An operator expansion for the elastic limit

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Abstract

A leading twist expansion in terms of bi-local operators is proposed for the structure functions of deeply inelastic scattering near the elastic limit $x \to 1$, which is also applicable to a range of other processes. Operators of increasing dimensions contribute to logarithmically enhanced terms which are supressed by corresponding powers of $1 - x$. For the longitudinal structure function, in moment (N) space, all the logarithmic contributions of order $\ln^k N/N$ are shown to be resummable in terms of the anomalous dimension of the leading operator in the expansion.

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a. Introduction: Of special interest in QCD hard processes are "elastic" limits, in which the mass of the final state is small compared to the momentum transfer. In such limits we can study, in principle, the transition from short-distance to long-distance dynamics. Typically, perturbative calculations receive logarithmic enhancements near the elastic limit. Interesting and phenomenologically relevant examples include (but are not limited to): the structure functions in deeply inelastic scattering (DIS) in the limit $x \to 1$ where $x = Q^2/2p \cdot q$, with $q^2 = -Q^2$ the momentum transfer and p the hadron's momentum, the Drell-Yan cross section at partonic threshold, $z \equiv Q^2/\hat{s} \to 1$, with Q the pair mass and $\hat{s}$ the partonic center-of-mass energy squared, and the thrust, T in e^+e^- annihilation for $T \to 1$, where $T \sim (m_1^2 + m_2^2)/s$, with m_i^2 the masses of the two final-state jets that characterize the elastic limit.

In each of these cases, logarithmic corrections at leading power have been resummed to all orders in perturbation theory at leading and next-to-leading logarithmic accuracy [1–3]; for example, the terms like $\alpha_s^n[\ln^m(1 - x)/(1 - x)]_+$, with $m \leq 2n - 1$, in the structure function F_2 in DIS. Until now, however, there has been no technique to resum higher-order logarithmic corrections beyond the leading power of $1 - T$, $1 - z$ or $1 - x$. The phenomenological importance of these corrections has been highlighted recently in the context of Higgs production [4]. They may also play an important role in renormalon phenomenology [5].

In this paper, we take a step toward the resummation of logarithmic corrections at nonleading powers in the elastic limit. We propose that logarithmic enhancements near the boundary of phase space can be analyzed systematically through a bi-local operator product expansion. Our procedure supplements the light-cone expansion in DIS at leading twist, by identifying new operators associated with leading and nonleading behavior of the coefficient functions as $x \to 1$. It is also applicable to processes, such as thrust, for which the light-cone expansion is not available. To be specific, we shall consider the example of logarithmic enhancements to the longitudinal structure function F_L, but we expect, following our comments above, numerous other applications.

For F_L power singularities at $x = 1$ are absent. In moment space, the leading power terms (in N) are of order $\alpha_s^n \ln^k N/N$. We shall argue that all such logarithmic contributions can be treated systematically within the formalism of large-x factorization and the bi-local operator product expansion introduced below. This expansion is distinct from, although consistent with, the light-cone expansion, because operators of different dimensionality all contribute to the leading power in Q, the difference being only that higher dimensional operators produce terms suppressed by higher powers of $1 - x$.

Every inclusive DIS observable is derived as a particular projection of the hadronic tensor

$$W_{\mu\nu}(p, q) = \frac{1}{4\pi} \int d^4y \, e^{-iq \cdot y} \, \langle p|j_\mu^\dagger(0) \, j_\nu(y)|p\rangle \,. \tag{1}$$

We shall denote such a four-dimensional Fourier transform, with respect to momentum k, as $\mathrm{FT}_k^{(4)}$ below. For our discussion, it is enough to consider only electromagnetic interactions. Generalization to the electroweak case is straightforward. Structure functions are obtained via the projections $F_{\mathrm{r}} = P_{\mathrm{r}}^{\mu\nu} W_{\mu\nu}$, for $\mathrm{r} = 2, L$, with

$$P_L^{\mu\nu} = \frac{8x^2}{Q^2}p^\mu p^\nu, \qquad P_2^{\mu\nu} = -\eta^{\mu\nu} + \frac{3}{2}P_L^{\mu\nu}\,. \tag{2}$$

The resummation of logarithmic corrections at leading power in F_2 as $x \to 1$ may be regarded as a consequence of the following factorization theorem [6]:

$$F_2(x, Q^2) = |H_2(Q^2)|^2 \int_x^1 dx' \int_0^{x'-x} dw \, J\left((x' - x - w)Q^2\right) \, V(w) \, \phi(x')$$
$$\equiv |H_2(Q^2)|^2 J \otimes V \otimes \phi \,, \tag{3}$$

where $\otimes$ denotes the convolution in the longitudinal momentum fraction. This theorem has been proven in the axial gauge in Ref. [1]. Below we shall define each of the factors that enter Eq. (3) in a manifestly gauge independent way. $H_2(Q^2)$ is the short distance hard scattering function on either side of the final state cut. V is the soft radiation function that contains all enhancements coming from on-shell propagation of low frequency partons. It is defined as

$$V(w) = \int dy^- e^{-iwy^- p \cdot \bar{v}} \, \langle 0 | \Phi_v^\dagger(0, -\infty) \, \Phi_{\bar{v}}(0, y^- \bar{v}) \, \Phi_v(y^- \bar{v}, -\infty) | 0 \rangle$$
$$\equiv \mathrm{FT}_{-wp}^{(1)} \, \langle 0 | \Phi_v^\dagger(0, -\infty) \, \Phi_{\bar{v}}(0, y^- \bar{v}) \, \Phi_v(y^- \bar{v}, -\infty) | 0 \rangle \,. \tag{4}$$

Here the light-like incoming direction is denoted by v^μ (+ direction) and the parity reflected by $\bar{v}^\mu$ (− direction), with $v \cdot \bar{v} = 1$. In the second line we have introduced a notation for Fourier transforms along the light cone. The Wilson line operator along the v light-cone direction is

$$\Phi_v(x + tv, -\infty) = P \exp\left(-ig_s \int_{-\infty}^t ds \, v^\mu A_\mu(x + sv)\right) \,. \tag{5}$$

ϕ is the parton distribution function that contains the collinear enhancements from the initial state. Its operator definition, for the scattering of quarks, is

$$\phi(x') = \mathrm{FT}_{x'p}^{(1)} \, \langle p | \psi(0) \, \slashed{v} \, \Phi_{\bar{v}}(0, y^-) \, \bar{\psi}(y) | p \rangle \otimes V^{-1} \,, \tag{6}$$

where V^{-1} is defined by $V^{-1} \otimes V = 1$. The role of V^{-1} is to remove soft contributions, which would otherwise be double-counted in both ϕ and V. Finally, J is a function that describes the outgoing "current jet", of invariant mass $(x' - x - w)Q^2$. The partons in J are almost collinear and moving in the $\bar{v}^\mu$ light-cone direction. The operator definition of J is

$$J\left((1 - z)Q^2\right) = \mathrm{FT}_{q+zp}^{(4)} \, \langle 0 | \Phi_v^\dagger(0, -\infty) \, \psi(0) \, \bar{\psi}(y) \, \Phi_v(y, -\infty) | 0 \rangle \otimes V^{-1}. \tag{7}$$

The Wilson lines make the functions manifestly gauge invariant. Whenever they do not explicitly appear in our expressions henceforth it is to be assumed that they are always there.

   b. *Operator expansion:* Let us now generalize the analysis of the leading contributions to F_2. We shall see that the method not only applies directly to F_L, but also opens the way to the analysis of nonleading singular corrections in F_2, and elsewhere.

   As $x \to 1$, $F_L(x, Q^2)$ depends on three scales: the momentum transfer Q, the hadronic mass scale of order Λ_{QCD}, and an intermediate scale $(1 - x)^{1/2}Q$, which is essentially the mass of the final state. We assume that, although x is close to unity, nevertheless $(1 - x)^{1/2}$

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$\gg \Lambda_{\text{QCD}}/Q$, so that the intermediate scale remains perturbative. At the same time, logarithmic behavior in $1 - x$ arises from the limit in which the final state becomes massless relative to Q. Logarithmic contributions to this limit are associated in perturbation theory with the momentum configurations illustrated by Fig. 1. The figure shows a cut diagram representation of the scattering process, in which the final state consists of a single "jet" of particles, moving in the $\bar{v}^\mu$ direction, in the notation of the previous section. Following the general analysis of Ref. [7], in these regions of momentum space the lines of any diagram fall into one of four categories. They are either "soft" lines (subdiagram S in the figure), with momenta that vanish relative to Q in all four components, "hard" lines, which are off-shell by order Q^2, or "jet-like" lines, with one momentum component of order Q, either in the v^μ (parallel to the incoming hadron) or $\bar{v}^\mu$ (part of the outgoing jet). Hard lines are all contained in the subdiagrams labelled H (in the amplitude) and $H^\dagger$ (in the complex conjugate) in Fig. 1. Note that the hard amplitudes are not specific to any structure function. The discussion so far is at the level of the tensor $W^{\mu\nu}$.

Because lines in H and $H^\dagger$ are off-shell by the largest momentum scale, Q, we may extend the factorization program somewhat, and treat them in terms of an "effective Hamiltonian", as an expansion in products of local fields,

$$H_{\text{eff}}^\nu = \sum_i C_i(Q, \mu)\, O_i^\nu(0)\,, \tag{8}$$

where μ is the factorization scale and the operators O_i^ν carry the quantum numbers of the electromagnetic current. The functions C_i are the usual coefficient functions that accompany the hard-scattering function. For the leading-power analysis of F_2 [1], we keep only the lowest-dimension operator, the original electromagnetic current $j^\nu(0) = \bar{\psi}(0)\gamma^\nu\psi(0)$, the matrix elements of whose product is then factorized into the parton distribution ϕ, Eq. (6), which uses up two of the quark fields in $j^\dagger j$, and the jet function J, Eq. (7), which uses up the other two.

Suppose that $P_r^{\mu\nu}$ projects out one of the structure functions. Then we can write

$$\begin{aligned}
P_r^{\mu\nu} W_{\mu\nu} &= P_r^{\mu\nu}\, \text{FT}_q^{(4)}\, \langle p|\, \bar{\psi}(0)\gamma_\mu\psi(0)\, \bar{\psi}(y)\gamma_\nu\psi(y)\, |p\rangle \\
&\approx \text{FT}_{xp}^{(1)}\langle p|\, \bar{\psi}(0)\, \psi(y^-\bar{v})\, |p\rangle \otimes \text{FT}_{q+xp}^{(4)}\, P_r^{\mu\nu}\gamma_\mu\, \langle 0|\psi(0)\, \bar{\psi}(y)\, |0\rangle\, \gamma_\nu\,.
\end{aligned} \tag{9}$$

Noting that the Wilson lines, Eq. (5), are needed to formulate the factorization in a gauge invariant fashion, we see that the leading order factorization of F_2 is thus reproduced. It is obvious that this procedure gives a vanishing result when the j's are projected with $P_L^{\mu\nu}$ in Eq. (2) to give F_L. All we have to do in this case, however, is to consider the operators of the next higher dimension in H_{eff}, Eq. (8). Upon using the equations of motion for both ψ and $\bar{\psi}$, the following candidates, all of dimension four, can be identified:

$$\begin{aligned}
O_{2a}^\mu &= \bar{v}^\mu\, \bar{\psi}\, \overleftarrow{\slashed{D}}_\perp\, \psi\,, \quad O_{2b}^\mu = v^\mu\, \bar{\psi}\, \overleftarrow{\slashed{D}}_\perp\, \psi\,, \\
O_{2c}^\mu &= \bar{\psi}\, \overleftarrow{D}_\nu\, \sigma_\perp^{\mu\nu}\psi\,, \quad O_{2d}^\mu = \bar{\psi}\, \overleftarrow{D}^\mu\, \psi\,.
\end{aligned} \tag{10}$$

It is easily seen that O_{2b} and O_{2c} have vanishing projection when contracted with $P_L^{\mu\nu}$. They do not contribute to F_L but can contribute to F_2 at the subleading level. O_{2d} has a

longitudinal projection that is proportional to $\bar{\psi}\, (v \cdot \overleftarrow{D})\psi$. After integration by parts this term vanishes by the equation of motion of ψ at $x = 1$. We therefore conclude that the only operator of dimension four that will contribute to F_L at leading level is O_{2a} with longitudinal projection

$$v_\mu O_{2a}^\mu = \bar{\psi}\; \overleftarrow{\slashed{D}}_\perp\, \psi\,. \tag{11}$$

Recall that the field ψ will be contracted with partons from the incoming state and that it is $\bar{\psi}\; \overleftarrow{\slashed{D}}_\perp$ that enters the jet function. The reason that the covariant derivative is associated with $\bar{\psi}$ and not on ψ comes from dimensional considerations, as explained below.

To match the above bi-local operators to the general cut diagram in Fig. 1, we need to decide which fields inside each O_2 will be contracted with partons from the incoming state, and which with the outgoing jet. A moment's thought shows that, for an incoming quark state, only the quark field ψ can be associated with the parton distribution. This is because dimensional considerations require $C_2(Q)$ to behave as Q^{-1}, up to logarithmic corrections. The only way to make up this suppression, and derive a result that is not power-suppressed in Q, is to integrate over final-state phase space. If fields of dimension $3/2 + d$ are emitted into the final state, the resulting phase space integral behaves as $[(1-x)Q^2]^d$, which precisely makes up for the dimensional suppression of C_2, for $d = 1$, and indeed for any of the higher-dimension coefficient functions in the effective Hamiltonian Eq. (8). No such compensation can arise when the extra fields are contracted with lines parallel to the initial state, because the initial state is not summed. Consider, then, a general operator O_i in the expansion of H_{eff},

$$O_i^\mu = \Xi_i^{\mu\dagger}\, \psi\,, \tag{12}$$

where Ξ_i^μ is some operator of dimension greater than $3/2$. The leading power contribution of O_i to $P_{\mathrm{r}}^{\mu\nu} W_{\mu\nu}$ as $x \to 1$ will be associated with the following factorization, which generalizes Eq. (9) directly:

$$\begin{aligned}
P_{\mathrm{r}}^{\mu\nu} W_{\mu\nu} &= P_{\mathrm{r}}^{\mu\nu}\, \mathrm{FT}_q^{(4)}\, \langle p|\, \bar{\psi}(0)\, \Xi_{\mu,i}(0)\, \Xi_{\nu,i}^\dagger(y)\, \psi(y)\, |p\rangle \\
&\approx \mathrm{FT}_{xp}^{(1)}\, \langle p|\, \bar{\psi}(0)\, \psi(y^-\bar{v})\, |p\rangle \otimes \mathrm{FT}_{q+xp}^{(4)}\, P_{\mathrm{r}}^{\mu\nu} \langle 0|\, \Xi_{\mu,i}(0)\, \Xi_{\nu,i}^\dagger(y)\, |0\rangle\,.
\end{aligned} \tag{13}$$

The second correlation function is a jet function and we see that at each level in the expansion of H_{eff} a new set of jet functions is generated that differ in their contributions by powers of $(1 - x)$ according to the dimensional counting rule given above.

c. The F_L factorization: Next we apply the above analysis to the $x \to 1$ behavior of F_L. Its leading behavior, which comes from O_{2a}^μ, is of course leading power in Q, but suppressed by a factor of $1 - x$ compared to F_2, which is just what we expect. In addition, it is also easy to see that for a gluonic state, the lowest-order operator, is even higher order, so that the contribution to F_L from gluons is actually finite in the limit $x \to 1$. The factorization formula for the leading corrections of F_L is

$$F_L(x, Q^2) = |H_L(Q^2)|^2 J' \otimes V \otimes \phi\,. \tag{14}$$

The factors V and ϕ are the same as in F_2 case, see Eqs. (4), (6). The new jet function J', generated by the O_{2a} operator is defined by

$$J'((1-z)Q^2) = \left(\frac{1}{4\pi} \frac{8x^2}{Q^2} \right) \mathrm{FT}^{(4)}_{q+zp} \langle 0|\Phi_v^\dagger(0,-\infty) \not{D}_\perp \psi(0) \not{D}_\perp \bar{\psi}(y) \Phi_v(y,-\infty)|0\rangle \otimes V^{-1}.$$

(15)

The conventional normalization factors of Eqs. (1), (2) have been included. Once the definition is given in terms of an effective operator, the anomalous dimension of J' can be computed in dimensionally regularized perturbation theory with massless partons. Details of this calculation will be published in a forthcoming paper. The following points, however, are worth noting here: First, one has to be careful to distinguish between the UV renormalization of the operator in J' that corresponds to genuine collinear enhancements, from the renormalization of the coupling constant. This is as easy task since the renormalization of the coupling is well known. Second, since J' starts at $\mathcal{O}(\alpha_s)$ whereas J, Eq. (7), starts at $\mathcal{O}(\alpha_s^0)$, the running coupling will mix with the anomalous dimension of J' already at the first non-trivial order $\mathcal{O}(\alpha_s^2)$.

The resummation of the leading corrections to F_L is a consequence of its factorization. First, the factorization formula is written in moment space in terms of the Mellin transformed factors:

$$\tilde{F}_L(N, Q^2, \epsilon) = \left| H_L\left(\frac{(p \cdot \bar{v})^2}{\mu^2}, \frac{(\bar{p} \cdot v)^2}{\mu^2}, \alpha_s(\mu^2) \right) \right|^2$$

$$\times \frac{1}{N} \tilde{J}'\left(\frac{Q^2}{N\mu^2}, \frac{(\bar{p} \cdot v)^2}{\mu^2}, \alpha_s(\mu^2) \right) \tilde{V}\left(\frac{Q^2}{N^2\mu^2}, \alpha_s(\mu^2) \right) \tilde{\phi}\left(\frac{Q^2}{N^2\mu^2}, \frac{(p \cdot \bar{v})^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right), \quad (16)$$

with $\bar{p}^\mu$ the jet momentum at the elastic limit. The Mellin transformation of the jet function J' is defined without including the overall $1/N$,

$$\frac{1}{N} \tilde{J}'\left(\frac{Q^2}{N\mu^2}, \frac{(\bar{p} \cdot v)^2}{\mu^2}, \alpha_s(\mu^2) \right) = \int_0^1 dz\, z^{N-1}\, J'\left(\frac{(1-z)Q^2}{x\mu^2}, \frac{(\bar{p} \cdot v)^2}{\mu^2}, \alpha_s(\mu^2) \right). \quad (17)$$

The factor of $1/N$ arises because, as seen from the dimensional counting of the previous section, the jet J' is of order $1-x$ relative to the F_2 jet J. It is this additional factor of $1-x$ that shows up as a power supresion $1/N$ in moment space. Once this is isolated, the renormalization group can be used to resum the logarithmic in N terms.

The resummation formula of the logarithmic corrections is derived using the, by now standard, methods that are applicable to a large range of processes near their elastic limits. Enhancements from soft gluon radiation exponentiate. The evolution of the incoming jet ϕ can also be written as an exponential. These two factors are common to both F_2 and F_L and we shall not reproduce them here (for review and details see Ref. [6]). We concentrate on the differences between the two strucutre functions at leading power in N. The renormalization group equation satisfied by the jet function J' is

$$\frac{d}{d\ln \mu^2} \ln \tilde{J}'\left(\frac{Q^2}{N\mu^2}, \frac{(\bar{p} \cdot v)^2}{\mu^2}, \alpha_s(\mu^2) \right) = -\frac{1}{2}\gamma_{J'}(\alpha_s(\mu^2)). \quad (18)$$

The anomalous dimension $\gamma_{J'}$ is novel and characteristic of F_L at the leading power level. Up to corrections of $\mathcal{O}(\alpha_s^2)$ it turns out to be [8]

$$\gamma_{J'}(\alpha_s) = \frac{\alpha_s}{\pi} \left[\frac{9}{2} C_F - 2C_A - 4\zeta(2) \left(C_F - \frac{C_A}{2} \right) \right]. \tag{19}$$

The initial condition for Eq. (18) follows from the lowest order contribution to F_L, which comes entirely from J'. It is

$$\tilde{J}'(\alpha_s(\mu^2)) = C_F \frac{\alpha_s(\mu^2)}{\pi} + \mathcal{O}(\alpha_s^2(\mu^2)). \tag{20}$$

The final result for $\tilde{F}_L$ may be written in a variety of exponentiated forms depending on which anomalous dimensions we wish to include in the exponent and what initial conditions we choose for the factors H_L, $\tilde{J}'$, $\tilde{V}$ and $\tilde{\phi}$. All these equivalent forms resum the logarithmic corrections at leading power, i.e. all terms of order $\ln^k N/N$. Furthermore, we can exploit the fact that apart from their respective final state jets, F_2 and F_L are similar. This means that the moments of the coefficient functions in the light-cone expansion $\tilde{C}_2$ and $\tilde{C}_L$ are connected in a precise way, i.e. they differ only in terms that depend on γ_J and $\gamma_{J'}$ respectively. In dimensionally regularized perturbation theory this relation is

$$\tilde{C}_L = \frac{\tilde{F}_L(N, Q^2, \epsilon)}{\tilde{F}_2(N, Q^2, \epsilon)} \tilde{C}_2. \tag{21}$$

The ratio of the two structure functions is infrared finite and can be obtained from their exponentiated forms. A consequence of Eqs. (20) and (21) is that, given $\tilde{C}_2$ to some order in α_s, we can readily predict $\tilde{C}_L$ to one order higher in α_s. To $\mathcal{O}(\alpha_s^2)$, using the result in Eq. (19), we obtain in the $\overline{\text{MS}}$ scheme

$$\begin{aligned}
\tilde{C}_L &= \frac{\alpha_s}{\pi} \frac{C_F}{N} \\
&+ \left(\frac{\alpha_s}{\pi} \right)^2 \frac{C_F}{2N} \left(\gamma_K^{(1)} \ln^2 \frac{N}{N_0} - (\gamma_{J'}^{(1)} - \frac{1}{2}\beta_1) \ln \frac{N}{N_0} \right),
\end{aligned} \tag{22}$$

with γ_K the cusp anomalous dimension [1–3] and $N_0 = e^{-\gamma_E}$. This is in agreement with the full $\mathcal{O}(\alpha_s^2)$ calculation of C_L [9,10].

In conclusion, we emphasize that the operator product expansion presented in this letter can be used to provide a systematic treatment of the logarithmic enhancements at the edges of phase space at both leading and subleading powers in moment space for a large number of hard processes in QCD. Further details and applications will be discussed in future publications.

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FIGURES

FIG. 1. Momentum configurations that produce logarithmic enhancements near the elastic limit $x \to 1$ in DIS.