

# Relation between quark masses and weak mixings

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## ABSTRACT

Simple transformation formulas between fermion matrices and observables, and numerical values of quark matrices, are obtained on a particular weak basis with one quark matrix diagonal and the other with vanishing elements 1-1, 1-3 and 3-1, and with only the element 2-2 complex. When we choose  $M_u$  diagonal, then  $M_d$  shows intriguing numerical properties which suggest a four parameter description of it, which implies  $V_{us} \simeq \sqrt{m_d/m_s}$ ,  $V_{cb} \simeq (3/\sqrt{5})(m_s/m_b)$  and  $V_{ub} \simeq (1/\sqrt{5})(\sqrt{m_d m_s}/m_b)$ . Few comments on mass-mixing relations are added.

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In the standard model Lagrangian [1], written in a general weak basis, quark mass matrix elements are not explicitly related to physical observables, that is quark masses and weak mixings. The problem of finding such a relation, without extra symmetries, has been addressed in [2, 3, 4, 5]. In particular, in [2] it was shown that it is always possible to find a weak basis where the quark mass matrices have the nearest neighbor interaction form and depend on twelve real parameters. Two of these twelve parameters are arbitrary [3] and related to the phase convention of the weak mixing matrix [4]. Then, in [5], it was shown that it is always possible to set one quark matrix in the diagonal form and the other in a form with zero entries in positions 1-1, 2-2 and 3-1, and with only the element 1-2 complex. In such a way mass matrices contain ten real parameters, exactly the same number of physical observables, six quark masses and three mixing angles and one phase. This corresponds to the choice of a *minimal parameter basis* [6]. As one mass matrix is chosen to be diagonal, it is relatively easy to obtain exact transformation formulas between mass matrices and observables. Other minimal parameter bases are considered in [7, 8]. Here we describe a further minimal parameter basis, which shows interesting properties and on which transformation formulas are simple.

In fact it is also always possible [9] to choose a weak basis for which

$$M_d = \text{diag}(m_d, m_s, m_b) \tag{1}$$

and

$$M_u = \begin{pmatrix} 0 & M_{12} & 0 \\ M_{21} & M_{22} & M_{23} \\ 0 & M_{32} & M_{33} \end{pmatrix} \tag{2}$$

(or  $M_u$  is diagonal and  $M_d$  has the form (2)).

On this basis the relation between mass matrices and observables is given by

$$M_u M_u^+ = K^+ \cdot \text{diag}(m_u^2, m_c^2, m_t^2) \cdot K \equiv X^u \tag{3}$$

where  $K$  is the Cabibbo-Kobayashi-Maskawa (CKM) matrix [10]. In the case of  $M_u$  diagonal we have instead

$$M_d M_d^+ = K \cdot \text{diag}(m_d^2, m_s^2, m_b^2) \cdot K^+ \equiv X^d. \quad (4)$$

By writing  $M_{ij} = m_{ij} e^{ir_{ij}}$  we can reconstruct the usual representations of  $K$  [11] by means of three non vanishing phases  $r_{12}$ ,  $r_{22}$  and  $r_{23}$ . The product  $M_u M_u^+$  is then given by

$$\begin{pmatrix} m_{12}^2 & m_{12} m_{22} e^{i(r_{12}-r_{22})} & m_{12} m_{32} e^{ir_{12}} \\ m_{12} m_{22} e^{-i(r_{12}-r_{22})} & m_{21}^2 + m_{22}^2 + m_{23}^2 & m_{22} m_{32} e^{ir_{22}} + m_{23} m_{33} e^{ir_{23}} \\ m_{12} m_{32} e^{-ir_{12}} & m_{22} m_{32} e^{-ir_{22}} + m_{23} m_{33} e^{-ir_{23}} & m_{32}^2 + m_{33}^2 \end{pmatrix}. \quad (5)$$

and the transformation formulas between masses and mixings in  $X$  and mass matrix elements in  $M$  are written in a very simple form

$$m_{12} = \sqrt{X_{11}^u} \quad (6)$$

$$m_{22} = |X_{12}^u|/m_{12} \quad (7)$$

$$m_{32} = |X_{13}^u|/m_{12} \quad (8)$$

$$m_{33} = \sqrt{X_{33}^u - m_{32}^2} \quad (9)$$

$$r_{12} = \text{phase}(X_{13}^u) \quad (10)$$

$$r_{22} = r_{12} - \text{phase}(X_{12}^u) \quad (11)$$

$$M_{23} = (X_{23}^u - m_{22} m_{32} e^{ir_{22}})/m_{33}$$

$$m_{23} = |M_{23}| \quad (12)$$

$$r_{23} = \text{phase}(M_{23}) \quad (13)$$

$$m_{21} = \sqrt{X_{22}^u - m_{22}^2 - m_{23}^2}. \quad (14)$$

In the case of  $M_u$  diagonal the same formulas hold with  $X^u \rightarrow X^d$ . With a phase transformation of quark fields,

$$M_{u,d} \rightarrow \text{diag}(e^{-ir_{12}}, e^{-ir_{23}}, 1) \cdot M_{u,d} \cdot \text{diag}(e^{ir_{23}}, 1, 1), \quad (15)$$

only a phase  $r'_{22} = r_{22} - r_{23}$  remains in the element 2-2, and we obtain, using numerical values of quark masses at  $\mu = M_Z$  as in [7] ( $m_u = 0.00233$ ,  $m_c = 0.677$ ,  $m_t = 181$ ,  $m_d = 0.00469$ ,  $m_s = 0.0934$ ,  $m_b = 3.00 \text{ GeV}$ ) and mixings as in [11] (with  $\delta = 1.35$ ),

$$M_u = \begin{pmatrix} 0 & 1.591 & 0 \\ 0.011 & 7.118 e^{1.334i} & 0.269 \\ 0 & 180.1 & 17.02 \end{pmatrix} \text{ GeV}, \quad (16)$$

and if instead we choose  $M_u$  to be diagonal,

$$M_d = \begin{pmatrix} 0 & 0.024 & 0 \\ 0.021 & 0.105 e^{-1.205i} & 0.106 \\ 0 & 1.333 & 2.685 \end{pmatrix} \text{ GeV}. \quad (17)$$

We can see that in (16), due to the large value of the top quark mass, the biggest matrix element is not 3-3, as in (17), but the element 3-2. This feature is different from the basis in [5] where the biggest element is the element 3-3 either if  $M_u$  or  $M_d$  is diagonal. Moreover, the numerical values in (17) suggest to take  $M_u$  diagonal and

$$M_d = \begin{pmatrix} 0 & a & 0 \\ a & b e^{i\varphi} & b \\ 0 & c & 2c \end{pmatrix}, \quad (18)$$

where  $a$ ,  $b$  and  $c$  are of order  $10^{-2}$ ,  $10^{-1}$  and  $1 \text{ GeV}$ , respectively. From (18) we obtain the approximate expression

$$M_d \simeq \begin{pmatrix} 0 & \sqrt{m_d m_s} & 0 \\ \sqrt{m_d m_s} & m_s e^{i\varphi} & m_s \\ 0 & m_b/\sqrt{5} & 2m_b/\sqrt{5} \end{pmatrix}. \quad (19)$$

In the heavy quark limit  $m_b \gg m_s, m_d$  we have the effective matrix for the two lightest down quarks

$$M_d \simeq \begin{pmatrix} 0 & \sqrt{m_d m_s} \\ \sqrt{m_d m_s} & m_s \end{pmatrix} \quad (20)$$

which gives the famous relation [12, 13]

$$V_{us} \simeq \sqrt{\frac{m_d}{m_s}}. \quad (21)$$

In the chiral limit  $m_d \ll m_s, m_b$  we have instead, for the two heaviest down quarks

$$M_d \simeq \begin{pmatrix} m_s & m_s \\ m_b/\sqrt{5} & 2m_b/\sqrt{5} \end{pmatrix} \quad (22)$$

and when we diagonalize the Hermitian matrix

$$M_d M_d^\dagger \simeq \begin{pmatrix} m_s^2 & 3m_s m_b/\sqrt{5} \\ 3m_s m_b/\sqrt{5} & m_b^2 \end{pmatrix} \quad (23)$$

we obtain the relation

$$V_{cb} \simeq \frac{3}{\sqrt{5}} \frac{m_s}{m_b}, \quad (24)$$

which gives  $V_{cb} = 0.042$  to be compared with the experimental value  $0.041 \pm 0.005$  [11]. Finally, taking the full matrix

$$M_d M_d^\dagger \simeq \begin{pmatrix} m_d m_s & m_s \sqrt{m_d m_s} e^{-i\varphi} & m_b \sqrt{m_d m_s/5} \\ m_s \sqrt{m_d m_s} e^{i\varphi} & m_s (m_d + 2m_s) & m_s m_b (e^{i\varphi} + 2)/\sqrt{5} \\ m_b \sqrt{m_d m_s/5} & m_s m_b (e^{-i\varphi} + 2)/\sqrt{5} & m_b^2 \end{pmatrix} \quad (25)$$

we have the relation

$$V_{ub} \simeq \frac{1}{\sqrt{5}} \frac{\sqrt{m_d m_s}}{m_b}, \quad (26)$$

which gives  $V_{ub} = 0.003$  to be compared with the experimental range  $0.002 \div 0.005$  [11]. From (21),(24) and (26) we yield also

$$\frac{V_{ub}}{V_{cb}} \simeq \frac{1}{3} V_{us}. \quad (27)$$

Setting  $x = \sqrt{m_d/m_s}$  and  $y = \sqrt{m_s/m_b}$ , we have  $V_{us} \simeq x$ ,  $V_{cb} \simeq (3/\sqrt{5})y^2$  and  $V_{ub} \simeq (1/\sqrt{5})xy^2$  which means that, on this basis, weak mixings, apart from numerical coefficients not so different from 1, are generated by square roots of quark

mass ratios  $x$  and  $y$ . Of course  $x \sim y \sim \lambda$  leads to the Wolfenstein parametrization [14] of the CKM matrix. On the basis with  $M_d$  diagonal and  $M_u$  given by (2) such simple features are lost. Nevertheless weak mixings appear related to up quark ratios (for example  $V_{cb} \simeq 11 m_c/m_t$ ). Then, from the paper [15], where the relations  $V_{us} \simeq \sqrt{m_d/m_s}$ ,  $V_{cb} \simeq m_s/m_b$ , but  $V_{ub} \simeq \sqrt{m_u/m_t}$  were inferred, and our work, we argue that choosing different weak bases we can accordingly obtain different relations between mixings and masses, each of them in agreement with experimental data. Hence, each weak basis may be useful to describe some features of fermion masses and mixings. As a last remark on the basis considered here, we observe that, as written in footnote 6 of [2], in left-right symmetric models [16] both  $M_u$  and  $M_d$  can always take the form (2).

In conclusion, we have obtained very simple formulas for relating fermion matrices to observables, and numerical values of quark matrices on a basis with one quark mass matrix diagonal and the other with three zeros in positions 1-1, 1-3 and 3-1. Such numerical values suggest a simple form for  $M_d$  which does imply relations (21),(24),(26), and (27). Moreover, on this basis weak mixings have a simple expression.

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