

A Consistent Calculation of Bubble-Nucleation Rates

Alessandro Strumia

*Dipartimento di Fisica, Università di Pisa and
INFN, Sezione di Pisa, I-56127 Pisa, Italia*

and

Nikolaos Tetradis

*Scuola Normale Superiore,
Piazza dei Cavalieri 7, I-56126 Pisa, Italia*

Abstract

We present a consistent picture of tunnelling in field theory. Our results apply both to high-temperature field theories in four dimensions and to zero-temperature three-dimensional ones. Our approach is based on the notion of a coarse-grained potential U_k that incorporates the effect of fluctuations with characteristic momenta above a given scale k . U_k is non-convex and becomes equal to the convex effective potential for $k \rightarrow 0$. We demonstrate that a consistent calculation of the nucleation rate must be performed at non-zero values of k , larger than the typical scale of the saddle-point configuration that dominates tunnelling. The nucleation rate is exponentially suppressed by the action S_k of this saddle point. The pre-exponential factor A_k , which includes the fluctuation determinant around the saddle-point configuration, is well-defined and finite. Both S_k and A_k are k -dependent, but this dependence cancels in the expression for the nucleation rate. This picture breaks down in the limit of very weakly first-order phase transitions, for which the pre-exponential factor compensates the exponential suppression.

1 Introduction

The consistent description of first-order phase transitions is a difficult problem which has attracted the attention of statistical and particle physicists for a long time (for a review see ref. [1] and references therein). Our present understanding of these phenomena is based on the work of Langer on nucleation theory [2]. His formalism has been applied to relativistic field theory by Coleman [3] and Callan [4] and extended by Affleck [5] and Linde [6] to finite temperature. The basic quantity in this approach is the nucleation rate, which gives the probability per unit time and volume to nucleate a certain region of the stable phase (the true vacuum) within the metastable phase (the false vacuum). The rate is exponentially suppressed by the free energy of the critical bubble, which is a static configuration (usually assumed to be spherically symmetric) within the metastable phase whose interior consists of the stable phase. This configuration has a certain radius that can be determined from the parameters of the underlying theory. Bubbles slightly larger than the critical one expand rapidly, thus converting the metastable phase into the stable one. Possible deformations of the critical bubble generate a static pre-exponential factor in the nucleation rate. The leading contribution to this factor has the form of a fluctuation determinant. Another dynamical prefactor determines the fast growth rate of the bubbles that are slightly larger than the critical one [2, 7]. In this work we concentrate on the calculation of the static prefactor. We are, therefore, mostly concerned with the rate of nucleation of critical bubbles. Their real-time evolution after their nucleation is a separate question, which we hope to address in a future publication.

The nucleation rate per unit volume I (probability of nucleation of a critical bubble per unit time and volume) for a four-dimensional field theory at temperature T , in the limit that thermal fluctuations dominate over quantum fluctuations, is given by [4]–[6]

$$I = \frac{E_0}{2\pi} \left(\frac{S}{2\pi} \right)^{3/2} \left| \frac{\det'[\delta^2\Gamma/\delta\phi^2]_{\phi=\phi_b}}{\det[\delta^2\Gamma/\delta\phi^2]_{\phi=0}} \right|^{-1/2} \exp(-S). \quad (1.1)$$

Here Γ is the free energy of the system for a given configuration of the field ϕ that acts as the order parameter of the problem. The rescaled free energy of the critical bubble is $S = \Gamma_b/T = [\Gamma(\phi_b(r)) - \Gamma(0)]/T$, where $\phi_b(r)$ is the spherically-symmetric bubble configuration and $\phi = 0$ corresponds to the false vacuum. The fluctuation determinants are evaluated either at $\phi = 0$ or around $\phi = \phi_b(r)$. The prime in the fluctuation determinant around the bubble denotes that the three zero eigenvalues of the operator $[\delta^2\Gamma/\delta\phi^2]_{\phi=\phi_b}$ have been removed. Their contribution generates the factor $(S/2\pi)^{3/2}$ and the volume factor that is absorbed in the definition of I (nucleation rate per unit volume). The quantity E_0 is the square root of the absolute value of the unique negative eigenvalue. This last contribution appears only for the high-temperature theory [5]. It is absent in the expression for the quantum-tunnelling rate in the zero-temperature three-dimensional theory.

The free energy of the critical bubble can be easily determined either analytically or numerically. The bubble is the dominant saddle-point configuration that interpolates between the two vacua. Its profile is determined by a differential equation, which can be integrated numerically or even solved analytically in simple cases. The calculation of the fluctuation determinants in the prefactor is a more difficult task. They can be brought in a more manageable form if one employs spherical coordinates. However, the true difficulty concerns the ultraviolet divergences that are inherent in their calculation. An appropriate regularization scheme must be employed in order to control them [8]–[10].

The situation becomes even more complicated in the case of radiatively induced first-order phase transitions. These are a consequence of the appearance of a new vacuum state in the theory as a result of the integration of (quantum or thermal) fluctuations [11]. In field theory the free energy (more precisely the thermodynamic potential) density of a system for homogeneous configurations is usually identified with the temperature-dependent effective potential. A radiatively induced first-order phase transition appears in theories for which the tree-level potential has only one minimum, while a second minimum appears at the level of radiative corrections, usually computed within a perturbative scheme. This approach, however, faces two fundamental difficulties:

- a) The effective potential, being the Legendre transform of the generating functional for the connected Green functions, is a convex function of the field. Consequently, it does not seem to be the appropriate quantity for the study of tunnelling, as no structure with more than one minima separated by a barrier exists¹.
- b) The fluctuation determinants in the expression for the nucleation rate have a form completely analogous to the one-loop correction to the potential. The question of double-counting the effect of fluctuations (in the potential and the prefactor) must be properly addressed.

In this paper we demonstrate that all the above issues can be resolved through the implementation of the notion of coarse graining in the formalism. The appropriate quantity for the description of the physical system is the *effective average action* [13], which is the generalization in the continuum of the blockspin action of Kadanoff [14]. The dependence of this action on the coarse-graining scale k is described by an exact flow equation [15] typical of the Wilson approach to the renormalization group [16]. The formalism has been applied with success to second-order phase transitions. A complete picture has emerged for the phase transitions in a variety of scalar models, with a reliable determination of both non-universal quantities (such as critical temperatures) and universal ones (such as critical exponents, the equation of state and crossover curves) [17]–[22]. The generalization of the formalism to gauge theories [23] has led to the study of second-order and first-order phase transitions for the Abelian and non-Abelian Higgs models [24]–[27], with implications for the electroweak phase transition [27]. The method has also been applied to the chiral phase transition [28]. The framework for the discussion of nucleation rates in first-order phase transitions has been set in ref.s [29, 30]².

We first summarize the basic notions in the calculation of nucleation rates in our approach. We define the bare theory at some high scale Λ that can be identified with the ultraviolet cutoff. The renormalized theory at lower scales k is described in terms of the effective average action Γ_k , which can be interpreted as the coarse-grained free energy at a given scale k . Fluctuations with characteristic momenta $q^2 \gtrsim k^2$ are integrated out and their effect is incorporated in Γ_k . The k dependence of Γ_k is determined by an exact flow equation. This can be translated into evolution equations for the invariants appearing in a derivative expansion of the action. We consider only the effective average potential U_k and a standard kinetic term and neglect higher derivative terms in the action. The validity of our assumption is discussed in the next section. At scales k below the temperature T , the theory can be described in terms of an effective three-dimensional action at zero temperature [17, 27]. This dimensional reduction indicates the absence of explicit time dependence for the parameters of the theory. It is a consequence of our implicit assumption that the high-frequency modes of the system are in thermal equilibrium and their time dependence can be averaged out. On the contrary, the low-frequency modes have real-time dynamics that are related to the behaviour of the system during and after nucleation. In this work we concentrate on static properties of the system, such as the characteristics of the critical bubble and the nucleation rate. For such quantities, the description in terms of an effective three-dimensional theory is sufficient. The study of dynamical questions, such as the expansion rate of bubbles slightly larger than the critical one, requires a study of the full four-dimensional theory, at least for the low-frequency modes.

We determine the form of the potential U_k at scales k below the temperature through a numerical solution of the evolution equation. We are interested in theories for which U_k has two minima separated by a barrier for low values of k . This structure may exist already at the level of the bare potential, or appear as the result of the integration of the high-frequency modes. When k^2 becomes smaller

¹ It has been argued in ref. [12] that the appropriate quantity for the study of tunnelling is the generating functional of the 1PI Green functions (calculated perturbatively), which differs from the effective potential in the non-convex regions. However, as we show in the following, the consistent picture must rely on the notion of coarse graining and on the separation of the high-frequency fluctuations that are responsible for the non-convexity of the potential, from the low-frequency ones that are relevant for tunnelling. Such notions cannot be easily implemented in the context of perturbation theory.

²For a related work see ref. [31].

than the typical positive curvature of the convex parts of U_k , the massive modes that induce the evolution of the potential decouple. As a result the convex regions of the potential stop evolving. However, the non-convex part (the barrier) continues to evolve. Full convexity is approached in the limit $k \rightarrow 0$ [32, 19, 20]. It is a consequence of the integration of configurations that interpolate between the minima in the functional integral that defines U_k [32, 33]. All explicit information about tunnelling is lost in the resulting Maxwell construction for the effective potential.

It is clear from the above that the calculation of the nucleation rate must be performed at a non-zero value of k , such that configurations that interpolate between the minima are not taken into account. This value must be chosen so that the convex parts of the potential have stopped evolving significantly, while a well-defined barrier still exists. For a range of values of k that satisfy this requirement we perform the calculation of the nucleation rate. The profile and the free energy of the bubble are determined in the standard way. The evaluation of the fluctuation determinants in the prefactor is again performed following standard techniques, but with an important modification. An ultraviolet cutoff of order k is imposed, such that fluctuations with characteristic momenta $q^2 \gtrsim k^2$ are not included. The reason is that the effect of such fluctuations is already incorporated in U_k . This modification resolves two of the serious problems mentioned earlier. The pre-exponential factor is now finite and no double-counting of the fluctuations takes place.

However, an important issue arises at this point. The scale k was introduced in the problem as a mere calculational tool. If our approach makes sense, *the choice of k should not affect physical parameters* such as the nucleation rate. The remarkable outcome of our study is that this expectation is confirmed. Despite the significant k dependence of the free energy of the bubble and the prefactor, the nucleation rate is k -independent to a good accuracy. A residual small k dependence can be interpreted as a measure of the contribution of the next order in the saddle-point approximation for fluctuations around the bubble.

In the following sections we present the details of the calculation outlined above for a theory of a real scalar field. In order not to obscure the essential physics by complicating the model too much, we have made some simplifications. We do not discuss the evolution of Γ_k for $k \gtrsim T$. For readers who are interested in the details of the mechanism of dimensional reduction in our approach, detailed discussions can be found in refs [17, 22, 27] for a variety of models. We start the evolution at a scale k_0 sufficiently below the temperature of the system, so that the dynamics is three-dimensional to a good approximation. As an initial condition we consider a potential U_{k_0} with two inequivalent minima separated by a barrier. This form of the potential is determined by the bare potential U_Λ and the integration of fluctuations between the scales Λ and k_0 . Some of these fluctuations may correspond to additional massive degrees of freedom that decoupled above the scale k_0 . In the next section, we describe in detail the initial form of the potential we use. We integrate the evolution equation for the effective three-dimensional theory starting at the scale k_0 , and perform the calculation of the nucleation rate as described above. A significant evolution, with a substantial variation of the form of the potential, may take place between k_0 and the range of scales where we compute the nucleation rate.

In the following section we derive the evolution equation for the potential and describe the initial condition for its integration. In section 3 we discuss the formalism we employ for the calculation of the nucleation rate. Our results are presented in section 4 and our conclusions are given in section 5.

2 Evolution equation for the potential

In this section we summarize the formalism of the effective average action for a theory of a real scalar field ϕ and derive the evolution equation for the potential. We discuss the effective three-dimensional theory that results from the dimensional reduction of a high-temperature four-dimensional theory at scales below the temperature. The temperature can be absorbed in a redefinition of the field and its

potential, so that these have dimensions appropriate for an effective three-dimensional theory

$$\begin{aligned}\phi &= \frac{\phi_4}{\sqrt{T}} \\ U(\phi) &= \frac{U_4(\phi_4, T)}{T}.\end{aligned}\tag{2.1}$$

In this way, the temperature does not appear explicitly in our expressions. This has the additional advantage of permitting the straightforward application of our results to the problem of quantum tunnelling in a three-dimensional theory at zero temperature.

2.1 Intuitive derivation

Before presenting the rigorous derivation, it is instructive to derive the evolution equation for the potential based on an intuitive argument, along the lines of ref.s [13, 17]. We start by considering the Z_2 -symmetric scalar model, in Euclidean three-dimensional space. The one-loop effective potential is given by the expression

$$\begin{aligned}U_k^{(1)}(\rho) &= V(\rho) + \frac{1}{2} \ln \det [P_k + V'(\rho) + 2V''(\rho)\rho] \\ &= V(\rho) + \frac{1}{2} \int_{\Lambda} \frac{d^3q}{(2\pi)^3} \ln [P_k(q) + V'(\rho) + 2V''(\rho)\rho],\end{aligned}\tag{2.2}$$

where $V(\rho)$ is the bare potential. In order to be consistent with the conventions in previous publications, we have defined the variable

$$\rho = \frac{1}{2}\phi^2,\tag{2.3}$$

which we frequently use in this section. Primes denote derivatives with respect to ρ : $V'(\rho) = dV/d\rho$. In terms of this variable, the mass term of the scalar field is $d^2V/d\phi^2 = V'(\rho) + 2V''(\rho)\rho$. The inverse propagator $P_k(q)$ in momentum space for a massless field is given by $P_{k=0}(q) = q^2$ in perturbation theory. We assume that the momentum integration is regulated by an ultraviolet cutoff Λ .

We would like to introduce an effective infrared cutoff k for the low-frequency modes, so that the momentum integration in eq. (2.2) does not receive contributions from modes with characteristic momenta $q^2 \lesssim k^2$. The simplest way to achieve this is through the addition of a mass term k^2 to the perturbative inverse propagator, so that for a massless field

$$P_k(q) = q^2 + k^2.\tag{2.4}$$

The potential now depends on k , as indicated by the subscript in eq. (2.2).

The next step is to derive an evolution equation for the change of U_k with the scale k and follow the evolution for $k \rightarrow 0$. For this purpose we take the logarithmic derivative with respect to k and substitute U_k for V in the right-hand side of eq. (2.2). The intuitive justification for this replacement is based on the fact that the new contributions to the momentum integration, when k is lowered by a small amount Δk , come from the region $k - \Delta k < q < k$. The relevant mass term and couplings that should appear in the evolution equation are the renormalized ones at the scale k (which, for the scalar field, are related to derivatives of U_k) and not the bare ones. This ‘‘renormalization-group improvement’’ results in the evolution equation

$$\frac{\partial U_k'(\rho)}{\partial t} = -\frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \frac{\partial P_k}{\partial t} \frac{3U_k''(\rho) + 2U_k'''(\rho)\rho}{[P_k(q) + U_k'(\rho) + 2U_k''(\rho)\rho]^2}\tag{2.5}$$

$$= -\frac{k^2}{8\pi} \frac{3U_k''(\rho) + 2U_k'''(\rho)\rho}{\sqrt{k^2 + U_k'(\rho) + 2U_k''(\rho)\rho}},\tag{2.6}$$

where $t = \ln(k/\Lambda)$, with Λ identified with the ultraviolet cutoff of the theory. We have derived the evolution equation for $U'_k(\rho)$ (and not $U_k(\rho)$) because this is the easiest to integrate numerically. For $k = \Lambda$ the infrared and ultraviolet cutoffs coincide, and no integration of fluctuations takes place. This determines the initial conditions for the solution of eq. (2.6) as $U_\Lambda(\rho) = V(\rho)$. In the opposite limit, $k \rightarrow 0$, one recovers the effective potential $U(\rho) \equiv U_0(\rho)$.

One could look for an iterative solution of eq. (2.5). The first iteration results in the equation

$$U^{(1)}(\rho) = U_k(\rho) + \frac{1}{2} \ln \frac{\det [P_{k=0} + U'_k(\rho) + 2U''_k(\rho)\rho]}{\det [P_k + U'_k(\rho) + 2U''_k(\rho)\rho]} \quad (2.7)$$

for the effective potential. This expression can be compared with eq. (2.2) with $k = 0$. The two equations have the same structure, but the bare potential $V(\rho)$ is replaced by the k -dependent potential $U_k(\rho)$. Also, the determinant resulting from the radiative corrections is replaced by a ratio of determinants. The numerator of this ratio is what one would expect from eq. (2.2) with $k = 0$. The denominator involves the inverse propagator P_k and effectively removes the fluctuations with characteristic momenta $q^2 \gtrsim k^2$. This is justified by the fact that the effect of these fluctuations has already been incorporated in $U_k(\rho)$. There is no need for the addition of an extra regulator Λ as in eq. (2.2). Its role is played by k , which acts as an effective ultraviolet cutoff in the calculation of the effective potential from $U_k(\rho)$.

The above observation will be important in the following section where nucleation rates will be computed. The first part of our procedure will be to integrate the evolution equation (2.5) numerically, from a scale k_0 , where we choose the initial form of the potential $U_{k_0}(\rho)$, down to a non-zero scale k . This effective integration of the high-frequency modes generates the appropriate potential that describes the dynamics of low-frequency modes with $q^2 \lesssim k^2$. Tunnelling will be discussed in the context of the low-energy theory through the standard saddle-point approximation [2, 3]. The first correction to the leading semiclassical result involves the usual ‘‘one-loop’’ form of a fluctuation determinant $\det [P_{k=0} + \mathcal{O}]$, where \mathcal{O} is an operator to be defined in the next section. A consistent treatment of the effect of fluctuations, which avoids double-counting the high-frequency modes, can be obtained if this correction is modified to $\det [(P_{k=0} + \mathcal{O}) / (P_k + \mathcal{O})]$ as in eq. (2.7).

2.2 Rigorous derivation

The evolution equation for the potential can be derived within a more rigorous approach through the formalism of the effective average action [13, 15, 18]. The effective average action Γ_k , for a theory described by a bare action S , results from the effective integration of degrees of freedom with characteristic momenta larger than a given infrared cutoff k . Its dependence on the scale k is described by an exact flow equation. In this subsection we summarize the formalism for the case of a Z_2 -symmetric theory of a real scalar field in Euclidean three-dimensional space. A detailed discussion can be found in ref.s [15, 18].

We specify the action together with some ultraviolet cutoff Λ , so that the theory is properly regulated. We add to the kinetic term a piece that has the following form in momentum space

$$\Delta_k S[\chi] = \frac{1}{2} \int d^3q R_k(q) \chi^*(q) \chi(q), \quad (2.8)$$

where $\chi^*(q) = \chi(-q)$. The function R_k is used to prevent the propagation of modes with characteristic momenta $q^2 \lesssim k^2$. As a result, the inverse propagator for the action $S + \Delta S$ has a minimum $\sim k^2$. The modes with $q^2 \gg k^2$ are unaffected, while the low-frequency modes with $q^2 \ll k^2$ are cut off:

$$\lim_{q^2 \rightarrow 0} R_k \sim k^2. \quad (2.9)$$

We emphasize at this point that many alternative choices of R_k are possible.

We subsequently introduce sources and define the generating functional for the connected Green functions for the action $S + \Delta S$. Through a Legendre transformation we obtain the generating functional for the 1PI Green functions $\tilde{\Gamma}_k[\phi]$, where ϕ is the expectation value of the field χ in the presence of sources. The use of the modified propagator for the calculation of $\tilde{\Gamma}_k$ results in the effective integration of only the fluctuations with $q^2 \gtrsim k^2$. Finally, the effective average action is obtained by removing the infrared cutoff

$$\Gamma_k[\phi] = \tilde{\Gamma}_k[\phi] - \frac{1}{2} \int d^3q R_k(q) \phi^*(q) \phi(q). \quad (2.10)$$

For k equal to the ultraviolet cutoff Λ ³, Γ_k becomes equal to the bare action S (no effective integration of modes takes place), while for $k \rightarrow 0$ it tends towards the effective action Γ corresponding to S (all the modes are included). The interpolation of Γ_k between the bare and the effective action makes it a very useful field-theoretical tool. The means for practical calculations is provided by an exact flow equation⁴, which describes the response of the effective average action to variations of the infrared cutoff ($t = \ln(k/\Lambda)$) [15]:

$$\frac{\partial}{\partial t} \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \left\{ (\Gamma_k^{(2)}[\phi] + R_k)^{-1} \frac{\partial}{\partial t} R_k \right\}. \quad (2.11)$$

Here $\Gamma_k^{(2)}$ is the second functional derivative of Γ_k with respect to ϕ .

Making use of the Z_2 symmetry, we parametrize the effective average action as

$$\Gamma_k = \int d^3x \left\{ U_k(\rho) + \frac{1}{2} \partial^\mu \phi : Z_k(\rho, -\partial^2) : \partial_\mu \phi + \dots \right\}, \quad (2.12)$$

where the normal ordering indicates that the derivative operators are always on the right. The dots stand for invariants that involve more derivatives of the field. In order to turn the flow equation for the effective average action into equations for U_k , Z_k , etc, we have to evaluate the trace in eq. (2.11) for properly chosen background field configurations. For the evolution equation for U_k we have to expand around a constant field configuration. We find [15, 18]

$$\frac{\partial}{\partial t} U_k(\rho) = \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \frac{\partial R_k(q)/\partial t}{Z_k(\rho, q^2)q^2 + R_k(q) + U'_k(\rho) + 2\rho U''_k(\rho)}. \quad (2.13)$$

The above equation is an exact evolution equation for the potential. Its solution, however, requires information on the wave-function renormalization $Z_k(\rho, q^2)$. Throughout this paper we shall set $Z_k(\rho, q^2) = 1$, which corresponds to the first order of the derivative expansion of eq. (2.12). This is expected to be a good approximation because the size of Z_k is related to the anomalous dimension of the field, which is small for this model ($\eta \simeq 0.035$ for the three-dimensional theory we consider). For $\eta = 0$ the kinetic term in the k -dependent inverse propagator must be exactly proportional to q^2 both for $q^2 \rightarrow 0$ and $q^2 \rightarrow \infty$. Several studies have confirmed the smallness of the corrections arising from the deviation of $Z_k(\rho, q^2)$ from 1 [18, 35, 21]. Within our approximation, eq. (2.13) reproduces eq. (2.5) through the definition

$$P_k(q) = q^2 + R_k(q). \quad (2.14)$$

For the choice

$$R_k(q) = k^2, \quad (2.15)$$

eq. (2.6) is obtained.

³ For scales $k \gtrsim T$ the high-temperature behaviour of the four-dimensional theory is relevant. For this reason our discussion in terms of an effective three-dimensional theory is not sufficient. In this work, however, we never discuss the evolution at such high scales, but work at scales $k \leq k_0 \lesssim T$ instead. We refer to the ultraviolet cutoff Λ only at this point for reasons of completeness of the presentation. For a full discussion of the evolution at large momentum scales and the mechanism of dimensional reduction, see ref.s [17, 22, 27].

⁴ For other versions of exact renormalization group equations, see ref.s [16, 34].

We should point out that the choice of eq. (2.15) for the cutoff function $R_k(q)$ may be problematic. For example, in four dimensions it results in divergent integrals in the right-hand side of evolution equations such as eq. (2.13). Even for the three-dimensional case that we are considering, the right-hand side of eq. (2.13) involves an irrelevant ρ -independent divergent constant, which disappears in the evolution equation for $U'_k(\rho)$. For these reasons, in most cases it is preferable to work with a cutoff function of the form

$$R_k(q) = \frac{q^2 \exp(-q^2/k^2)}{1 - \exp(-q^2/k^2)}, \quad (2.16)$$

for which integrals such as the one in the right-hand side of eq. (2.13) are finite. However, such a cutoff function corresponds to a non-local operator in position space, which would make the calculations of the next section impossible. Because of this, we employ the cutoff function of eq. (2.15) in this paper and emphasize that care must be taken in the extension of our discussion to the case of full four-dimensional dynamics.

2.3 Scale-invariant form of the evolution equation

It is convenient to cast the evolution equation (2.6) in a form that does not explicitly depend on the scale k . This makes the identification of possible fixed points easier. For this reason we define the dimensionless quantities

$$\begin{aligned} u_k(\tilde{\rho}) &= \frac{U_k(\rho)}{k^3} \\ \tilde{\rho} &= \frac{\rho}{k}. \end{aligned} \quad (2.17)$$

Primes on u_k denote derivatives with respect to $\tilde{\rho}$. We can now rewrite the evolution equation for the potential as

$$\frac{\partial u'_k}{\partial t} = -2u'_k + \tilde{\rho}u''_k + \frac{1}{8\pi^2}(3u''_k + 2\tilde{\rho}u'''_k)L_1^3(u'_k + 2u''_k\tilde{\rho}). \quad (2.18)$$

The non-trivial solution of the above equation with $\partial u'_k/\partial t = 0$ corresponds to the Wilson-Fisher fixed point that determines the dynamics of the second-order phase transition in the Z_2 -symmetric theory [17, 18, 21].

The dimensionless function $L_1^3(w)$ is given by

$$\begin{aligned} L_1^3(w) &= -\frac{1}{2\pi k} \int d^3q \frac{\partial P_k}{\partial t} (P_k + w)^{-2} \\ &= -\frac{1}{k} \int_0^\infty dx \sqrt{x} \frac{\partial P_k}{\partial t} (P_k + w)^{-2}, \end{aligned} \quad (2.19)$$

with $x = q^2$. This has been discussed extensively in ref.s [13, 18, 32] (for various forms of the infrared-regulating function R_k and in various dimensions). It has the interesting property that it falls off for large values of w following a power law. As a result it introduces a threshold behaviour for the contributions of massive modes to the evolution equation. The third term in the right-hand side of the evolution equation (2.18) includes the L_1^3 function with the mass of the ϕ field divided by k^2 as its argument. When the scale k^2 crosses below the running squared mass this contribution vanishes and the massive mode decouples. As a result the evolution of $U_k(\rho)$ stops. The function $L_1^3(w)$ also has a pole at $w = -1$. This property induces the convexity of the potential in the limit $k \rightarrow 0$. The argument of $L_1^3(w)$ in the non-convex regions is given by the negative curvature of the potential divided by k^2 . As the pole cannot be crossed, the curvature follows k to zero, thus inducing the convexity of the effective potential. For the choice of eq. (2.15) for the cutoff function, we obtain

$$L_1^3(w) = -\frac{\pi}{\sqrt{1+w}}, \quad (2.20)$$

in agreement with eq. (2.6).

Two algorithms for the numerical integration of eq. (2.18) have been presented in detail in ref. [35]. The comparison of the two methods provides a good check on possible systematic numerical errors. The two algorithms give results that agree at the 0.3% level. We expect the numerical solution to be an approximation of the solution of the partial differential equation (2.18) with the same level of accuracy.

2.4 Explicit breaking of the Z_2 symmetry

Up to this point we have not discussed the breaking of the Z_2 symmetry that could lead to vacuum instability. The formalism of the previous subsections is completely Z_2 -invariant and has to be modified in order to account for the symmetry-breaking effects. However, we shall work within a framework for which the modifications are minimal. As we explained in the introduction, we consider an effective three-dimensional theory that results from the dimensional reduction of a high-temperature four-dimensional one. We define this theory at a scale $k_0 \lesssim T$. The form of U_k is determined by the bare potential U_Λ and the integration of fluctuations between the scales Λ and k_0 . Some of these fluctuations may correspond to additional massive degrees of freedom that decoupled above the scale k_0 . We choose a form of the potential that breaks the Z_2 symmetry, while permitting the presence of a non-trivial evolution at scales $k \leq k_0$.

Following ref. [36], we consider theories that are described by potentials of the form

$$U_{k_0}(\phi) = \frac{1}{2}m_{k_0}^2\phi^2 + \frac{1}{6}\gamma_{k_0}\phi^3 + \frac{1}{8}\lambda_{k_0}\phi^4. \quad (2.21)$$

By a variable shift

$$\sigma = \phi + \frac{\gamma_{k_0}}{3\lambda_{k_0}} \quad (2.22)$$

we can bring this potential to the form

$$U_{k_0}(\sigma) = c_{k_0} - J_\gamma\sigma + \frac{1}{2}\mu_{k_0}^2\sigma^2 + \frac{1}{8}\lambda_{k_0}\sigma^4, \quad (2.23a)$$

with

$$J_\gamma = \frac{\gamma_{k_0}}{3\lambda_{k_0}}m_{k_0}^2 - \frac{\gamma_{k_0}^3}{27\lambda_{k_0}^2}, \quad (2.23b)$$

$$\mu_{k_0}^2 = m_{k_0}^2 - \frac{\gamma_{k_0}^2}{6\lambda_{k_0}}. \quad (2.23c)$$

The right-hand side of the exact flow equation for the effective average action of the σ field is not affected by the linear term $-J_\gamma\sigma$ or the constant c_{k_0} . Therefore, the evolution equation for the potential can be integrated using the Z_2 -symmetric formalism. The potential of ϕ is recovered through the relation

$$U_k(\phi) = c_{k_0} - J_\gamma\sigma + U_k^{Z_2}(\sigma) = c_{k_0} - J_\gamma \cdot \left(\phi + \frac{\gamma_{k_0}}{3\lambda_{k_0}} \right) + U_k^{Z_2} \left(\phi + \frac{\gamma_{k_0}}{3\lambda_{k_0}} \right), \quad (2.24)$$

where $U_k^{Z_2}$ is the potential of the Z_2 -symmetric model with mass term $\mu_{k_0}^2$ and quartic coupling λ_{k_0} . In ref. [36] it has been verified that the above procedure gives the same result as the straightforward integration of the evolution equation (2.18) with an initial condition given by eq. (2.21).

In the following we integrate numerically eq. (2.18) for Z_2 -symmetric models. Different types of evolution can be obtained by keeping λ_{k_0} constant and varying the mass term $\mu_{k_0}^2$. More specifically, the Wilson-Fisher fixed point of the three-dimensional theory can be approached for a certain (negative) value $(\mu_{k_0}^2)_{cr}$. This fixed point determines the properties of the second-order phase transition of the Z_2 -symmetric theory. Through the appropriate choice of γ_{k_0} , we can vary the difference in the energy density between the minima of the potential according to eq. (2.24). Arbitrarily weakly first-order phase transitions can be studied in the limit $\mu_{k_0}^2 \rightarrow (\mu_{k_0}^2)_{cr}$, $\gamma_{k_0} \rightarrow 0$.

3 Calculation of the nucleation rate

3.1 General formalism

As we discussed in the previous sections, we study tunnelling in a theory of a real scalar field with a Euclidean action

$$\Gamma_k = \int d^3x \left\{ \frac{1}{2} (\partial^\mu \phi) (\partial_\mu \phi) + U_k(\phi) \right\} \quad (3.1)$$

and a coarse-graining scale $k \neq 0$. The parameters in the above action are effective three-dimensional ones defined according to eqs. (2.1). The potential $U_k(\phi)$ has two minima corresponding to vacua with different vacuum energy densities: the stable (true) minimum is located at $\phi = \phi_t$ and the unstable (false) one at $\phi = \phi_f = 0$.

The unrenormalized decay rate per unit volume from the false minimum towards the true one is given by [3, 4, 8]

$$I = \frac{E_0}{2\pi} \left(\frac{S_k}{2\pi} \right)^{3/2} \left| \frac{\det'[\delta^2 \Gamma_k / \delta \phi^2]_{\phi=\phi_b}}{\det[\delta^2 \Gamma_k / \delta \phi^2]_{\phi=0}} \right|^{-1/2} \exp(-S_k). \quad (3.2)$$

This is analogous to eq. (1.1) after the absorption of the explicit factors of T in the redefinition of the field and potential and the introduction of a coarse-graining scale. Notice that if the factor $E_0/2\pi$ is removed, the above expression reproduces the quantum-tunnelling rate of the zero-temperature three-dimensional theory [4, 5]. As this factor gives only a small contribution to the total rate, our discussion applies to quantum tunnelling as well within a good approximation.

The nucleation rate is exponentially suppressed by the action S_k (the rescaled free energy) of the bubble configuration $\phi_b(r)$. This is an SO(3)-invariant solution of the classical equations of motion which interpolates between the local maxima of the potential $-U_k(\phi)$. It satisfies the equation

$$\frac{d^2 \phi_b}{dr^2} + \frac{2}{r} \frac{d\phi_b}{dr} = U'_k(\phi_b), \quad (3.3)$$

with the boundary conditions $\phi_b \rightarrow 0$ for $r \rightarrow \infty$ and $d\phi_b/dr = 0$ for $r = 0$. The action S_k of the bubble is given by

$$S_k = 4\pi \int_0^\infty \left[\frac{1}{2} \left(\frac{d\phi_b(r)}{dr} \right)^2 + U_k(\phi_b(r)) \right] r^2 dr \equiv S_k^t + S_k^v, \quad (3.4)$$

where the kinetic and potential contributions, S_k^t and S_k^v respectively, satisfy

$$\frac{S_k^v}{S_k^t} = -\frac{1}{3}. \quad (3.5)$$

The pre-exponential factor corresponds to the first correction to the semiclassical approximation in the saddle-point method. The numerator is the fluctuation determinant around the bubble

$$\det' \left[\delta^2 \Gamma_k / \delta \phi^2 \right]_{\phi=\phi_b} = \det' \left[-\partial^2 + U''_k(\phi = \phi_b(r)) \right], \quad (3.6)$$

while the denominator is the fluctuation determinant around the false vacuum $\phi = \phi_f = 0$

$$\det \left[\delta^2 \Gamma_k / \delta \phi^2 \right]_{\phi=0} = \det \left[-\partial^2 + U''_k(\phi = 0) \right]. \quad (3.7)$$

In this section we revert to the standard way of denoting the derivatives with respect to ϕ with primes. This should not be confusing as we always indicate the quantity with respect to which we differentiate as the argument of the function. The differential operator $-\partial^2 + U''_k(\phi_b(r))$ has three zero modes (the three spatial translations of the bounce). The prime over the determinant indicates that these modes have to be omitted in its calculation. Their contribution generates the factor $(S_k/2\pi)^{3/2}$ in eq. (3.2)

and the volume factor that is absorbed in the definition of I (nucleation rate per unit volume). The quantity E_0 is the square root of the absolute value of the unique negative eigenvalue of the above operator.

The pre-exponential factor defined in eq. (3.2) is in general ultraviolet-divergent and an appropriate regularization scheme must be employed. Within our approach, the form of the regularization is dictated by the discussion at the end of subsection 2.1. The effect of the high-frequency modes has been incorporated in the form of the coarse-grained potential $U_k(\phi)$, which is obtained through the integration of the evolution equation (2.18). In order not to double-count this effect, fluctuation determinants computed within the low-energy theory must be replaced by a ratio of determinants, in complete analogy to eq. (2.7). This implies that a consistent expression for the nucleation rate is given by

$$\begin{aligned}
I &\equiv A_k \exp(-S_k) \\
A_k &= \frac{E_0}{2\pi} \left(\frac{S_k}{2\pi}\right)^{3/2} \times \\
&\quad \left| \frac{\det' [-\partial^2 + U_k''(\phi_b(r))]}{\det [-\partial^2 + R_k(-\partial^2) + U_k''(\phi_b(r))]} \frac{\det [-\partial^2 + R_k(-\partial^2) + U_k''(0)]}{\det [-\partial^2 + U_k''(0)]} \right|^{-1/2}. \quad (3.8)
\end{aligned}$$

The infrared-regulating function R_k has been discussed in the previous section (see eq. (2.14)). The simplest choice for this function, which is necessary for the feasibility of the computation of the next section, is given by eq. (2.15), i.e. $R_k = k^2$. We point out that only the operator $-\partial^2 + U_k''(\phi_b(r))$ has negative and zero eigenvalues that require special treatment in eq. (3.8).

3.2 Details of the numerical computation

From this point on, we use the form $R_k = k^2$ for the infrared-regulating function. The profile of the bubble can be easily computed with the “shooting” method [37]. We integrate eq. (3.3) numerically, starting at $r = 0$ with a value of ϕ near the true minimum ϕ_t and $d\phi/dr = 0$. We then adjust the initial value of ϕ so that the boundary condition $\phi_b \rightarrow 0$ for $r \rightarrow \infty$ is satisfied. (In practice this condition is satisfied at a value r_∞ sufficiently larger than the typical size of the bubble.)

The computation of the fluctuation determinants is more complicated. The differential operators that appear in eq. (3.8) have the general form

$$\mathcal{W}_{\kappa\alpha} = -\partial^2 + m_\kappa^2 + \alpha W_k(r), \quad (3.9a)$$

where

$$m_\kappa^2 \equiv U_k''(0) + \kappa k^2, \quad (3.9b)$$

$$W_k(r) \equiv U_k''(\phi_b(r)) - U_k''(0), \quad (3.9c)$$

with $\kappa, \alpha = 0$ or 1 . Since the $\mathcal{W}_{\kappa\alpha}$ operators are SO(3) symmetric, it is convenient to use spherical coordinates (r, θ, φ) and express the eigenfunctions ψ in terms of spherical harmonics: $\psi(r, \theta, \varphi) = u(r)/r Y_{\ell m}(\theta, \varphi)$ [8, 9]. Here ℓ and m are the usual angular quantum numbers. The Laplacian operator ∂^2 takes the form

$$-\partial^2 \rightarrow \frac{1}{r} \left[-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} \right] r \equiv -\frac{1}{r} \nabla_\ell^2 r, \quad (3.10)$$

so that

$$\begin{aligned}
\det \mathcal{W}_{\kappa\alpha} &= \prod_{\ell=0}^{\infty} (\det \mathcal{W}_{\ell\kappa\alpha})^{2\ell+1} \\
\mathcal{W}_{\ell\kappa\alpha} &= -\nabla_\ell^2 + m_\kappa^2 + \alpha W_k(r). \quad (3.11)
\end{aligned}$$

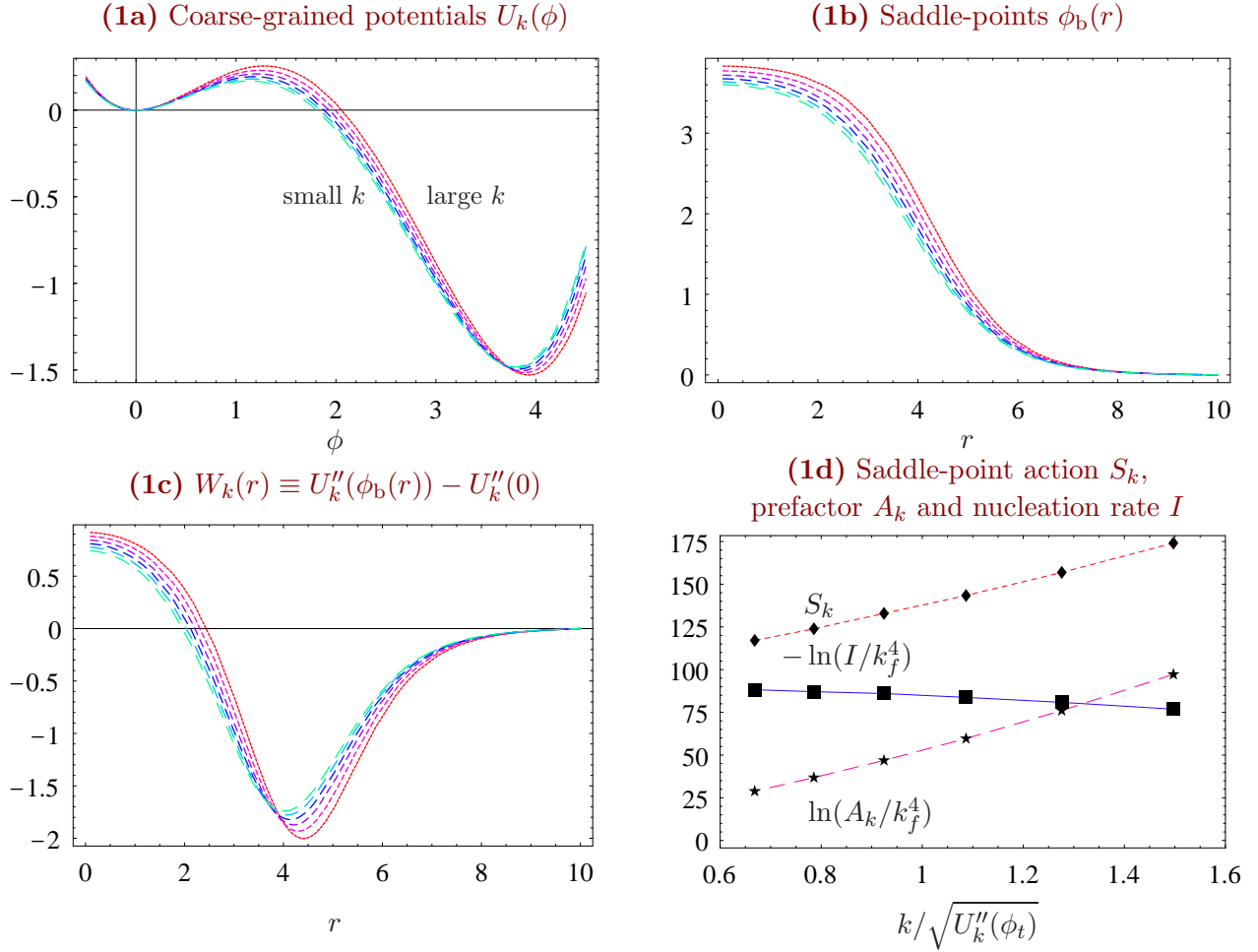


Figure 1: The steps in the computation of the nucleation rate for a model with $\mu_{k_0}^2/k_0^2 = -0.05$, $\lambda_{k_0}/k_0 = 0.1$, $\gamma_{k_0}/k_0^{3/2} = -0.0634$. The dimensional quantities are given in units of $k_f = 0.223 k_0$.

We recall that $\det \mathcal{W}_{\ell\kappa\alpha}$ is defined as the product of all eigenvalues λ that lead to solutions of $\mathcal{W}_{\ell\kappa\alpha} u(r) = \lambda u(r)$, with the function $u(r)$ vanishing at $r = 0$ and $r \rightarrow \infty$. The computation of such complicated determinants is made possible by a powerful theorem [38, 8] that relates ratios of determinants to solutions of ordinary differential equations. In particular, we have

$$g_{\ell\kappa} \equiv \frac{\det \mathcal{W}_{\ell\kappa 1}}{\det \mathcal{W}_{\ell\kappa 0}} = \frac{\det[-\nabla_\ell^2 + m_\kappa^2 + 1 \cdot W_k(r)]}{\det[-\nabla_\ell^2 + m_\kappa^2 + 0 \cdot W_k(r)]} = \frac{y_{\ell\kappa 1}(r \rightarrow \infty)}{y_{\ell\kappa 0}(r \rightarrow \infty)}, \quad (3.12)$$

where $y_{\ell\kappa\alpha}(r)$ is the solution of the differential equation

$$\left[-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + m_\kappa^2 + \alpha W_k(r) \right] y_{\ell\kappa\alpha}(r) = 0, \quad (3.13)$$

with the behaviour $y_{\ell\kappa\alpha}(r) \propto r^{\ell+1}$ for $r \rightarrow 0$. Such equations can be easily solved numerically with Mathematica [39]. In the “free case” ($\alpha = 0$) it is possible to obtain the exact analytical solution⁵

$$y_{\ell\kappa 0} \propto i_{\ell+1/2}(m_\kappa r) / \sqrt{m_\kappa r}. \quad (3.14)$$

⁵The function i_n is the standard Bessel I function, defined as $i_n(z) = \text{BesselI}[n, z]$ in Mathematica notation [39]. For $n = \ell+1/2$ a semi-integer, i_n can be expressed in terms of elementary functions as $\sqrt{z} i_{\ell+1/2}(z) = P_+^\ell(1/z)e^z + P_-^\ell(1/z)e^{-z}$, where P_\pm^ℓ are polynomials of degree ℓ . For $z \rightarrow \infty$ (or, more precisely, for $z \gg \ell$) $i_{\ell+1/2}(z) \rightarrow e^z / \sqrt{2\pi z}$.

The final expression for the nucleation rate, appropriate for an efficient numerical computation, is

$$\begin{aligned}
I &= \frac{1}{2\pi} \left(\frac{S_k}{2\pi} \right)^{3/2} \exp(-S_k) c_0 c_1 \prod_{\ell=2}^{\infty} c_\ell, \\
c_0 &= \left(\frac{E_0^2 g_{01}}{|g_{00}|} \right)^{1/2}, \quad c_1 = \left(\frac{g_{11}}{g'_{10}} \right)^{3/2}, \quad c_\ell = \left(\frac{g_{\ell 1}}{g_{\ell 0}} \right)^{(2\ell+1)/2}.
\end{aligned} \tag{3.15}$$

The factors c_ℓ for $\ell \geq 2$ can be computed in a straightforward way through eqs. (3.12), (3.13) as we explained above. The calculation of c_1 is more complicated because of the necessity to eliminate the zero eigenvalues in g'_{10} . This can be achieved by replacing the operator $\mathcal{W}_{10\alpha}$ with $\mathcal{W}_{10\alpha} + \epsilon U_k''(0)$ and evaluating g'_{10} as [8]

$$g'_{10} = \frac{1}{U_k''(0)} \lim_{\epsilon \rightarrow 0} \left[\frac{1}{\epsilon} \frac{\det \mathcal{W}_{101}(\epsilon)}{\det \mathcal{W}_{100}} \right]. \tag{3.16}$$

The (unique) negative eigenvalue $-E_0^2$ of \mathcal{W}_{001} can be obtained by solving the equation $\mathcal{W}_{001}u = -E_0^2 u$ and using the shooting method to determine the value of E_0^2 that ensures the correct boundary condition $u(r \rightarrow \infty) = 0$.

As a final remark, we give the explicit expression for c_ℓ in the limit of large ℓ . It can be obtained by solving the differential equations (3.13), using first-order perturbation theory in W_k [8]. In terms of

$$w_n \equiv - \int_0^\infty r^n W_k(r) dr, \tag{3.17a}$$

we find

$$g_{\ell 0}^{(2\ell+1)/2}, g_{\ell 1}^{(2\ell+1)/2} \rightarrow \exp(-w_1/2) \left[1 + \mathcal{O}(\ell^{-2}) \right], \tag{3.17b}$$

$$c_\ell \rightarrow 1 + k^2 \frac{w_3}{4\ell^2} + \mathcal{O}(\ell^{-4}). \tag{3.17c}$$

We have checked that our numerical solution reproduces the above behaviour for large ℓ . The expression (3.15) for the nucleation rate is finite, as can be easily checked by considering the identity

$$\prod_{\ell=2}^{\infty} \left(1 + \frac{D^2}{\ell^2} \right) = \frac{\sinh D\pi}{D\pi(1 + D^2)}. \tag{3.18}$$

This last expression is very useful for the numerical computation. To obtain an accurate result we need to compute the exact value of c_ℓ only up to $\ell = 10\text{--}100$. Large $\ell > 30$ are necessary only for large values of k and/or large couplings in the potential.

4 Results

The various steps in our calculation are summarized in fig. 1, for a theory described by the potential of eqs. (2.21), (2.23c) with $\mu_{k_0}^2 = -5 \cdot 10^{-2} k_0^2$, $\gamma_{k_0} = -6.34 \cdot 10^{-2} k_0^{3/2}$, $\lambda_{k_0} = 0.1 k_0$. The dimensionful scale k_0 is related to the temperature T of the system. It is determined by our assumption that the effective three-dimensional description becomes valid at k_0 . Previous studies of dimensional reduction in the context of the effective average action [17, 27] indicate that $k_0 \simeq T$. It must be pointed out at this point that the two-minimum structure is often a consequence of the integration of fluctuations of effective three-dimensional degrees of freedom (additional scalar or gauge fields). In such cases the scale k_0 must be taken sufficiently small for this structure to emerge [27]–[29]⁶.

In fig. 1a we present the evolution of the potential $U_k(\phi)$ as the scale k is lowered. We have shifted the metastable vacuum to $\phi = 0$. The solid line corresponds to $k/k_0 = 0.513$, while the line with

⁶The calculation of the pre-exponential factor must take into account fluctuations associated with these additional degrees of freedom [40].

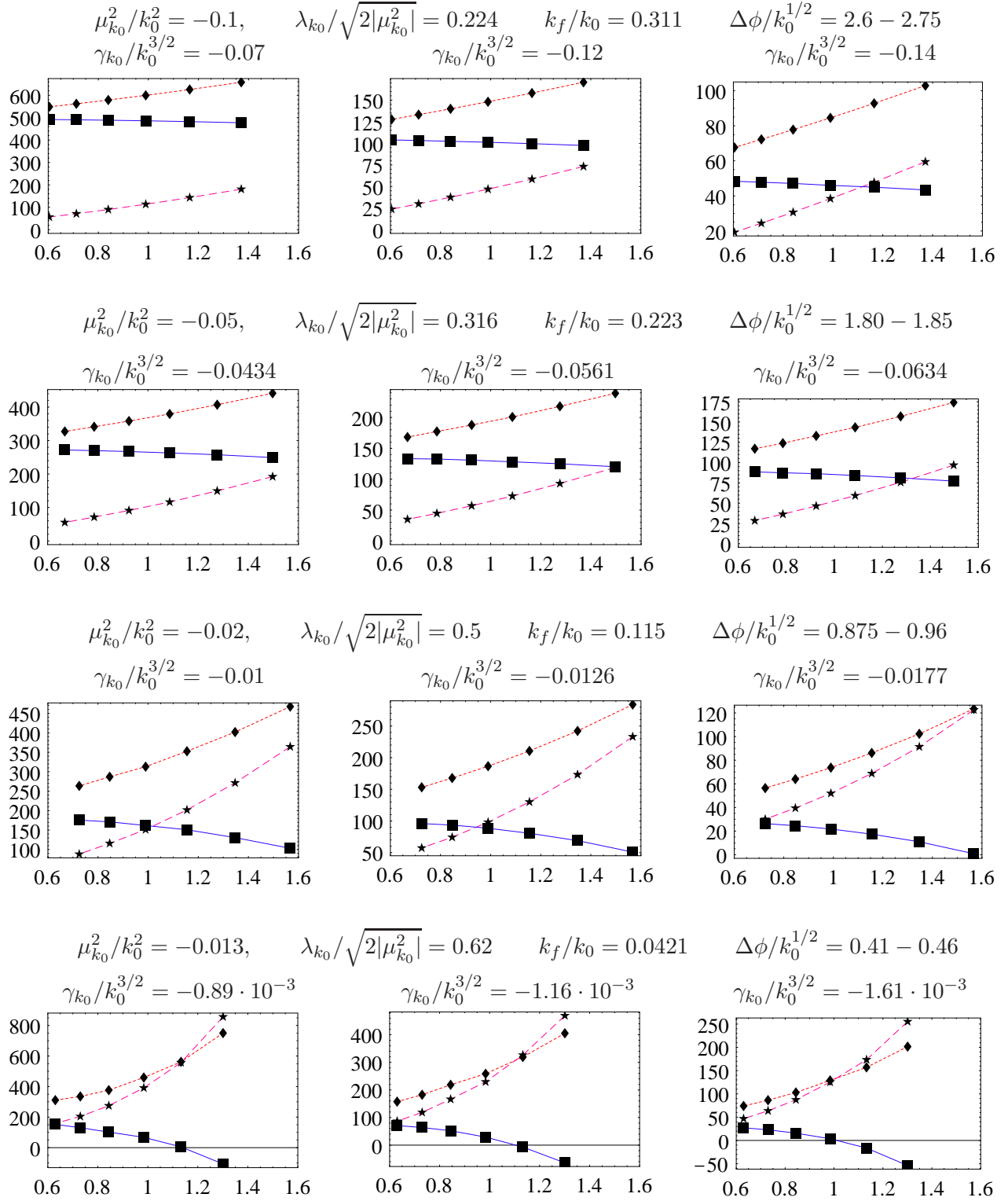


Figure 2: The behaviour of the nucleation rate I for several values of the parameters of the model with $\lambda_{k_0}/k_0 = 0.1$. We show the values of S_k (diamonds), $\ln(A_k/k_f^4)$ (stars) and $-\ln(I/k_f^4)$ (squares) as a function of $k/\sqrt{U_k''(\phi_t)}$.

longest dashes (that has the smallest barrier height) corresponds to $k_f/k_0 = 0.223$. At the scale k_f the negative curvature at the top of the barrier is slightly larger than $-k_f^2$. This means that the pole of the function $L_1^3(w)$ at $w = -1$ is approached in eq. (2.18). This is the point in the evolution of the potential where configurations that interpolate between the minima start becoming relevant in the functional integral that defines the coarse-grained potential [32, 33]. For this reason, we stop the evolution at this point. The potential and the field have been normalized with respect to k_f , so that they are of order 1. We observe that, as k is lowered, the absolute minimum of the potential settles at a non-zero value of ϕ , while a significant barrier separates it from the metastable minimum at $\phi = 0$. The profile of the critical bubble $\phi_b(r)$ is plotted in fig. 1b in units of k_f for the same sequence of scales. For $k \simeq k_f$ the characteristic length scale of the bubble profile and $1/k$ are comparable. This is expected, because the form of the profile is determined by the barrier of the potential, whose curvature is $\simeq -k^2$ at this point. This is an additional indication that we should not proceed to coarse-graining scales below k_f . We observe a significant variation of the value of the field ϕ in the interior of the bubble for different k . This is reflected in the form of the quantity $W_k(r)$, defined in eq. (3.9c), which we plot in fig. 1c.

Our results for the nucleation rate are presented in fig. 1d. The horizontal axis corresponds to $k/\sqrt{U_k''(\phi_t)}$, i.e. the ratio of the scale k to the square root of the positive curvature of the potential at the absolute minimum. The latter quantity gives the mass of the field at the absolute minimum. Typically, when k crosses below this mass (corresponding to the value 1 on the horizontal axis) the massive fluctuations of the field start decoupling and the evolution of the convex parts of the potential slows down and eventually stops. The dark diamonds give the values of the action S_k of the critical bubble at the scale k . We observe a strong k dependence of this quantity, which is expected from the behaviour in figs 1a–1c. The stars in fig. 1d indicate the values of $\ln(A_k/k_f^4)$. Again a strong k dependence is observed. More specifically, the value of A_k decreases for decreasing k . This is expected, because k acts as the effective ultraviolet cutoff in the calculation of the fluctuation determinants in A_k . For smaller k , fewer fluctuations with wavelengths above an increasing length scale $\sim 1/k$ contribute explicitly to the fluctuation determinants. The dark squares give our results for $-\ln(I/k_f^4) = S_k - \ln(A_k/k_f^4)$. It is remarkable that the k dependence of this quantity disappears as k crosses below $\sqrt{U_k''(\phi_t)}$ and approaches k_f . The small residual dependence on k can be used to estimate the contribution of the next order in the expansion around the saddle point. It is reassuring that this contribution is expected to be smaller than $\ln(A_k/k_f^4)$.

This behaviour confirms our expectation that the nucleation rate should be independent of the scale k that we introduced as a calculational tool. It also demonstrates that all the configurations plotted in fig. 1b give equivalent descriptions of the system, at least for the lower values of k . The implication is that the critical bubble should not be identified just with the saddle point of the semiclassical approximation, whose action is scale dependent. It is the combination of the saddle point and its possible deformations in the thermal bath that has physical meaning.

In fig. 2 we present the calculation of the nucleation rate for several values of the parameters of the model. Each row is computed for initial potentials U_{k_0} with the same values of $\mu_{k_0}^2$ and λ_{k_0} . It also has the same value of k_f . Thus, it corresponds to the same Z_2 -symmetric theory in the limit $\gamma_{k_0} = 0$. Different values of γ_{k_0} are used for the three calculations in each row, so that the saddle points have different profiles and, therefore, different nucleation rates are predicted. Moving down the sequence of rows, the value of λ_{k_0} is kept fixed, while $|\mu_{k_0}^2|$ is reduced. The effective dimensionless coupling $\lambda_{k_0}/\sqrt{2|\mu_{k_0}^2|}$ of the Z_2 -symmetric theory increases, and the resulting potentials have more pronounced barriers relative to the location of the minima. This indicates that the effect of fluctuations should be enhanced. The last row corresponds to a Z_2 -symmetric theory that starts approaching the Wilson-Fisher fixed point during the evolution of the potential, before deviating towards the phase with symmetry breaking or the symmetric one [18, 21, 27, 35]. We have tried to keep the values of the predicted nucleation rates comparable in each column of fig. 2. However, the discontinuity in the

field expectation value during the first-order phase transition $\Delta\phi = \phi_t$ decreases as we move down each column. Thus, the strength of the phase transition diminishes.

The most striking aspect of the comparison of the results in each column concerns the relative values of S_k and $\ln(A_k/k_f^4)$. In the first column the contribution of the prefactor to the nucleation rate is much smaller than that of the action of the saddle point. The main role of the prefactor is to remove the k -dependence from I/k_f^4 . As we move down each column, the difference between S_k and $\ln(A_k/k_f^4)$ diminishes. In the last row the two quantities are comparable. This confirms our expectation that the effects of fluctuations should be enhanced in more weakly first-order phase transitions. The second observation concerns the k -dependence of the predicted nucleation rate. In the last row the contribution from the prefactor fails to cancel completely the k dependence of the action of the saddle point. In more quantitative terms, when the prefactor is taken into account the k dependence of the nucleation rate is reduced by a factor ~ 10 in the first row, while only by a factor ~ 2 in the last one. The reason for the above behaviour is clear. In the last row and for $k = k_f$, the nucleation rate is roughly equal to or smaller than the contribution from the prefactor. Thus, the effect of the next order in the expansion around the saddle point is important and can no longer be neglected. This indicates that there is a limit for the validity of Langer's picture of homogeneous nucleation [2]. For sufficiently weakly first-order phase transitions the saddle point of the semiclassical approximation is overwhelmed by the fluctuations around it. As a result, one can no longer rely on a picture based on the semiclassical approximation.

5 Summary and conclusions

In this paper we addressed the problem of the calculation of nucleation rates for first-order phase transitions in the context of high-temperature field theories. The most commonly employed field-theoretical tool in such investigations is the effective potential, usually computed within a perturbative scheme. A first-order phase transition is expected if two minima are present in the effective potential, separated by a barrier. The nucleation rate is calculated through an expansion around the saddle point of the functional integral that interpolates between the two minima. The rate is given by an expression that involves an exponential suppression by the action of the saddle point, and a pre-exponential factor that includes the fluctuation determinant around the saddle-point configuration.

Several major obstacles must be overcome before a consistent description can be obtained. The most obvious one is a consequence of the convexity of the effective potential. This precludes the discussion of tunnelling, as no barrier exists between the minima. All relevant information is washed out by the Maxwell construction. In most studies, a perturbative approximation to the generating functional of the 1PI Green functions is used instead of the effective potential. Such a quantity has non-convex parts, but it also has imaginary parts that are difficult to interpret. For radiatively induced first-order phase transitions (a common occurrence in field theory) there are more conceptual difficulties. It is not clear which fluctuations of the system generate the two-minimum structure of the potential and which are associated with the pre-exponential factor in the calculation of the nucleation rate. It is difficult to resolve in a clear way the issue of double-counting the effect of fluctuations. At the technical level, a serious issue concerns the ultraviolet divergences that appear in the calculation of the pre-exponential factor. They must be cancelled by counterterms of the original action, in a way consistent with the calculation of the potential that determines the action of the saddle point.

In this work we followed an approach that resolves all the above issues. It relies on the introduction of a coarse-graining scale k in the problem. This scale separates the high-frequency fluctuations of the system, which may be responsible for the presence of the second minimum through the Coleman-Weinberg mechanism, from the low-frequency ones which are relevant for tunnelling. The appropriate tool for the calculation of the nucleation rate is the coarse-grained free energy at a non-zero value of k . The pre-exponential factor is well-defined and finite, as the scale k acts as an ultraviolet cutoff in the calculation of the fluctuation determinants. This is a natural consequence of the fact that all fluctuations with typical momenta above k are already incorporated in the form of the coarse-grained

free energy.

We employed the formalism of the effective average action Γ_k [13, 15, 18], which we identified with the coarse-grained free energy. As a starting point, we considered Γ_k for a real scalar field at a scale k_0 below the temperature, such that the theory has an effective three-dimensional description. We approximated Γ_k by a standard kinetic term and a potential with two minima given by eq. (2.21). We assumed that this form of the potential results from the bare potential U_Λ after the integration of (quantum and thermal) fluctuations between the scales Λ and k_0 . Some of these fluctuations may correspond to additional massive degrees of freedom that decoupled above the scale k_0 . If the two-minimum structure is a consequence of the integration of fluctuations of effective three-dimensional degrees of freedom (additional scalar or gauge fields) the scale k_0 must be taken sufficiently small for this structure to emerge [27]–[30].

We computed the form of the potential U_k at scales $k \leq k_0$ by integrating an evolution equation derived from an exact flow equation for Γ_k . U_k is non-convex for non-zero k , and approaches convexity only in the limit $k \rightarrow 0$. The nucleation rate must be computed for k larger than the scale k_f at which the functional integral in the definition of U_k starts receiving contributions from field configurations that interpolate between the two minima. This happens when $-k^2$ becomes approximately equal to the negative curvature at the top of the barrier [32]. For $k > k_f$ the typical length scale of a thick-wall critical bubble is $\gtrsim 1/k$.

We performed the calculation of the nucleation rate for a range of scales above and near k_f . We found that the saddle-point configuration has an action S_k with a significant k dependence. For strongly first-order phase transitions, the nucleation rate $I = A_k \exp(-S_k)$ is dominated by the exponential suppression. The main role of the prefactor A_k , which is also k -dependent, is to remove the scale dependence from the total nucleation rate. The implication of our results is that the critical bubble should not be identified just with the saddle point of the semiclassical approximation. It is the combination of the saddle point and its possible deformations in the thermal bath (accounted for by the fluctuation determinant in the prefactor) that has physical meaning.

For progressively more weakly first-order phase transitions, the difference between S_k and $\ln(A_k/k_f^4)$ diminishes. This indicates that the effects of fluctuations become more and more enhanced. At the same time a significant k -dependence of the predicted nucleation rate develops. The reason for the above deficiency is clear. When the nucleation rate is roughly equal to or smaller than the contribution from the prefactor, the effect of the next order in the expansion around the saddle point is important and can no longer be neglected. This indicates that there is a limit for the validity of Langer’s picture of homogeneous nucleation [2]. For sufficiently weakly first-order phase transitions the saddle point of the semiclassical approximation is dominated by the fluctuations around it. Despite the presence of a discontinuity in the order parameter, one can no longer rely on a picture based on the semiclassical approximation. An alternative picture must be developed for the description of the physical system [41].

Finally, we point out that our results are relevant for the question of quantum tunnelling in the zero-temperature three-dimensional theory. Only a small numerical factor differentiates between our expressions and the ones that determine the quantum-tunnelling rate in that case [4, 5]. The essential qualitative conclusions remain unaffected.

Acknowledgements We would like to thank C. Wetterich for many helpful discussions. The work of N.T. was supported by the E.C. under TMR contract No. ERBFMRX-CT96-0090.

References

- [1] P.A. Rikvold and B.M. Gorman, *Ann. Rev. Comput. Phys. I*, edited by D. Stauffer, (World Scientific, Singapore, 1994) p. 149.
- [2] J. Langer, *Ann. Phys.* **41**, 108 (1967); *ibid.* **54**, 258 (1969); *Physica* **73**, 61 (1974).
- [3] S. Coleman, *Phys. Rev. D* **15**, 2929 (1977).
- [4] C.G. Callan and S. Coleman, *Phys. Rev. D* **16**, 1762 (1977).
- [5] I. Affleck, *Phys. Rev. Lett.* **46**, 388 (1981).
- [6] A.D. Linde, *Nucl. Phys. B* **216**, 421 (1983).
- [7] L.P. Csernai and J.I. Kapusta, *Phys. Rev. D* **46**, 1379 (1992).
- [8] W.N. Cottingham, D. Kalafatis and R. Vinh Mau, *Phys. Rev. B* **48**, 6788 (1993).
- [9] J. Baacke and V.G. Kiselev, *Phys. Rev. D* **48**, 5648 (1993); J. Baacke, *ibid.* **52**, 6760 (1995).
- [10] J. Kripfganz, A. Laser and M.G. Schmidt, *Nucl. Phys. B* **433**, 467 (1995).
- [11] S. Coleman and E. Weinberg, *Phys. Rev. D* **7**, 1888 (1973).
- [12] E. Weinberg and A. Wu, *Phys. Rev. D* **36**, 2474 (1987).
- [13] C. Wetterich, *Nucl. Phys. B* **352**, 529 (1991); *Z. Phys. C* **57**, 451 (1993); *ibid.* **60**, 461 (1993).
- [14] L.P. Kadanoff, *Physics* **2**, 263 (1966).
- [15] C. Wetterich, *Phys. Lett. B* **301**, 90 (1993).
- [16] K.G. Wilson, *Phys. Rev. B* **4**, 3174 and 3184 (1971); K.G. Wilson and I.G. Kogut, *Phys. Rep.* **12**, 75 (1974); F.J. Wegner, in: *Phase Transitions and Critical Phenomena*, vol. 6, eds. C. Domb and M.S. Green (Academic Press, New York, 1976).
- [17] N. Tetradis and C. Wetterich, *Nucl. Phys. B* **398**, 659 (1993); *Int. J. Mod. Phys. A* **9**, 4029 (1994).
- [18] N. Tetradis and C. Wetterich, *Nucl. Phys. B* **422**, 541 (1994).
- [19] M. Reuter, N. Tetradis and C. Wetterich, *Nucl. Phys. B* **401**, 567 (1993).
- [20] N. Tetradis and D. Litim, *Nucl. Phys. B* **464**, 492 (1996).
- [21] J. Berges, N. Tetradis and C. Wetterich, *Phys. Rev. Lett.* **77**, 873 (1996).
- [22] S. Bornholdt, N. Tetradis and C. Wetterich, *Phys. Lett. B* **348**, 89 (1995); *Phys. Rev. D* **53**, 4552 (1996); S. Bornholdt, P. Büttner, N. Tetradis and C. Wetterich, preprint CERN-TH/96-67, cond-mat/9603129.
- [23] M. Reuter and C. Wetterich, *Nucl. Phys. B* **391**, 147 (1993); *ibid.* **408**, 91 (1993); *ibid.* **417**, 181 (1994); *ibid.* **427**, 291 (1994).
- [24] D. Litim, C. Wetterich and N. Tetradis, *Mod. Phys. Lett. A* **12**, 2287 (1997).
- [25] B. Bergerhoff, D. Litim, S. Lola and C. Wetterich, *Int. J. Mod. Phys. A* **11**, 4273 (1996); B. Bergerhoff, F. Freire, D. Litim, S. Lola and C. Wetterich, *Phys. Rev. B* **53**, 5734 (1996).

- [26] B. Bergerhoff and C. Wetterich, Nucl. Phys. B **440**, 171 (1995); preprints HD-THEP-95-37, hep-ph/9508352 and HD-THEP-96-51, hep-ph/9611462.
- [27] N. Tetradis, Nucl. Phys. B **488**, 92 (1997).
- [28] J. Berges, D.U. Jungnickel and C. Wetterich, preprint HD-THEP-97-20, hep-ph/9705474.
- [29] J. Berges, N. Tetradis and C. Wetterich, Phys. Lett. B **393**, 387 (1997).
- [30] J. Berges and C. Wetterich, Nucl. Phys. B **487**, 675 (1997).
- [31] M. Alford and J. March-Russell, Nucl. Phys. B **417**, 527 (1994).
- [32] A. Ringwald and C. Wetterich, Nucl. Phys. B **334**, 506 (1990); N. Tetradis and C. Wetterich, Nucl. Phys. B **383**, 197 (1992).
- [33] J. Alexandre, V. Branchina and J. Polonyi, preprint cond-mat/9803007.
- [34] F.J. Wegner and A. Houghton, Phys. Rev. A **8**, 401 (1973); S. Weinberg, Critical phenomena for field theorists, in Erice Subnuc. Phys. 1 (1976); J. Polchinski, Nucl. Phys. B **231**, 269 (1984); A. Hasenfratz and P. Hasenfratz, Nucl. Phys. B **270**, 687 (1986); J.F. Nicoll and T.S. Chang, Phys. Lett. A **62**, 287 (1977); T.S. Chang, J.F. Nicoll and J.E. Young, Phys. Lett. A **67**, 287 (1978); U. Ellwanger, Phys. Lett. B **335**, 364 (1994).
- [35] J. Adams, J. Berges, S. Bornholdt, F. Freire, N. Tetradis and C. Wetterich, Mod. Phys. Lett. A **10**, 2367 (1995).
- [36] S. Seide and C. Wetterich, preprint HD-THEP-98-20, cond-matt/9806372.
- [37] W.H. Press, B.P. Flannery, S.A. Teukolsky and W.T. Vetterling, *Numerical Recipes: The Art of Scientific Computing* (University Press, Cambridge, 1988).
- [38] S. Coleman, in *The Whys of Subnuclear Physics*, Proceedings of the International School, Erice, Italy, 1977, ed. by A. Zichichi, Subnuclear Series Vol. 15 (Plenum, New York, 1979).
- [39] S. Wolfram, *The Mathematica book*, 3rd ed. (Wolfram Media/Cambridge University Press, 1996).
- [40] W. Buchmüller, Z. Fodor, T. Helbig and D. Walliser, Ann. Phys. **234**, 260 (1994); A. Kusenko, Phys. Lett. B **358**, 47 (1995).
- [41] M. Gleiser, E.W. Kolb and R. Watkins, Nucl. Phys. B **364**, 411 (1991); M. Gleiser and E.W. Kolb, Phys. Rev. Lett. **69**, 1304 (1992); Phys. Rev. D **48**, 1560 (1993); N. Tetradis, Z. Phys. C **57**, 331 (1993); G. Gelmini and M. Gleiser, Nucl. Phys. B **419**, 129 (1994); M. Gleiser, Phys. Rev. Lett. **73**, 3495 (1994); Phys. Rev. D **49**, 2978 (1994); E.J. Copeland, M. Gleiser and H.-R. Müller, Phys. Rev. D **52**, 1920 (1995); M. Gleiser, A. Heckler and E.W. Kolb, Phys. Lett. B **405**, 121 (1997); J. Borrill and M. Gleiser, Nucl. Phys. B **483**, 416 (1997).