Linear Sigma Model and Chiral Symmetry at Finite Temperature

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The chiral phase transition is investigated within the framework of the linear sigma model at finite temperature. We concentrate on the meson sector of the model and calculate the finite temperature effective potential in the Hartree approximation by using the Cornwall-Jackiw-Tomboulis formalism of composite operators. The effective potential is calculated for $N = 4$ involving the usual sigma and three pions and in the large N approximation involving $N - 1$ pion fields. In the $N = 4$ case we have examined the theory both in the chiral limit and with the presence of a symmetry breaking term which generates the pion masses. In both cases we have solved the system of the resulting gap equations for the thermal effective masses of the particles numerically and we have investigated the evolution of the effective potential. In the $N = 4$ case there is indication of a first order phase transition and the Goldstone theorem is not satisfied. The situation is different in the general case using the large N approximation, the Goldstone theorem is satisfied and the phase transition is of the second order. For this analysis we have ignored quantum effects and we used the imaginary time formalism for calculations.

I. INTRODUCTION

The study of matter at very high temperatures and densities and of the phase transitions which take place between the different phases is very interesting from several points of view and it has been the subject of intense study the latest years since it is relevant to particle physics, astrophysics and cosmology. According the standard big bang model it is believed that a series of phases transitions happened at the early stages of the evolution of the universe, the QCD phase transition being one of them [1–6]. There is present hope that it could also be possible to probe the underlying physics of QCD in the laboratory in experiments involving relativistic heavy ions collisions. These experiments are planned for the near future and the results could be possible to shed some light in questions like the restoration or not of chiral symmetry, the nature of quark gluon plasma and the physics of neutron stars [7].

Our aim is to study the chiral symmetry of QCD which is spontaneously broken by the small current quark masses. A powerful method in approaching questions like the restoration of spontaneously broken symmetries is to construct order parameters which characterise the state of symmetry of the system under consideration. These quantities are zero in the one phase but not in the other. A classic example of an order parameter is the magnetisation of a ferromagnetic substance, it is non-zero below the Curie temperature, but it disappears at temperatures higher than that. The system undergoes a transition from an asymmetric, ordered state with non zero magnetisation at low temperature to a symmetric disordered state with zero magnetisation at high temperatures well above the Curie point. We usually encounter two types of phase transitions. In transitions of the first order the order parameter "jumps" discontinuously from its value in the one phase to that in the other (usually zero). In contrast during second order transitions the order parameter vanishes continously.

An important order parameter which is related to the chiral phase transition of QCD is the quark condensate, a measure of the density of quark-antiquark pairs that have condensed into the same quantum mechanical state. They fill the lowest energy state -the vacuum of QCD- and as a result the chiral symmetry is broken, since there is no invariance under chiral transformations [8]. We expect that if we raise the temperature the quark condensate will disappear and the theory will be chirally symmetric.

As in many other cases in physics, in order to deal with the chiral phase transition we can use effective models to describe the physical situation. In the case of chiral symmetry a model with the correct chiral properties is the *linear sigma model*, a theory of fermions (quarks or nucleons) interacting with mesons [9]. This model has been used extensively as an effective theory in low energy phenomenology of QCD describing the physics of mesons and it is

well suited for a study of the chiral phase transition [7,8]. We review briefly the meson sector of the model and make contact with the pion phenomenology in section 3.

In studies of phase transitions the finite temperature effective potential is an important and popular theoretical tool. Early stages of applying similar techniques goes back at seventies when Kirzhnits and Linde [10] were the first proposed that symmetries broken at zero temperature could be restored at finite (high enough) temperatures. Subsequent work by Weinberg [11], Dolan and Jackiw [12] as well as many others resulted a wide adoption of the effective potential as the basic tool in such studies. The finite temperature effective potential $V(\phi, T)$ is defined through an effective action $\Gamma(\phi)$ which is the generating functional of the one particle irreducible graphs and it has the meaning of the free energy density of the system under consideration.

A generalised version is the effective potential $V(\phi, G)$ for composite operators introduced by Cornwall, Jackiw and Tomboulis (CJT) [13]. According to their formalism a generalised version of the effective action is introduced, which in contrary to the usual effective action $\Gamma(\phi)$ depends not only on $\phi(x)$ but on $G(x, y)$ as well. These two quantities are to be realized as the possible expectation values of a quantum field $\phi(x)$ and the time ordered product of the field operator $T\phi(x)\phi(y)$ respectively. In this case the effective action $\Gamma(\phi, G)$ is the generating functional of the two particle irreducible vacuum graphs (a graph is called "two particle irreducible" if it does not become disconnected upon opening two lines [13]). This formalism was originally written at zero temperature but it has been extended at finite temperature by Amelino-Camelia and Pi where it was used for investigations of the effective potential of the $\lambda\phi^4$ theory [14] and gauge theories [15].

Physical solutions demand minimization of the effective action with respect to both ϕ and G [13,14]. As a result the CJT effective potential should satisfy the stationarity requirments

$$\frac{dV(\phi, G)}{d\phi} = 0 \tag{1}$$

and

$$\frac{dV(\phi, G)}{dG} = 0 \ . \tag{2}$$

Then the conventional effective potential results as $V(\phi) = V(\phi; G_0(\phi))$ at the solution $G(\phi) = G_0(\phi)$ of the second equation.

There is an advantage in using the CJT method to calculate the effective potential in Hartree approximation. According to reference [14], using an ansantz for a "dressed propagator" we need to evaluate only one graph that of the "double bubble" in fig. 1a, instead of summing the infinite class of "daisy" and "super daisy" graphs, given in figs. 1b, 1c respectively, using the usual tree level propagators.
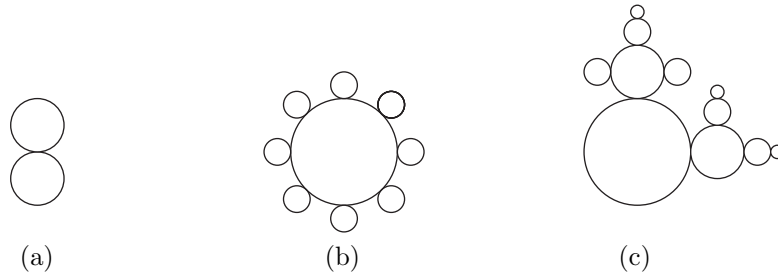

(a)   (b)   (c)

FIG. 1. The double bubble and examples of daisy and superdaisy diagrams

We demonstrate the advantage of this method in the next section where we calculate the finite temperature effective potential for one scalar field with quartic self-interaction. The calculation of the effective potential by using the CJT formalism is reviewed in great detail in references [14,16] but in order to illustrate the method for the calculation of the effective potential for the linear sigma model at finite temperature we will reproduce here the basic steps.

In our calculations we use the imaginary time formalism, known as Matsubara formalism [12,17–19]. According to this technique we work in Euclidean space-time and use the same Feynman rules as at zero temperature but evaluating momentum space integrals we replace integration over the time component k_4 with a summation over discrete frequencies which means that in the case of bosons $k_4 = 2\pi i nT, \ \ n = 0, \pm 1, \pm 2, \pm 3, \dots$. This is encoded into the following relationship

$$\int \frac{d^4k}{(2\pi)^4} f(k) \longrightarrow \frac{1}{\beta} \sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3} f(2\pi i nT, \mathbf{k}) \equiv \int_\beta f(2\pi i nT, \mathbf{k}) \ , \tag{3}$$

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where β is the inverse temperature, $\beta = 1/k_B T$, and as usual Boltzmann's constant is taken $k_B = 1$. For the shake of simplicity in what follows we have introduced a subscript β to denote integration and summation over the Matsubara frequency sums.

The outline of the remaining sections is as follows. In next section 2 we calculate the effective potential for the $\lambda\phi^4$ theory using the CJT method. In section 3 we apply the same technique to calculate the effective potential for the linear sigma model in the Hartree approximation and we solve numerically the system of the resultant gap equations both in the chiral limit and with the presence of a linear term which breaks the chiral symmetry of the Lagrangian and generates the pion observed masses. We have repeated these steps for the generalised version of the linear sigma model with $N-1$ pion fields in the large N approximation. Finally, in section 4 we outline our results and conclusions.

II. THE $\lambda\Phi^4$ THEORY

The Euclidean Lagrange density for a single scalar field with quartic self interaction interaction is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{1}{2}m^2\Phi^2 - \frac{\lambda}{24}\Phi^4 \ , \tag{4}$$

where the "mass squared" m^2 is considered as a negative parameter, in order to realise the spontaneous breaking of symmetry. By shifting the field as $\Phi \to \Phi + \phi$ the "classical potential" takes the form

$$U(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{24}\phi^4 \ , \tag{5}$$

and the interaction Lagrangian which describes the vertices of the shifted theory is given by

$$\mathcal{L}_{int}(\sigma;\phi) = -\frac{\lambda}{6}\phi\Phi^3 - \frac{\lambda}{24}\Phi^4 \ . \tag{6}$$

The tree-level propagator which corresponds to the above Lagrangian density is

$$\mathcal{D}^{-1}(\phi;k) = k^2 + m^2 + \frac{1}{2}\lambda\phi^2 \ . \tag{7}$$

According to CJT formalism [14] the finite temperature effective potential is given by

$$\begin{aligned} V(\phi, G) = U(\phi) &+ \frac{1}{2}\int_\beta \ln G^{-1}(\phi;k) \\ &+ \frac{1}{2}\int_\beta [\mathcal{D}^{-1}(\phi;k)G(\phi;k) - 1] \\ &+ V_2(\phi, G) \ , \end{aligned} \tag{8}$$

where $U(\phi)$ is the "classical potential" given by equation (5) and $V_2(\phi, G)$ represents the infinite sum of the two-particle irreducible vacuum graphs. We are going to evaluate the effective potential in the Hartree approximation which corresponds to that the leading contribution to last term $V_2(\phi, G)$ of the effective potential comes from the "double bubble" diagram given in fig. 1a [13,14]. Therefore the effective potential results as

$$\begin{aligned} V(\phi, G) = \frac{1}{2}m^2\phi^2 &+ \frac{1}{24}\lambda\phi^4 + \frac{1}{2}\int_\beta \ln G^{-1}(\phi;k) \\ &+ \frac{1}{2}\int_\beta [(k^2 + m^2 + \frac{1}{2}\lambda\phi^2)G(\phi;k) - 1] + \frac{1}{8}\lambda\left[\int_\beta G(\phi;k)\right]^2 \ . \end{aligned} \tag{9}$$

Minimizing the effective potential with respect to "dressed propagator" $G(\phi;k)$ we obtain a gap equation

$$G^{-1}(\phi;k) = k^2 + m^2 + \frac{1}{2}\lambda\phi^2 + \frac{1}{2}\lambda\int_\beta G(\phi;k) \ . \tag{10}$$

The solution $G_0(\phi;k)$ of the gap equation is inserted back into the expression for the effective potential to give the potential as a function of ϕ. According to [13] using the propagator $G_0(\phi;k)$ for internal lines corresponds to summing all daisy and super-daisy diagrams using the usual tree level propagators as in [12].

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In order to proceed and according to reference [14], it is convenient to adopt the following form for the propagator $G(\phi; k)$

$$G(\phi; k) = \frac{1}{k^2 + M^2} , \tag{11}$$

where an "effective mass" $M = M(\phi; k)$ has been introduced. Then the gap equation for the propagator becomes an equation for the effective mass

$$M^2 = m^2 + \frac{\lambda}{2}\phi^2 + \frac{\lambda}{2}\int_\beta \frac{1}{k^2 + M^2} , \tag{12}$$

where it is obvious that in this approximation the effective mass M is momentum independent.

In terms of the solution $M_0(\phi)$ of the gap equation (12), the effective potential takes the form

$$V(\phi, M_0) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{24}\phi^4 + \frac{1}{2}\int_\beta \ln(k^2 + M_0^2)$$

$$-\frac{1}{2}(M_0^2 - m^2 - \frac{\lambda}{2}\phi^2)\int_\beta \frac{1}{k^2 + M_0^2} + \frac{1}{8}\lambda \left[\int_\beta \frac{1}{k^2 + M_0^2}\right]^2 . \tag{13}$$

Performing the Matsubara frequency sums as in [12], the logarithmic integral which appear into the above expression of the effective potential divides into a zero temperature part $Q_0(M)$ which is divergent and one non zero $Q_\beta(M)$ temperature part which is finite and can be written as

$$Q(M) = \frac{1}{2}\int_\beta \ln(k^2 + M^2) = Q_0(M) + Q_\beta(M)$$

$$= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\omega_k}{2} + \frac{1}{\beta}\int \frac{d^3\mathbf{k}}{(2\pi)^3} \ln[1 - \exp(-\beta\omega_k)] , \tag{14}$$

where $\omega_k = (\mathbf{k}^2 + M^2)^{1/2}$. In this above expression and in what follows we omit the subscript 0 of M. Similarly, the second integral is divided into two parts as well. One zero temperature part $F_0(M)$ and a finite temperature $F_\beta(M)$ part as

$$F(M) = \int_\beta \frac{1}{k^2 + M^2} = F_0(M) + F_\beta(M)$$

$$= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_k} + \frac{1}{\beta}\int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{\omega_k} \frac{1}{\exp(-\beta\omega_k) - 1} . \tag{15}$$

The second term vanishes at zero temperature, while the first term survives but it gives rise to divergences which can be carried out using appropriate renormalisation prescriptions [14,16]. If one is interested in temperature induced effects only, as it is our approximation, the divergent integrals can be ignored. In this case, by making a change of the integration variables the finite temperature part of $F(M)$ can be written as

$$F_\beta(M) = \frac{T^2}{2\pi^2}\int_0^\infty \frac{x^2 dx}{[x^2 + y^2]^{1/2}} \frac{1}{\exp[x^2 + y^2]^{1/2} - 1} , \tag{16}$$

where we have used a shorthand notation and $y = M/T$. Similarly the finite temperature part of the logarithmic integral becomes

$$Q_\beta(M) = \frac{T^4}{2\pi^2}\int_0^\infty dx x^2 \ln\left[1 - \exp(-[x^2 + y^2]^{1/2})\right] . \tag{17}$$

Then the finite temperature effective potential, ignoring quantum effects, can be written as

$$V(\phi, M) = \frac{1}{2}m^2\phi^2 + \frac{1}{24}\lambda\phi^4 + Q_\beta(M)$$

$$-\frac{1}{2}(M^2 - m^2 - \frac{\lambda}{2}\phi^2)F_\beta(M) + \frac{\lambda}{8}[F_\beta(M)]^2 . \tag{18}$$

We can obtain a more compact form if we make use of the gap equation (12),

$$V(\phi, M) = \frac{1}{2}m^2\phi^2 + \frac{1}{24}\lambda\phi^4 + Q_\beta(M) - \frac{1}{8}\lambda[F_\beta(M)]^2 . \tag{19}$$

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III. THE EFFECTIVE POTENTIAL OF THE LINEAR SIGMA MODEL

A. The Linear Sigma Model

As we mentioned in the introduction, the linear sigma model serves as a good low energy effective theory in order one to have some insight into QCD. The model was first introduced in the sixties [9] as a model for pion-nucleon interactions and has attracted much attention recently specially in studies involving the Disoriented Chiral Condensates (DCC's) [20–24]. The model is very well suited for describing the physics of pions in studies of chiral symmetry. Fermions are inserted in the model either as nucleons, if one is to study nucleon interactions or as quarks. The mesonic part of the model consists of four scalar fields, one scalar isoscalar field which is called the sigma σ field and the usual three pion fields π^0, $\pi^\pm$ which form a pseudo-scalar isovector. The fields form a four vector (σ, π_i), $i = 1, 2, 3$ which we regard as the chiral field and the model displays an $O(4)$ symmetry. The σ field can be used to represent the quark condensate, the order parameter for the chiral phase transition, since they exhibit the same behaviour under chiral transformations [8]. The pions are very light particles and can be considered approximately as massless Goldstone bosons. We use the model as an effective theory for QCD ignoring the fermion sector at the moment and concentrating on the meson sector.

The generalised version of the meson sector of the linear sigma model called the $O(N)$ or vector model and is based on a set of N real scalar fields. The $O(N)$ model Lagrangian can be written as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \boldsymbol{\Phi})^2 - \frac{1}{2}m^2\boldsymbol{\Phi}^2 - \frac{1}{6N}\lambda\boldsymbol{\Phi}^4 - \varepsilon\sigma \ , \tag{20}$$

and in the absence of the last term it remains invariant under $O(N)$ symmetry transformations for any $N \times N$ orthogonal matrix. In order our notation to be consistent with applications on pion phenomenology we can identify the Φ_1 with the σ field and the remaining $N-1$ components as the pion fields, that is $\boldsymbol{\Phi} = (\sigma, \pi_1, \ldots, \pi_{N-1})$. The last term, $\varepsilon\sigma$ into the above expression has been introduced in order to generate the observed masses of the pions. The contact with phenomenology is obtained by considering the the case $N = 4$. Then the Lagrangian of the model is given by

$$\mathcal{L} = \frac{1}{2}(\partial\sigma)^2 + \frac{1}{2}(\partial\boldsymbol{\pi})^2 - \frac{1}{2}m^2\sigma^2 - \frac{1}{2}m^2\boldsymbol{\pi}^2 - \frac{\lambda}{24}(\sigma^2 + \boldsymbol{\pi}^2)^2 - \varepsilon\sigma \ , \tag{21}$$

where $\varepsilon = f_\pi m_\pi^2$ and $f_\pi = 93$ MeV is the pion decay constant. At zero temperature and in order to be consistent with the pion observed mass of $m_\pi \approx 138$ MeV and the usually adopted sigma mass $m_\sigma \approx 600$ MeV we choose the coupling constant λ in our model to be

$$\lambda = \frac{3(m_\sigma^2 - m_\pi^2)}{f_\pi^2} \ . \tag{22}$$

The negative mass parameter m^2 has been introduced in order to obtain spontaneous breaking of symmetry and its value is chosen to be

$$-m^2 = (m_\sigma^2 - 3m_\pi^2)/2 > 0 \ . \tag{23}$$

As we have referred earlier, our aim is to use the linear sigma model as a model to study the chiral phase transition, so we need to calculate the effective potential. For this calculation we adopt the CJT formalism which has been presented earlier for the $\lambda\phi^4$ theory and calculate the effective potential in the Hartree and large N approximations.

B. Hartree approximation in the chiral limit $\varepsilon = 0$

In order to deal with the exact chiral limit first, the starting point is the Lagrangian (21) and we ignore the symmetry breaking term at the moment. By shifting the sigma field as $\sigma \to \sigma + \phi$ the "classical potential" results as

$$U(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{24}\phi^4 \ , \tag{24}$$

and the interaction Lagrangian which describes the vertices of the new theory takes the form

$$\mathcal{L}_{int}(\sigma; \phi) = -\frac{\lambda}{4!}\sigma^4 - \frac{\lambda}{4!}\pi^4 - \frac{\lambda}{12}\sigma^2\pi^2 - \frac{\lambda}{6}\phi\sigma^3 - \frac{\lambda}{6}\phi\sigma\pi^2 \ , \tag{25}$$

plus terms linear in the σ field and constants which we omit for simplicity. In our approximation we do not consider interactions given by the last two terms in the Lagrangian.

The tree level sigma and pion propagators corresponding to the above Lagrangian are

$$\mathcal{D}_\sigma^{-1}(\phi; k) = k^2 + m^2 + \frac{1}{2}\lambda\phi^2 \ , \tag{26}$$

$$\mathcal{D}_\pi^{-1}(\phi; k) = k^2 + m^2 + \frac{1}{6}\lambda\phi^2 \ . \tag{27}$$

We evaluate the effective potential in the Hartree approximation which means that we only need to calculate the "double bubble" diagrams as in $\lambda\phi^4$ theory. In the linear sigma model the corresponding effective potential at finite temperature can be written as

$$
\begin{aligned}
V(\phi, G) = U(\phi) &+ \frac{1}{2}\int_\beta \ln G_\sigma^{-1}(\phi; k) + \frac{3}{2}\int_\beta \ln G_\pi^{-1}(\phi; k) \\
&+ \frac{1}{2}\int_\beta [\mathcal{D}_\sigma^{-1}(\phi; k)G_\sigma(k) - 1] + \frac{3}{2}\int_\beta [\mathcal{D}_\pi^{-1}(\phi; k)G_\pi(\phi; k) - 3] \\
&+ V_2(\phi, G_\sigma, G_\pi) \ ,
\end{aligned}
\tag{28}
$$

where the first term $U(\phi)$ is the "classical potential" and the last term $V_2(\phi, G_\sigma, G_\pi)$ originates from the sum of "double bubble" diagrams [14]. There are four types of double bubbles as we show in fig. 2 and these contribute the following terms in the potential

$$V_2(\phi, G_\sigma, G_\pi) = 3\frac{\lambda}{24}\left[\int_\beta G_\sigma(\phi; k)\right]^2 + 15\frac{\lambda}{24}\left[\int_\beta G_\pi(\phi; k)\right]^2 + 6\frac{\lambda}{24}\left[\int_\beta G_\sigma(\phi; k)\right]\left[\int_\beta G_\pi(\phi; k)\right] \ . \tag{29}$$
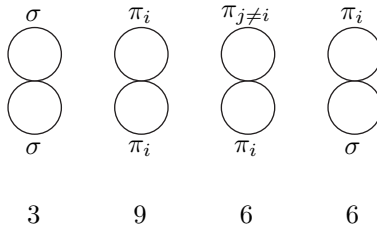

FIG. 2. The double bubble graphs which contribute to the effective potential for the O(4) linear sigma model in the Hatree approximation. The numbers show the weight of each type of bubble in the expression for the effective potential.

Minimizing the effective potential with respect to the "dressed" propagators, we get the following set of nonlinear gap equations

$$
\begin{aligned}
G_\sigma^{-1}(\phi; k) &= \mathcal{D}_\sigma^{-1}(\phi; k) + \frac{\lambda}{2}\int_\beta G_\pi(\phi; k) + \frac{\lambda}{2}\int_\beta G_\sigma(\phi; k) \\
G_\pi^{-1}(\phi; k) &= \mathcal{D}_\pi^{-1}(\phi; k) + \frac{5\lambda}{6}\int_\beta G_\pi(\phi; k) + \frac{\lambda}{6}\int_\beta G_\sigma(\phi; k) \ .
\end{aligned}
\tag{30}
$$

In order to proceed we can use the same ansatz for the dressed propagators as in the one field case [14]

$$G_{\sigma/\pi}^{-1} = k^2 + M_{\sigma/\pi}^2 \ , \tag{31}$$

and using the equations (26), (27), (30) and (31) we end up with the following system for the thermal effective masses

$$
\begin{aligned}
M_\sigma^2 &= m^2 + \frac{1}{2}\lambda\phi^2 + \frac{\lambda}{2}F(M_\sigma) + \frac{\lambda}{2}F(M_\pi) \\
M_\pi^2 &= m^2 + \frac{1}{6}\lambda\phi^2 + \frac{\lambda}{6}F(M_\sigma) + \frac{5\lambda}{6}F(M_\pi) \ .
\end{aligned}
\tag{32}
$$

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In these last two equations we have used a shorthand notation and $F(M)$ is given by

$$F(M) = \int_\beta \frac{1}{k^2 + M^2} \ . \tag{33}$$

As in $\lambda\phi^4$ theory the thermal effective masses are independent of momentum and functions of the order parameter ϕ and the temperature T.

By using these two equations the effective potential at finite temperature then can be written as

$$
\begin{aligned}
V(\phi, M) = {}& \frac{1}{2}m^2\phi^2 + \frac{1}{24}\lambda\phi^4 + \frac{1}{2}\int_\beta \ln(k^2 + M_\sigma^2) + \frac{3}{2}\int_\beta \ln(k^2 + M_\pi^2) \\
& - \frac{1}{2}(M_\sigma^2 - m^2 - \frac{1}{2}\lambda\phi^2)F(M_\sigma) - \frac{3}{2}(M_\pi^2 - m^2 - \frac{1}{6}\lambda\phi^2)F(M_\pi) \\
& + \frac{\lambda}{8}[F(M_\sigma)]^2 + \frac{5\lambda}{8}[F(M_\pi)]^2 + \frac{\lambda}{4}F(M_\sigma)F(M_\pi) \ .
\end{aligned}
\tag{34}
$$

Minimizing the effective potential with respect to the "dressed" propagators we have found the set of nonlinear gap equations for the effective particles' masses given by equation (32). In addition, by minimizing the potential with respect to the order parameter we obtain one more equation

$$0 = m^2 + \frac{1}{6}\lambda\phi^2 + \frac{\lambda}{2}F(M_\sigma) + \frac{\lambda}{2}F(M_\pi) \ . \tag{35}$$

In order to study the evolution of the potential as a function of temperature, we perform the Matsubara frequency sums [12] as in the one field case. There are some troubles concerning the renormalisation of the model [14,16,25,27,28]. At the level of our approximation we ignore quantum effects for the moment and keep only the finite temperature part of the integrals. Using a compact notation the finite temperature effective potential can be written in the form

$$
\begin{aligned}
V(\phi, M) = {}& \frac{1}{2}m^2\phi^2 + \frac{1}{24}\lambda\phi^4 + Q_\beta(M) + 3Q_\beta(M_\pi) \\
& - \frac{\lambda}{8}[F_\beta(M_\sigma)]^2 - \frac{5\lambda}{8}[F_\beta(M_\pi)]^2 - \frac{\lambda}{4}F_\beta(M_\sigma)F_\beta(M_\pi) \ ,
\end{aligned}
\tag{36}
$$

where in this last step we have used the gap equations (32). The expessions for $F_\beta(M)$ and $Q_\beta(M)$ are given by the equations (16) and (17).

In order to calculate the effective masses as functions of temperature we need to solve the system of equations (32) and (35). We first observe that if $\phi = 0$, which happens in the high temperature phase, the two equations become degenerate, the particles have the same mass and we have to solve only one equation

$$M^2 = m^2 + \lambda F_\beta(M) \ , \tag{37}$$

where obviously as in the expression of the effective potential we keep only the finite temperature part of the integral. This last equation can be used to define a "transition temperature" T_{c1}. This temperature is defined as the temperature where both particles become massless. Recall now that $F_\beta(M)$ is given by

$$F_\beta(M) = \frac{T^2}{2\pi^2}\int_0^\infty \frac{x^2 dx}{[x^2 + (M/T)^2]^{1/2}} \frac{1}{\exp[x^2 + (M/T)^2]^{1/2} - 1} \ , \tag{38}$$

where we show explicitly the dependence of $F_\beta(M)$ to mass and temperature. When the mass of the particles vanishes the integral reduces to the well known

$$I(x) = \int_0^\infty \frac{x dx}{e^x - 1} = \frac{\pi^2}{6} \ , \tag{39}$$

and the temperature T_{c1} is found to be

$$T_{c1} = \sqrt{2}\left(-\frac{6m^2}{\lambda}\right)^{1/2} \ . \tag{40}$$

But in defining our model parameters we have chosen that at zero temperature $\phi^2 = f_\pi^2 = -6m^2/\lambda$, where $f_\pi = 93$ MeV is the pion decay constant, so we find that $T_{c1} = \sqrt{2}f_\pi \approx 131.5$ MeV.

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In the low temperature phase we can eliminate ϕ and end up with the following nonlinear system

$$M_\sigma^2 = -2m^2 - \lambda F_\beta(M_\sigma) - \lambda F_\beta(M_\pi)$$

$$M_\pi^2 = -\frac{\lambda}{3} F_\beta(M_\sigma) + \frac{\lambda}{3} F_\beta(M_\pi) \ , \tag{41}$$
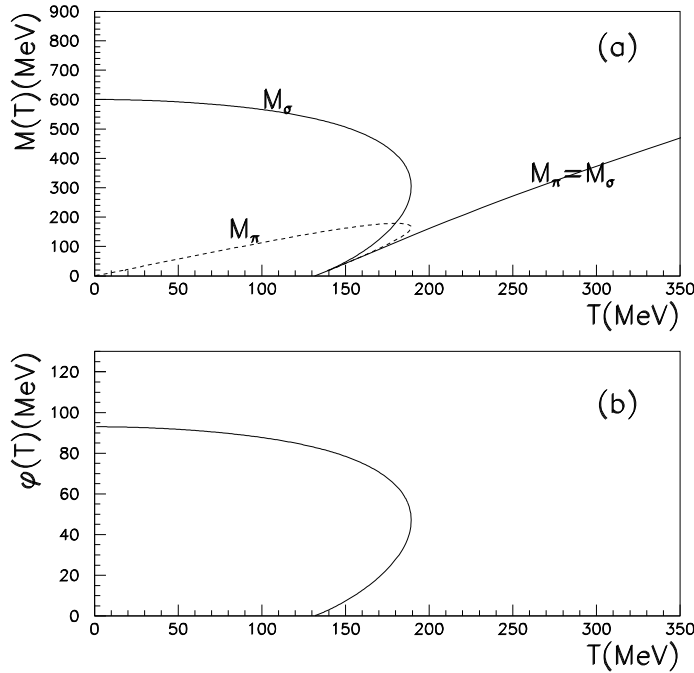
which we solve numerically and the solution is presented in fig. 3a.

FIG. 3. (a) Solution of the system of gap equations in the chiral limit. The sigma and pion effective masses are given as functions of temperature. (b) Evolution of the order parameter as a function of temperature.

As shown in fig. 3a, the temperature T_{c1} which is calculated numerically was found to be in excellent agreement with the value obtained by using the limit of the high temperature equations with degenerate masses.

At this point we can observe that there is an indication of a first order phase transition, because combining the first of the two equations in (32) with equation (35) we find that

$$M_\sigma^2 = \frac{1}{3} \lambda \phi^2 \ . \tag{42}$$

This last equation shows of course that the order parameter varies with temperature proportionally to the sigma mass. The temperature dependence of $\phi = \phi(T)$ is calculated by using the sigma mass as it was found by solving the system in eqn (41). This is given in fig. 3b where it is obvious that this approximation predicts a first order phase transition. This last observation coincides with the qualitative picture given by Baym and Grinstein in their early paper [25] where their "modified Hartree approximation" predicts a first order phase transition as well. However, in contrast to our approximation into their analysis they included quantum effects as well. Signals of a first order phase transition have also been reported in recent analyses by Randrup [26] and by Roh and Matsui [27].

In order to get more insight into the nature of the phase transition and verify that the transition is of the first order we can calculate the effective potential $V(\phi, T)$ as a function of the temperature and the order parameter. So, we first solve numerically the system of equations (32) and (43) (where of course we keep only the finite temrerature part of the integrals) and calculate the effective masses of the particles as functions the order parameter and the temperature. Finally the effective potential is calculated numerically by using these masses. The evolution of the potential for several temperatures is given in fig. 4. The shape of the potential confirms that a first order phase transition takes place, since it exhibits two degenerate minima at a temperature $T_c \approx 182$ MeV which is usually defined as

the transition temperature. The second minimum of the potential at $\phi \neq 0$ disappears at a temperature $T_{c2} \approx$ 187 MeV. The temperatures T_{c1} and T_{c2} are called (in condensed matter terminology) the lower and upper spinodal points respectively. Between these temperatures metastable states exist and the system can exhibit supercooling or superheating. For $T_{c1} < T < T_c$ the metastable states are centered around the origin since for $T_c < T < T_{c2}$ the metastable states occur for $\phi \neq 0$. When the system reaches T_{c1} or T_{c2} the curvature of the potential at the metastable minima vanishes. A discussion about first order phase transitions and more details about how these transitions procceed can be found in refs [4–6].
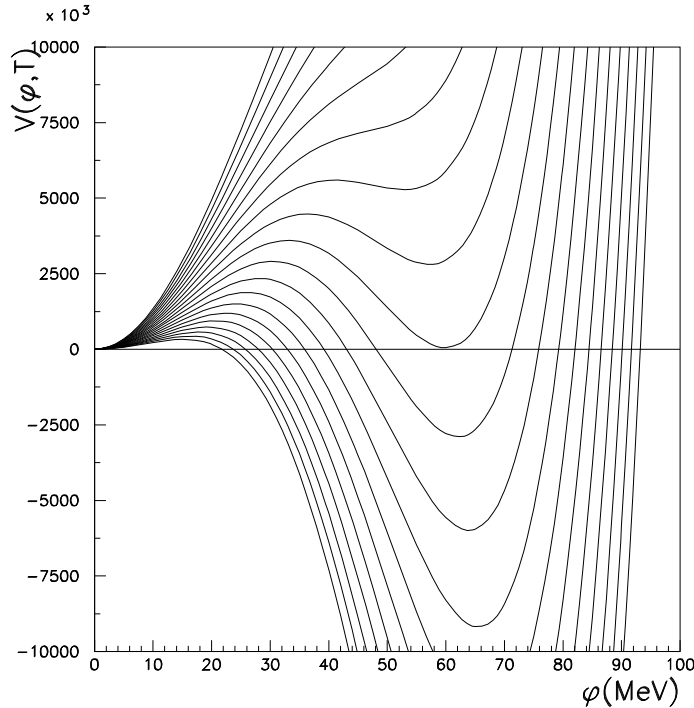

FIG. 4. Evolution of the effective potential $V(\phi, T)$ as a function of the order parameter ϕ for several temperatures in steps of 2 MeV. The two minima appear as degenerate at $T_c \approx 182$ MeV.

C. Hartree approximation in the broken symmetry case $\varepsilon \neq 0$

When $\varepsilon \neq 0$ the term linear in the sigma field into the Lagrangian generates the pion observed masses. This term is G independent and so minimization of the potential with respect to "dressed" propagators will give us the same set of gap equations for the effective masses as before. However, minimizing the potential with respect to ϕ we get the following equation

$$\left[m^2 + \frac{1}{6}\lambda\phi^2 + \frac{\lambda}{2}F_\beta(M_\sigma) + \frac{\lambda}{2}F_\beta(M_\pi) \right]\phi - \varepsilon = 0 \ . \tag{43}$$

In order to proceed we need to solve the nonlinear system of three equations (32) and (43). We first observe that at $T = 0$ the above equation becomes

$$M_\pi^2 = m^2 + \frac{1}{6}\lambda\phi^2 = \frac{\varepsilon}{\phi} = m_\pi^2 \ , \tag{44}$$

where m_π is the tree level pion mass. Then for $\phi = f_\pi$ we recover the relation between the pion mass at zero temperature and the symmetry breaking factor ε: $\varepsilon = f_\pi m_\pi^2$ where f_π is the pion decay constant. We solved the system of eqns (32) and (43) numerically and the solution is presented in fig. 5a. At low temperatures the pions appear with the observed masses but their mass increases with temperature since the sigma mass decreases. At high

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temperatures (higher than ~ 300 MeV) due to interactions in the thermal bath all particles appear to have the same effective mass.

The appearance of the symmetry breaking term into the Lagrangian modifies the evolution of the order parameter ϕ as well. As it is obvious in fig. 5b, as the temperature increases the order parameter decreases and at very high temperatures vanishes smoothly. But in this case the change is not a phase transition any more. We rather encounter a smooth crossover from a low temperature phase when the particles appear with different masses to a high temperature phase where the thermal contribution to the effective masses makes them degenerate.
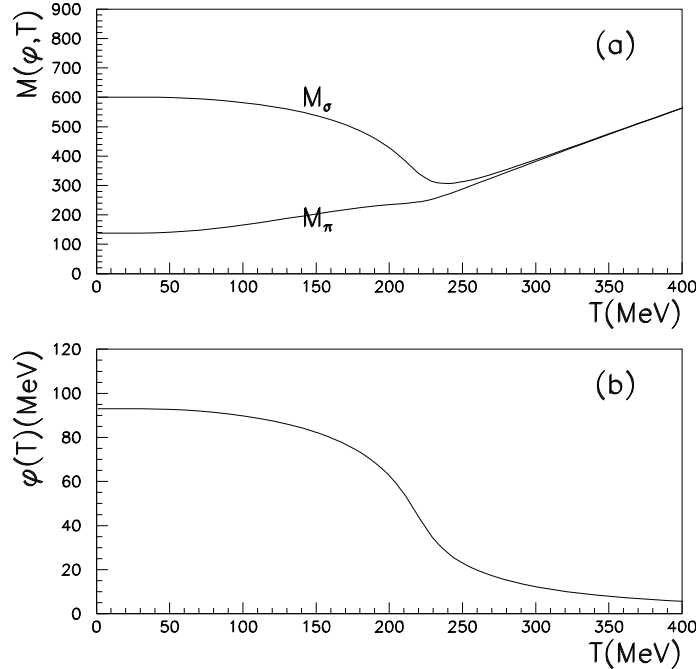
FIG. 5. (a) Solution of the system of gap equations in the case when $\epsilon \neq 0$. At low temperatures the pions appear with the observed masses. (b) Evolution of the order parameter as a function of temperature.

D. Large N approximation in the chiral limit $\varepsilon = 0$

The large N approximation of the linear sigma model has been studied recently by Amelino-Camelia [28] and our expressions are very similar to the ones obtained there, since the same method is used in both cases. However, in our approach we do not consider the renormalization of the model because in our approximation we take into account only finite temperature effects. Our analysis is in a sense complementary to that in [28] since we solve the system of gap equations and consider the effects of the symmetry breaking term (the last term in the Lagarangian (20)) which is omitted in reference [28].

The Lagrangian of the linear sigma model when we consider a large number $(N-1)$ pion fields is given in equation (20) and shifting the sigma field as $\sigma \rightarrow \sigma + \phi$, the tree level propagators are

$$D_\sigma^{-1}(\phi; k) = k^2 + m^2 + \frac{2\lambda}{N}\phi^2 \ , \tag{45}$$

$$D_\pi^{-1}(\phi; k) = k^2 + m^2 + \frac{2\lambda}{3N}\phi^2 \ . \tag{46}$$

Then the effective potential at finite temperature will appear as

$$V(\phi, M) = \frac{1}{2}m^2\phi^2 + \frac{1}{6N}\lambda\phi^4 + \frac{1}{2}\int_\beta \ln G_\sigma^{-1}(\phi; k) + \frac{N-1}{2}\int_\beta \ln G_\pi^{-1}(\phi; k)$$
$$+ \frac{1}{2}\int_\beta [\mathcal{D}_\sigma^{-1}(\phi; k)G_\sigma(\phi; k) - 1] + \frac{N-1}{2}\int_\beta [\mathcal{D}_\pi^{-1}(\phi; k)G_\pi(\phi; k) - (N-1)]$$
$$+ V_2(\phi, G_\sigma, G_\pi) , \tag{47}$$

where the last term originates from the double bubble diagrams and its contribution reads as

$$V_2(\phi, G_\sigma, G_\pi) = 3\frac{\lambda}{6N}\left[\int_\beta G_\sigma(k)\right]^2 + \frac{\lambda(N^2-1)}{6N}\left[\int_\beta G_\pi(\phi; k)\right]^2 + \frac{\lambda(N-1)}{6N}\left[\int_\beta G_\sigma(\phi; k)\right]\left[\int_\beta G_\pi(\phi; k)\right] . \tag{48}$$

The weight factors appearing in the above expression can be understood in a similar way as in the $O(4)$ case the only difference being the $N-1$ pion fields. Of course it is easy to see that we recover the previous case by simply substituting $N = 4$.

As in the case of $\lambda\phi^4$ and the $O(4)$ model we minimise the effective potential with respect to the dressed propagators and we get a set of gap equations. By using the same form for the dressed propagators as before, we end up with the following set of nonlinear gap equations for the thermal effective particle masses

$$M_\sigma^2 = m^2 + \frac{2\lambda}{N}\phi^2 + \frac{2\lambda}{N}F_\beta(M_\sigma) + \frac{2\lambda(N-1)}{3N}F_\beta(M_\pi)$$
$$M_\pi^2 = m^2 + \frac{2\lambda}{3N}\phi^2 + \frac{2\lambda}{3N}F_\beta(M_\sigma) + \frac{2\lambda(N+1)}{3N}F_\beta(M_\pi) , \tag{49}$$

where we only keep the finite temperature part of the integrals. As it is easy to observe, for $N = 4$ we obtain identical expressions for the system of gap equations as in the case of the $O(4)$ model. Then the effective potential will appear in the form

$$V(\phi, M) = \frac{1}{2}m^2\phi^2 + \frac{1}{6N}\lambda\phi^4 + \frac{1}{2}\int_\beta \ln(k^2 + M_\sigma^2) + \frac{(N-1)}{2}\int_\beta \ln(k^2 + M_\pi^2)$$
$$- \frac{1}{2}(M_\sigma^2 - m^2 - \frac{2\lambda}{N}\phi^2)F_\beta(M_\sigma) - \frac{N-1}{2}(M_\pi^2 - m^2 - \frac{2\lambda}{3N}\phi^2)F_\beta(M_\pi)$$
$$+ \frac{\lambda}{2N}[F_\beta(M_\sigma)]^2 + \frac{\lambda(N^2-1)}{6N}[F_\beta(M_\pi)]^2 + \frac{\lambda(N-1)}{3N}F_\beta(M_\sigma)F_\beta(M_\pi) . \tag{50}$$

In the large N approximation, which means that we ignore terms of $O(1/N)$, the system of the two equations (49) reduces to

$$M_\sigma^2 = m^2 + \frac{2\lambda}{N}\phi^2 + \frac{2\lambda}{3}F_\beta(M_\pi)$$
$$M_\pi^2 = m^2 + \frac{2\lambda}{3N}\phi^2 + \frac{2\lambda}{3}F_\beta(M_\pi) . \tag{51}$$

We have retained the terms quadratic in ϕ since ϕ depends on N as $\phi^2 = 3Nm^2/2\lambda$ and so these terms are of $O(1)$. In order to solve this system and be in "some contact" with phenomenology in the chiral limit, we set now $N = 4$ so at zero temperature the pions are massless and sigma has a mass $M_\sigma^2 = -2m^2$. Now our system is written as

$$M_\sigma^2 = m^2 + \frac{1}{2}\lambda\phi^2 + \frac{2\lambda}{3}F_\beta(M_\pi)$$
$$M_\pi^2 = m^2 + \frac{1}{6}\lambda\phi^2 + \frac{2\lambda}{3}F_\beta(M_\pi) , \tag{52}$$

and in order to solve it we proceed in a similar way as in Hartree approximation.

At very high temperatures the potential has only one minimum that at $\phi = 0$ and in this case the two equations become degenerate

$$M_\sigma^2 = M_\pi^2 = M^2 = m^2 + \frac{2\lambda}{3}F_\beta(M) . \tag{53}$$

This last equation actually defines the critical temperature. $F_\beta(M)$ is given by the same expression as in the $O(4)$ case. The mass of the particles vanishes at the critical temperature, so we can use the result of eqn. (38) to find that the critical temperature is at

$$T_c = \sqrt{3}\left(-\frac{6m^2}{\lambda}\right)^{1/2} = \sqrt{3}f_\pi \approx 161\text{MeV} . \tag{54}$$

Before proceeding to examine the low temperature phase we should note an observation which actually marks the significant difference between the Hartree approximation in the $N = 4$ case and the large N approximation. Minimizing the potential with respect to ϕ gives

$$\frac{dV(\phi, M)}{d\phi} = \frac{\partial V}{\partial \phi} = \phi\left[m^2 + \frac{2\lambda}{3N}\phi^2 + \frac{2\lambda}{3N}F_\beta(M_\sigma) + \frac{2\lambda(N-1)}{3N}F_\beta(M_\pi)\right] = 0 , \tag{55}$$

which in the large N approximation becomes

$$\frac{dV(\phi, M)}{d\phi} = \phi\left[m^2 + \frac{\lambda}{6}\phi^2 + \frac{2\lambda}{3}F_\beta(M_\pi)\right] = 0 . \tag{56}$$

Combining this last equation with the second of equations (52) above we observe that

$$\frac{dV(\phi, M)}{d\phi} = \phi M_\pi^2 = 0 . \tag{57}$$

Therefore the large N approximation implies that the pions should be massless in the low temperature phase in accordance with the Goldstone theorem.

This observation is reflected in the solution of the system of the gap equations as it is shown in fig. 6a. The pions at low temperatures appear as massless, but at high temperatures the thermal contribution to the effective masses make them degenerate with the sigma. The order parameter vanishes continuously in this case as it is shown in fig. 6b and this corresponds to a second order phase transition.
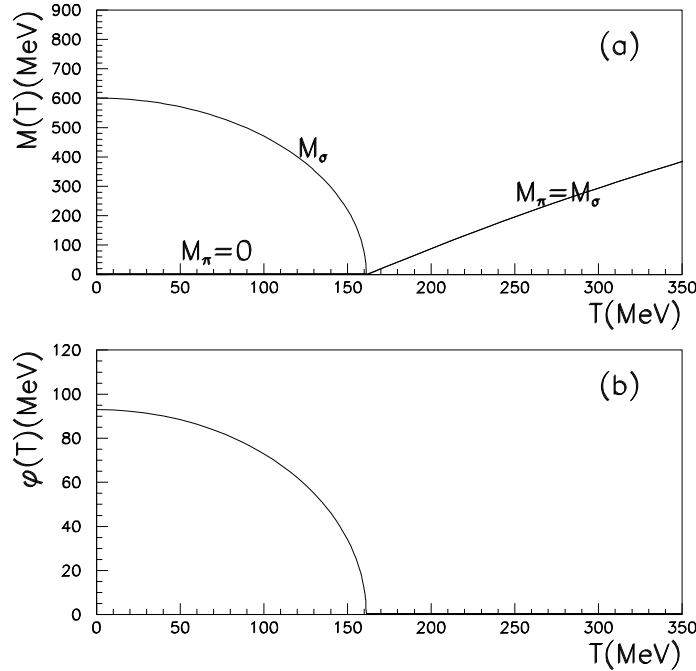

FIG. 6. (a) Solution of the system of gap equations in the large N approximation in the chiral limit. At low temperatures the pions appear as massless. (b) Evolution of the order parameter with temperature in the large N approximation.

E. Large N approximation in the broken symmetry case $\varepsilon \neq 0$

As already mentioned for the $O(4)$ case, the symmetry breaking term $\varepsilon\sigma$ has been introduced into the Lagrangian in order to generate the observed masses of the pions. The same can be done for the $O(N)$ model the only difference

being the $N-1$ pion fields. Inserting this term into the expression for the effective potential and differentiating with respect to ϕ we obtain, as in the $O(4)$ case, one more equation. In the large N approximation this is written as

$$\left[m^2 + \frac{1}{6}\lambda\phi^2 + \frac{2\lambda}{3}(F_\beta(M_\pi)) \right]\phi - \varepsilon = 0 \ . \tag{58}$$

We have solved this last system of equations (52) and (58) numerically and the solution is given in fig. 7. As in the $N=4$ case there is no longer any phase transition. We encounter again the crossover phenomenon between the low and high temperature phases, the difference being now that the change of the order parameter (fig. 7b) in the transition region is much more smooth in contrast the more "sharp" behaviour seen in the $N=4$ case in fig. 5b.
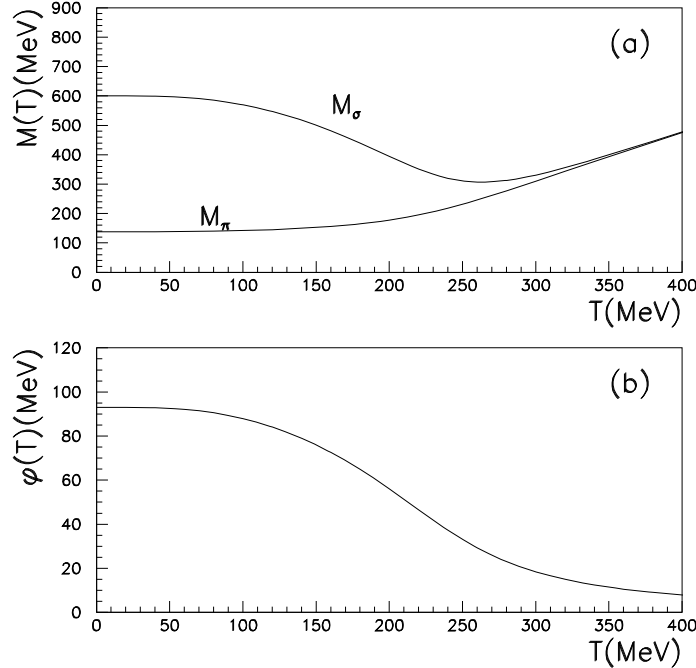

FIG. 7. (a) Solution of the system of gap equations in the large N approximation in the case of broken chiral symmetry $\varepsilon \neq 0$. At low temperatures the pions appear with the observed masses. (b) Evolution of the order parameter with temperature.

IV. CONCLUSIONS

We have studied the chiral phase transition using the linear sigma model. In order to get some insight into how the phase transition could proceed, we have calculated the finite temperature effective potential of this model in the Hartree and large N approximations using the CJT formalism of composite operators. The method proved to be very handy since we actually need to calculate only one diagram. In both cases we have solved numerically the system of gap equations and found the evolution with temperature of the effective thermal masses. In the Hartree approximation we find a first order phase transition, but in contrast to that, the large N approximation predicts a second order phase transition. This last observation seems to be in agreement with different approaches to the chiral phase transition based on the argument that the linear sigma model belongs in the same universality class as other models which are known to exhibit second order phase transitions [7,29]. However in the large N approximation the sigma contribution is ignored and this of course introduces errors when we calculate the critical temperature. In the case $N=4$ which is closer to phenomenology we could probably obtain a better approximation if we had considered the effects of interactions given by the last two terms in the Lagrangian (25). We are planning some investigations in order to include the effects of these terms in the calculation of the effective potential.

When we include the symmetry breaking term $\varepsilon\sigma$ which generates the pion observed masses, both in Hartree and large N approximations we found that there is no longer any phase transition. We rather observe a crossover

phenomenon where the change of the order parameter in the Hartree case occurs more rapidly in contrast to the more smooth behaviour exhibited by the large N approximation. Our observation confirms results reported recently by Chiku and Hatsuda [30] using a different approach. In their analysis they also report indication of first order phase transition in the chiral limit.

Of course as we have already pointed out, the linear sigma model is only an approximation to the real problem which is QCD, but the study of the chiral phase transition in the framework of this model could be a helpful guide to how one could tackle the original problem and get some insight in the physics invloved. We have used the imaginary time formalism which is adequate for studies at thermal equilibrium but if one is interested in studies of the dynamics of the phase transition, the real time formalism seems to be more convenient [3,18,19]. We are currently making some preliminary investigations in this direction.

ACKNOWLEDGMENTS

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