Chiral Disorder and QCD at Finite Chemical Potential

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We investigate the effects of a finite chemical potential μ in QCD viewed as a disordered medium. In the quenched approximation, $A_4 = i\mu$ induces a complex electric Aharonov-Bohm effect that causes the diagonal contribution to the quark return probability to vanish at $\mu = m_\pi/2$ (half the pion mass). In two-color QCD, the weak-localization contribution to the quark return probability remains unaffected causing a mutation in the spectral statistics. In full QCD, the complex electric flux is screened and the light quarks are shown to diffuse asymmetrically with a substantial decrease in the conductivity along the 'spatial' directions. Mean-field arguments suggest that a d=1 percolation transition may take place in the range $1.5\rho_0 < \rho < 3\rho_0$, where ρ_0 is nuclear matter density.

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1. QCD at finite chemical potential μ is still not well understood despite the many efforts invested by a number of groups in the past years. Lattice Monte-Carlo algorithms are difficult to implement at finite μ owing to the complex character of the measure [1]. Results from strong coupling arguments [2] and quenched simulations [1,3] are available but do not seem to be transparent physically. A number of theoretical constraints can be implemented using symmetry and data [4]. However, they are only reliable for densities typically of the order of nuclear matter density ρ_0. Results from constituent quark models at higher densities point at the possibility of a chiral transition at about 3 times nuclear matter density [5] and the occurrence of a diquark superconducting phase at even higher densities [6].

In this letter we would like to address the effects of a finite chemical potential on the chiral disorder of the QCD ground state. At $\mu = 0$ we have recently [7,8] shown that light quarks in a finite Euclidean volume V are in a diffusive mode, with a diffusion constant $D = 2F^2/\Sigma$ where F is the weak pion decay constant and $\Sigma = |\langle\overline{q}q\rangle|$ the light quark condensate. The effects of matter cause the medium to change thereby affecting the diffusion properties of the light quarks. In many ways our problem is similar to the problem of electrons in disordered metals under the influence of external sources [9].

2. The eigenvalue equation of the Dirac operator for fundamental quarks in a fixed gluon field A at finite chemical potential μ is

$$(i\overline{\nabla}[A] + i\mu\gamma_4)\, q_k = \lambda_k[A]\, q_k \ . \tag{1}$$

for the right-eigenfunctions, and

$$(i\overline{\nabla}[A] - i\mu\gamma_4)\, Q_k = \lambda_k^*[A]\, Q_k \ . \tag{2}$$

for the left-eigenfunctions. The eigenvalues are complex and paired by chiral symmetry. The set (q_k, Q_k) is biorthogonal. Generalizing the construction [7] for the case of a finite chemical potential, we may write the probability $p(t, \mu)$ for a light quark to start at $x(0)$ in V and return back to the same position $x(t)$ after a proper time duration t, as

$$p(t,\mu) = \frac{V^2}{N}\, e^{-2m|t|} \Big\langle |\langle x(0)|e^{i(i\overline{\nabla}[A]+i\mu\gamma_4)|t|}|x(0)\rangle|^2 \Big\rangle_A \ . \tag{3}$$

The averaging in (3) is over all gluon configurations using the unquenched QCD measure with massive (sea) quarks. The normalization in (3) is per state, where N is the total number of quark states in the four-volume V. Equation (3) can be resolved in terms of (1-2)

$$p(t) = \frac{V^2}{N} e^{-2m|t|} \sum_{j,k}$$
$$\times \Big\langle e^{i|t|(\lambda_j - \lambda_k^*)[A]} q_j(x) Q_j^*(x)\, Q_k(x) q_k^*(x) \Big\rangle_A \tag{4}$$

where the exponent $e^{-2m|t|}$ is solely due to the valence quark mass. We note that (4) is gauge-invariant and amenable to lattice Monte-Carlo simulation. It requires both the eigenvalues and eigenfunctions.

For analytical considerations, it is best to rewrite (3) in terms of the standard Euclidean propagators for the quark field,

$$p(t,\mu) = \frac{V^2}{N} \lim_{y\to x} \int \frac{d\lambda_1 d\lambda_2}{(2\pi)^2}\, e^{-i(\lambda_1 - \lambda_2)|t|}$$
$$\times \Big\langle \mathrm{Tr}\left(S(x,y;z_1,\mu)S^\dagger(x,y;z_2,\mu)\right) \Big\rangle_A \tag{5}$$

with $z_{1,2} = m - i\lambda_{1,2}$, and

$$S(x,y;z,\mu) = \langle x|\frac{1}{i\overline{\nabla}[A] + i\mu\gamma_4 + iz}|y\rangle \ . \tag{6}$$

Since the eigenvalues (1) are complex, it is important that $m > \max \mathrm{Im}\lambda_k$ in (5-6). For small μ the imaginary parts are of order μ^2 (second order perturbation theory) so it is enough to have $m > \mu^2$ in units where the infrared scale is of order 1. For large μ, m should be made large and

then reduced after integration. This will be understood throughout.

Setting $\lambda_{1,2} = \Lambda \pm \lambda/2$ and neglecting the effects of Λ in the averaging in (5) [7], we find that in the flavor symmetric limit, the correlation function in (5) relates to the 'baryonic' pion correlation function [10] after a proper analytical continuation of the current quark mass [7]. Specifically,

$$p(t,\mu) = \frac{EV^2}{2\pi N} \lim_{y \to x} \int \frac{d\lambda}{2\pi} e^{-i\lambda|t|} \mathbf{C}_{\pi_B}(x,y;z) \qquad (7)$$

where

$$\mathbf{1}^{ab} \, \mathbf{C}_{\pi_B}(x,y;z) =$$
$$\left\langle \text{Tr}\left(S(x,y;z,\mu) i\gamma_5 \tau^a S(y,x;z,-\mu) i\gamma_5 \tau^b \right) \right\rangle_A \qquad (8)$$

with $z = m - i\lambda/2$ and $E = \int d\Lambda$. For conventional pions both propagators in (8) carry μ with the same sign.

3. The effects of μ in (1) is that of a complex and constant 4-vector potential $A_4 = i\mu$. It breaks particle-antiparticle symmetry much like a vector potential breaks particle-particle (antiparticle-antiparticle) symmetry. It acts like a complex electric Aharonov-Bohm effect in the particle-antiparticle channel. The particle-antiparticle system breaks apart for μ typically of the order of the binding energy (about the pion mass). This phenomenon is reminiscent of the destruction of heavy-mesons by chromo-electric fields [11], and charge or spin density waves by transverse electric fields [12], although not identical since in our case the 'electric field' is zero.

In the quenched approximation, the only dependence on μ in (8) is that shown in the external propagators. In the semi-classical approximation and for three colors (for two colors see below) it acts as a 'complex' flux on the 'diffusons' (particle-antiparticle) [7]. For $z = m$, the long paths contributions to (8) are given by

$$\mathbf{C}_{\pi_B}(x,y;m) \approx \frac{1}{V} \sum_Q e^{iQ\cdot(x-y)} \frac{\Sigma^2}{F^2} \frac{1}{\tilde{Q}^2 + m_\pi^2} \qquad (9)$$

with $Q_\alpha = n_\alpha 2\pi/L$ and $\tilde{Q}_\alpha = Q_\alpha + 2i\mu\delta_{4\alpha}$ in $V = L^4$. The factor 2 in front of A_4 reflects on the fact that the fluxes *add* in the 'diffuson'. Using the Gell-Mann Oakes Renner (GOR) relation $F^2 m_\pi^2 = m\Sigma$, and the analytical continuation $m \to m - i\lambda/2$, we find

$$\mathbf{C}_{\pi_B}(x,y;z) \approx \frac{1}{V} \sum_Q e^{iQ\cdot(x-y)} \frac{2\Sigma}{-i\lambda + 2m + D\tilde{Q}^2} \qquad (10)$$

with the diffusion constant $D = 2F^2/\Sigma$ [7]. Inserting (10) into (7), we observe that the 'diffusion' pole in the lower part of the complex plane depends critically on the value of the chemical potential μ.

In the zero mode approximation $n_\alpha = 0$ or for large times $t > \tau_{\text{erg}} = L^2/D$, the quark return probability is

$$p(t,\mu) \approx \theta(m_\pi - 2\mu) \, e^{-D(m_\pi^2 - 4\mu^2)|t|} \qquad (11)$$

where we have used $E/\Delta = N$ and $\varrho = 1/\Delta V$, with $\Sigma = \pi\varrho$, according to the Banks-Casher relation. Here Δ is the mean interlevel spacing between the eigenvalues for $\mu = 0$. The occurrence of the step-function theta in (11) reflects on the fact that the diffusion pole moves from the lower-half to the upper-half of the complex λ-plane. For $t > \tau_{\text{erg}}$ the quark return probability vanishes for $\mu = m_\pi/2$ in the quenched approximation. Physically, this means that the complex electric flux splits the quark-antiquark pair in the quenched approximation a situation reminiscent of the magnetic fluxes in type-I superconductors [13]. This result is consistent with current quenched lattice simulations [3] and the results of schematic chiral random matrix models for finite μ [14–16].

We note that in the double scaling limit $Dm_\pi^2 t_H \sim mV \ll 1$ and $D\mu^2 t_H \sim \mu^2 V \ll 1$, (11) is about 1 and universal. This regime is amenable to a random matrix model analysis and signals the onset of a new universality for the complex eigenvalues of (1). It can be modeled using a reduction to 0-dimension.

4. For two-color QCD the situation is special since in this case the Dirac operator possesses an additional symmetry [17] due to the pseudo-real nature of the SU(2) representations. In the diffusive picture of the QCD vacuum this means that the quark return probability receives contributions from both 'diffusons' (diagonal) and 'cooperons' (interference) paths in the semi-classical approximation [7] (and references therein). The 'cooperons' are just the weak-localization contribution to the quark return probability [9]. In the standard description of diffusion they follow from the interference between the classical loops traveled in opposite directions (coherent backscattering) and reflect on the time-reversal invariance of the underlying microscopic Hamiltonian. While the 'diffusons' sense 2 flux lines, the 'cooperons' are flux-blind. A rerun of the above arguments now give

$$\mathbf{C}_{\pi_B}(x,y;z) \approx +\frac{1}{V} \sum_Q e^{iQ\cdot(x-y)} \frac{2\Sigma}{D\tilde{Q}^2 + 2m - i\lambda}$$
$$+\frac{1}{V} \sum_Q e^{iQ\cdot(x-y)} \frac{2\Sigma}{DQ^2 + 2m - i\lambda} \qquad (12)$$

instead of (9). The first term is the 'diffuson' contribution, and the second term the 'cooperon' contribution. Inserting (12) into (7) yields

$$p(t,\mu) \approx e^{-Dm_\pi^2|t|} \left(\theta(m_\pi - 2\mu) \, e^{+4D\mu^2|t|} + 1 \right) \qquad (13)$$

in the zero mode approximation or $t > \tau_{\text{erg}} = L^2/D$. For $\mu > m_\pi/2$ the 'diffuson' contribution (first term) drops and we are only left with the 'cooperon' contribution which is of order $e^{-2m|t|}$. The latter is of order 1 and universal for $\mu^2 < m \ll 1/V$. The transition to the 'cooperon' phase is simply a transition to the superconducting phase in this case.

In the universal regime and for small current quark masses, the chemical potential is $\mu \ll 1/\sqrt{V}$ and small. Hence, the complex eigenvalues λ_k carry an imaginary part of order $1/V$ which is of the order of the microscopic level spacing for the 'unperturbed' real parts. If we focus on the level-correlations between only the real parts of λ's in the microscopic limit $x = V\lambda \sim 1$ we expect a mutation in the level correlations from the orthogonal to unitary ensemble. The mutation follows a migration of part of the quark levels from the real axis to the complex plane under the influence of the tiny chemical potential. The spectral rigidity $\Sigma_2(N, \mu)$ for the real parts of λ's can be estimated using semi-classical arguments [7]. The result is [18]

$$\Sigma_2(N, \mu) \approx \theta(m_\pi - 2\mu)\frac{1}{2\pi^2}\ln\left(1 + \frac{N^2}{\tilde{\alpha}^2}\right)$$
$$+ \frac{1}{2\pi^2}\ln\left(1 + \frac{N^2}{\alpha^2}\right) \quad (14)$$

where $N = E/\Delta \gg 1$, $\tilde{\alpha} = D(m_\pi^2 - 4\mu^2)/2\Delta$ and $\alpha = 2m/\Delta$. The level spacing $\Delta = 1/\varrho V$ is taken to be that of the $\mu = 0$ limit. The behavior (14) can be addressed using current quenched lattice Monte-Carlo simulations in QCD [1,3].

5. In unquenched QCD, the vacuum supports quark-antiquark pairs. The 'baryonic' pion correlations are screened by pair creation, rendering the quark-antiquark system blind to the complex electric Aharonov-Bohm flux (constant A_4). As a result, the correlations in (8) are primarily that of a quark-antiquark in a vacuum for zero nucleon density with $\mu \leq m_N/3$ where m_N is the nucleon mass (ignoring binding energies).

At finite nucleon density, the quark return probability follows from a pertinent analytical continuation of the pion propagator in matter. In a mean-field approximation we have [19,20]

$$\mathbf{C}_{\pi_B}(x, y; m) \approx \frac{1}{V}\sum_Q e^{iQ\cdot(x-y)}\frac{(1-\alpha\rho)^2}{(1-\beta\rho)}\frac{\Sigma^2}{F^2}$$
$$\times \left(Q_4^2 + \left(\frac{1-\gamma\rho}{1-\beta\rho}\right)\vec{Q}^2 + \left(\frac{1-\alpha\rho}{1-\beta\rho}\right)m_\pi^2\right)^{-1} \quad (15)$$

with $Q_\sigma = n_\sigma 2\pi/L$ in $V = L^4$. Here $\alpha = \langle N|\overline{q}q|N\rangle/\Sigma$ measures the strength of the pion-nucleon sigma term relative to the scalar condensate, with $1/\alpha \sim 3\rho_0$. The parameters $\beta \sim \alpha$ and $\gamma \sim 2\alpha$ relate to the S-wave pion-nucleon scattering lengths [4,19]. The leading density approximation follows from the mean-field analysis by keeping only the leading term in the nucleon density ρ [4,19]. In the space-like regime under consideration there is no imaginary contribution to (15).

Using the GOR relation $F^2 m_\pi^2 = m\Sigma$, and the analytical continuation $m \rightarrow m - i\lambda/2$, we may rewrite (15) as

$$\mathbf{C}_{\pi_B}(x, y; z) \approx \frac{1}{V}\sum_Q e^{iQ\cdot(x-y)}$$
$$\times \frac{2\Sigma(1-\alpha\rho)}{-i\lambda + 2m + D_4 Q_4^2 + D_S\vec{Q}^2} \quad (16)$$

with temporal and spatial diffusion coefficients

$$D_4 = D\frac{1-\beta\rho}{1-\alpha\rho},$$
$$D_S = D\frac{1-\gamma\rho}{1-\alpha\rho}. \quad (17)$$

Hence $D_4 \sim D$ and ρ independent, while D_S vanishes for $\rho \sim 1/\gamma \sim 1/(2\alpha) \sim 1.5\rho_0$ in the mean-field approximation [21]. At this point the quark density of states at zero virtuality is about $\varrho(\mu) \sim (1-\alpha\rho)\Sigma \sim \varrho/2$ by the Banks-Casher relation [22]. Using the Kubo-formula we conclude that the conductivity vanishes along the spatial directions $\sigma_S = D_S\varrho = 0$. This is not a metal-insulator transition as the conductivity $\sigma_4 = D_4\varrho \neq 0$ is still non-zero. It can be regarded as an 'asymmetric' percolation transition from d=4 to d=1, with a diffusive quark return probability [8] of the form

$$p(t, \mu) \approx \frac{e^{-2m|t|}}{\sqrt{4\pi E_4|t|}} \quad (18)$$

where we have used $E/\Delta_* = N \gg 1$ and a density dependent level spacing $\Delta_*/\Delta \sim 1/(1-\alpha\rho)$. Here $E_4 = D_4/L^2$ is the 'temporal' Thouless energy as opposed to $E_S = D_S/L^2$ the 'spatial' Thouless energy. A similar phenomenon takes place at finite temperature [8].

At this stage there are two courses of action: The conductivity σ_4 vanishes continuously with the depletion of the number of quark states at zero virtuality corresponding to a vanishing of the quark density of states at $\rho \sim 3\rho_0$. (In fact this is what happens if only the leading density approximation were used.) This transition is likely of second-order or higher and would be in overall agreement with some constituent quark model results [5]. Alternatively, it may terminate abruptly for a density $1.5\rho_0 < \rho < 3\rho_0$ through a d=1 percolation transition. This is intuitively more appealing if we were to proceed from the high-density region backward, and support the idea that a nucleon Fermi-surface in 3-dimensions correspond to an array of 'rods' in d=4 Euclidean space (nucleon-worldlines) forcing the conductivity to be essentially 1-dimensional by Pauli-blocking. Ideas in favor of a percolation transition at finite density have been also stressed recently by Satz [23] using different arguments.

6. We have shown that the disordered properties of the QCD ground state are quantitatively altered by a finite chemical potential μ. In quenched QCD the effects of $A_4 = i\mu$ are analogous to that of a complex electric Aharonov-Bohm effect, causing the 'baryonic' quark-antiquark pair to accumulate 2 flux lines and rupture at $\mu = m_\pi/2$. This result is in agreement with quenched

lattice simulations [1,3]. In two-color QCD, the quark-quark and antiquark-antiquark pairs are flux-blind. As a result, the weak-localization contribution to the quark return probability remains unaffected. In the universal limit $\mu^2 < m \ll 1/V$, the quark spectrum in two-color QCD exhibits a change in the spectral statistics from an orthogonal to unitary ensemble again at $\mu = m_\pi/2$.

In the unquenched approximation the electric flux is screened and the diffusion becomes asymmetric. The light quarks take longer time to diffuse along the spatial directions owing to the presence of a Fermi surface. This asymmetry is commensurate with the softening in the pion dispersion relation at $\rho \sim \rho_0$ due to pion-nucleon S-wave rescattering. As a result, the bulk Ohmic conductivity of the system becomes quasi 1-dimensional, with a potential for a d=1 percolation transition in the density range $1.5\rho_0 < \rho < 3\rho_0$.

Most of our results can be numerically checked by analyzing the quark return probability in the ergodic and diffusive regime at finite chemical potential in lattice QCD or in continuum models such as the instanton liquid model [4,24].

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