

which is a well-known approximate result. Had we used the improved version of $V^{(W)}$ [6], it will be seen that, except for $\Gamma_{123}^{(W)}$, all $\Gamma^{(W)}$'s contain the same imaginary part, $-iJ$.

The above analysis shows that, to leading order in λ^2 , the (x, y) parameters are given in the hierarchical order $x_1 = O(1)$, $x_2 = O(\lambda^6)$, $x_3 = O(\lambda^6)$, $y_1 = O(\lambda^4)$, $y_2 = O(\lambda^2)$ and $y_3 = O(\lambda^6)$. These results are obtained with the tacit assumption that, in the parametrization $V^{(W)}$, the values (A, ρ, η) are all of order one and are unrelated to each other. When we incorporate recent measurements, it turns out that there are correlations and detailed structures which have interesting implications. This we will discuss in the following.

We begin by improving upon our earlier analysis in Ref. [2], and write down more precise relations between the (x, y) parameters and the angles (α, β, γ) of the unitarity triangle. From the unitarity condition,

$$V_{11}V_{13}^* + V_{21}V_{23}^* + V_{31}V_{33}^* = 0, \quad (20)$$

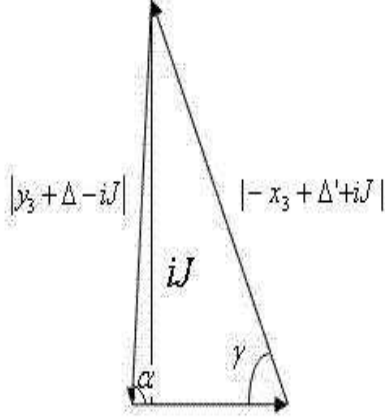


Figure 1: Rescaled unitarity triangle with base $|V_{11}|^2|V_{13}|^2$ and sides as labelled. Also, $\Delta = |V_{13}|^2|V_{31}|^2 = O(\lambda^{12})$, $\Delta' = |V_{13}|^2|V_{21}|^2 = O(\lambda^8)$.

we obtain a rephasing invariant equation by multiplying by $V_{11}^*V_{13}$. The result is a triangle in the complex plane whose height is (exactly) iJ , with one side given by

$$V_{11}^*V_{13}V_{31}V_{32}^* = y_3 + |V_{13}|^2|V_{31}|^2 - iJ. \quad (21)$$

This triangle is plotted in Fig.1.

Since $|V_{13}|^2|V_{31}|^2 \leq O(\lambda^{12})$, together with the estimate (to be discussed later) $y \approx O(\lambda^8)$, we have, from Fig. 1,

$$\tan \alpha = -(J/y_3)(1 + O(\lambda^4)). \quad (22)$$

Similarly, using Ref.[2], we find

$$\tan \beta = (J/x_2)(1 + O(\lambda^4)), \quad (23)$$

$$\tan \gamma = (J/x_3)(1 + O(\lambda^2)). \quad (24)$$

Thus, to a high degree of accuracy, the angles (α, β, γ) are the phase angles of $(\Gamma_{321}, \Gamma_{231}^*, \Gamma_{312}^*)$. (Note that, in Ref.[2], Eq.(60) has a wrong sign, and the statement thereafter contains a typo.)

Experimentally, β has been rather precisely measured, while α and γ are not so well determined. To assess their effects on the (x, y) parameters, we make use of currently available global fits [7,8]. Because the two analyses yield similar results, we will only quote the values from Ref.[7], for simplicity. The values of (α, β, γ) , in degrees, are given by

$$\alpha = 94_{-10}^{+12}, \quad \beta = 24 \pm 2, \quad \gamma = 62_{-12}^{+10}. \quad (25)$$

With these we find

$$x_2/J = 2.24 \pm 0.2, \quad (26)$$

$$x_2/x_3 = 4.3_{-1.3}^{+2.7}, \quad (27)$$

$$y_3/x_2 = 0.03_{-0.07}^{+0.1}, \quad (28)$$

and, using $\gamma = \pi - (\alpha + \beta)$ to correlate γ and α ,

$$y_3/x_3 = 0.13_{-0.4}^{+0.2}. \quad (29)$$

These results suffer from large uncertainties. However, it is clear that the parameters x_2, J, x_3, y_3 , are not of the same order. In particular, we find

$$y_3/x_2 = O(\lambda^2). \quad (30)$$

In fact, the result $y_3/x_2 = O(\lambda^2)$ can be expressed in a variety of forms when we write $y_3 = Re\Gamma_{321}$ in different parametrizations.

In the Wolfenstein parametrization,

$$Re\Gamma_{321} = A^2\lambda^6[\rho - (\rho^2 + \eta^2)]. \quad (31)$$

For $V^{(s)}$, we have

$$\begin{aligned} \text{Re}\Gamma_{321} &= \text{Re}[s_{13}(c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta})(e^{-i\delta}s_{12}s_{23} - c_{12}c_{23}s_{13})] \\ &\cong s_{13}(c_{\delta}s_{12}s_{23} - s_{13}). \end{aligned} \quad (32)$$

Using Eq.(10), $|V_{ub}|^2 = x_3 - y_3$, $|V_{td}|^2 = x_2 - y_3$, it follows that

$$\text{Re}\Gamma_{321} = \frac{1}{2}[-|V_{td}|^2 - |V_{ub}|^2 + |V_{us}|^2|V_{cb}|^2], \quad (33)$$

where we have used the constraint (Eq.(7)) ($x_2 + x_3 \cong |V_{us}|^2|V_{cb}|^2$, with $x_1 \cong 1$). Thus, the condition $y_3/x_2 = O(\lambda^2)$ takes on a variety of forms:

$$1) \quad -\tan\beta/\tan\alpha = O(\lambda^2); \quad (34)$$

$$2) \quad [\rho - (\rho^2 + \eta^2)]/(1 - \rho) = O(\lambda^2); \quad (35)$$

$$3) \quad \cot^2\delta(c_{\delta}s_{12}s_{23} - s_{13})/s_{13} = O(\lambda^2); \quad (36)$$

$$4) \quad \frac{1}{2}(|V_{us}|^2|V_{cb}|^2 - |V_{td}|^2 - |V_{ub}|^2)/|V_{td}|^2 = O(\lambda^2). \quad (37)$$

Here, in 3), we have used the approximate relations $\delta = \gamma$, $\tan\beta \tan\gamma = 1$ to obtain $x_2 = x_3 \tan^2\delta = s_{13}^2 \tan^2\delta$. All of these relations are reasonably well satisfied by using the values of the global CKM fits [7]. They are, approximately, $\rho = 0.19 \pm 0.08$, $\eta = 0.36 \pm 0.05$; $s_{12} = 0.226 \pm 0.002$, $s_{13} = (3.9 \pm 0.3) \times 10^{-3}$, $s_{23} = (41 \pm 1) \times 10^{-3}$, $\delta = 62 \pm 10$; $|V_{us}| = 0.23 \pm 0.002$, $|V_{ub}| = (3.9 \pm 0.3) \times 10^{-3}$, $|V_{cb}| = (41 \pm 1) \times 10^{-3}$, $|V_{td}| = (8.3 \pm 0.8) \times 10^{-3}$. Eqs.(34-37) exhibit the intriguing correlations amongst the V_{CKM} parameters. Without them y_3 would be of the same order as x_2 .

The above analysis can be summarized as an expansion in λ^2 :

$$(-y_2, -y_1, x_2, y_3) = (\lambda^2, A^2\lambda^4, B^2\lambda^6, C\lambda^8), \quad (38)$$

with

$$A^2 = 0.64 \pm 0.05 \quad (39)$$

$$B^2 = 0.5 \pm 0.1. \quad (40)$$

The parameter C is very poorly determined. We use the above relations and make a rough estimate to find

$$C = 0.3 \pm 1. \quad (41)$$

Thus, a convenient parametrization of V_{CKM} is to use the set (y_2, y_1, x_2, y_3) . It gives rise to an expansion in powers of λ^2 , in contrast to the use of $V^{(W)}$ or $V^{(s)}$, whose matrix elements can be regarded as expansions in powers of λ . The difference originates from rephasing considerations. For, when one uses rephasing invariants (which are what enter into physical quantities) constructed out of $V^{(W)}$ or $V^{(s)}$, such as $|V_{ij}|^2$ or $V_{\alpha i} V_{\beta j} V_{\alpha j}^* V_{\beta i}^*$ or (x_i, y_j) , it is seen that the expansion parameters is λ^2 , and not λ . Thus, e.g., $|V_{ij}|^2$ have values which are of order $O(1), O(\lambda^2), O(\lambda^4), O(\lambda^6)$. However, the combination y_3 (Eq.(33)) contains a cancellation, resulting in $y_3 = O(\lambda^8)$. It is this feature which distinguishes the (x, y) set from other parametrizations.

The result $y_3 = O(\lambda^8)$ has another interesting consequence. To the extent that all available measurements on V_{CKM} are only accurate to $O(\lambda^6)$, we should be able to set $y_3 = 0$ and fit the extant data on V_{CKM} by three parameters, (y_2, y_1, x_2) . This will be presented in Fig.2. The inputs are taken from those of the well-known unitarity triangle construction [7,8,9] in terms of ρ and η (since we are only interested in leading order effects, we take $\bar{\rho} = \rho, \bar{\eta} = \eta$). With $y_2 = -A^2 \lambda^4, y_1 = -\lambda^2, y_3 = 0, x_1 = 1 - \lambda^2$, all of the physical quantities $(\epsilon_K, |V_{ub}|^2, |V_{td}|^2, \tan \beta)$ can be converted from functions of (ρ, η) into those of (x_2, x_3) using $A^2 \lambda^6 (1 - 2\rho) = x_2 - x_3, A^2 \lambda^6 \eta = \sqrt{x_2 x_3}$. In addition, we use $J^2 = x_2 x_3$ and the constraint $x_2 + x_3 \cong y_1 y_2$. The values of the physical quantities are taken from Ref.[9]. It is seen that there is a shaded region in the (x_2, x_3) plane where all constraints meet. Fig.2 represents a

three parameter (y_2, y_1, x_2) fit (with $x_3 = y_1 y_2 - x_2$) to existing data on V_{CKM} . Qualitatively, we can understand the viability of a three parameter fit as follows. The parameter η , as determined in Ref.[7], has errors of order $O(\lambda^2)$. Setting $y_3 = 0$, according to Eq.(35), amounts to eliminating η and using $\eta = \sqrt{\rho - \rho^2}$. This relation is satisfied to $O(\lambda^2)$. Thus, Fig.2 represents a fit with the parameter set $(\lambda^2, A^2 \lambda^4, \rho, \eta = \sqrt{\rho - \rho^2})$, and should be robust at the $O(\lambda^2)$ level, as demonstrated. Similar results also follow if we use other parametrizations. The above analysis also suggests that whether a non-vanishing y_3 is needed has to wait until more precision measurements become available.

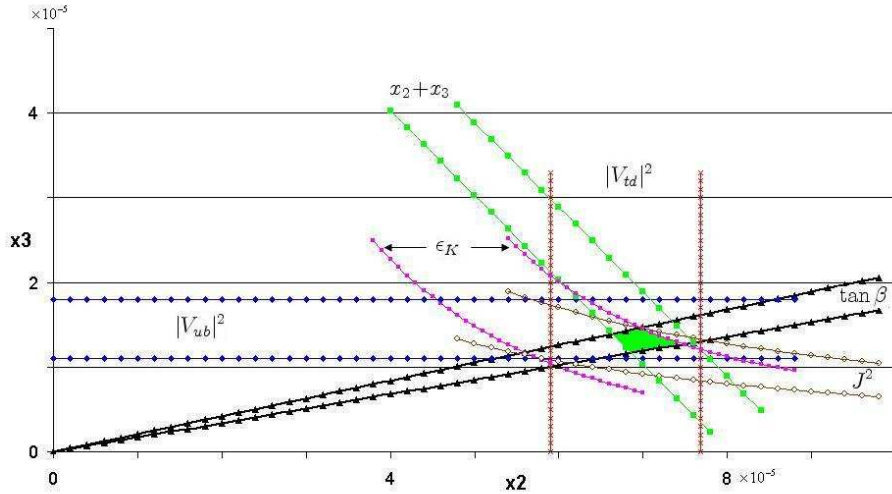


Figure 2: A three parameter fit to V_{CKM} , with fixed $y_1 (= -A^2 \lambda^4)$, $y_2 (= -\lambda^2)$, and the constraint $x_2 + x_3 = y_1 y_2$. Values of the physical quantities are taken from Ref.[9].

We close with a few concluding remarks. In this paper we argue that, although our present knowledge on V_{CKM} is still far from precise, it never-

theless offers considerable evidence of intriguing correlations, as in Eqs.(34-37). We can take advantage of this by using a particular set of parameters, (y_2, y_1, x_2, y_3) , which gives rise to a natural expansion in powers of λ^2 . It is shown that, at the current level of accuracy, we can set $y_3 = O(\lambda^8) \cong 0$ and obtain a three parameter fit to V_{CKM} . Within the currently available data set, the proper way to estimate a “best” value for y_3 would be to do a global fit with the (x, y) variables. However, this is beyond the scope of the present paper. Of course, we would have a better handle in the future, when more and better measurements are performed.

One important motivation in analyzing V_{CKM} is to check its consistency by over-constraining its parameters, with an eye for discrepancies which might originate from “new physics”. It is our hope that a parametrization which has a distinct hierarchy can help to identify features that will stand out, so that the success or failure of a model can be better assessed.

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