

Compact analytical form for a class of three-loop vacuum Feynman diagrams

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Abstract

We present compact, fully analytical expressions for singular parts of a class of three-loop diagrams which cannot be factorized into lower-loop integrals. As a result of the calculations we obtain the analytical expression for the three-loop effective potential of the massive $O(N)$ φ^4 model presented recently by J.-M.Chung and B.K.Chung, Phys. Rev.D**56**, 6508 (1997).

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Though Quantum Field Theory already has a long history and a number of different approaches, Feynman diagrams (FD) are still the main source of its dynamical information. Vacuum (bubble) FD (without external momenta) considered here have several points of applications. First of all, it is the evaluation of the effective potentials (for the recent study see [1] and references therein) and renormalization group characteristics (β -functions and anomalous dimensions) of quantum field models and specific operators (for example, anomalous dimensions of operators in the Wilson expansion). The second important place of the applicability is the calculations of various processes (essentially in Standard Model), where there is a possibility to neglect most of the masses and momenta and to calculate only some Taylor coefficients of multipoint FD. These coefficients are bubbles having different (sometimes quite big) powers of propagators (i.e. indices of propagators). Using different recurrence relations [2]-[4] it is possible usually to represent these Taylor coefficients as sets of very simple (usually one-loop) bubbles and several so-called master integrals, which cannot be factorized into sums of lower-loop integrals. These recurrence relations are some partial cases of the relation [5, 6] for a general n -point (sub)graph with masses of its lines m_1, m_2, \dots, m_n , line momenta $p_1, p_2 = p_1 - p_{12}, p_n = p_1 - p_{1n}$ and indices j_1, j_2, \dots, j_n , respectively,

$$0 = \int d^D p_1 \frac{\partial}{\partial p_1^\mu} \left(\prod_{i=1}^n c_i^{j_i} \right)^{-1} \quad (1)$$

$$= \int d^D p_1 \left(\prod_{i=1}^n c_i^{j_i} \right)^{-1} \left[D - 2j_1 \left(1 - \frac{m_1^2}{c_1} \right) - \sum_{i=2}^n j_i \left(1 - \frac{m_1^2 + m_i^2 + p_{1i}^2 - c_1}{c_i} \right) \right],$$

where $c_k = p_k^2 + m_k^2$ are the propagators of n -point (sub)graph. The equation (1) is based on the rule of integration by part [7] and has been discovered in the process of the calculation of propagator- and vertex-type diagrams (in [5] for the case $n = 3$) and of n -point diagrams (in [6] for the arbitrary n).

Because the diagram with the index $(i + 1)$ of the propagator c_i may be represented as the derivative (on the mass m_i), Eq.(1) leads to the differential equations (in principle, to partial differential equations) for the initial diagram (having the index i , respectively). This approach which is based on the Eq.(1) and allows to construct the (differential) relations between diagrams has been named as Differential Equations Method (DEM). For most interested cases (where the number of the masses is limited) these partial differential equations may be represented through original differential equation¹, which is usually simpler to analyse. For the bubbles, DEM has been applied in [9, 10].

In the case of large number of diagrams (for example, in the calculation of various processes of Standard Model) it is convenient [3, 4] to use Eq.(1) without representating the diagrams in its r.h.s. as derivatives of initial ones. Then Eq.(1) gives the connections between different diagrams decreasing essentially the number of complicated integrals. These integrals (i.e. master diagrams) may be calculated [11] with help of DEM as it was discussed above.

1. In the present article we consider a class of three-loop bubbles (see Fig. 1) having two different masses: the masses of thick lines are one-third of the masses of the bold lines. The diagrams are interested because they cannot be factorized into lower-loop integrals (see discussions in [12]). Moreover, some of them ($J(a)$, $K(a)$, $L(a)$ and $M(a)$) give contributions (see [1, 12]) to the three-loop effective potential of the massive $O(N)$ φ^4 -model. The diagrams have been studied recently in [12] using the methods developed in the articles [13, 14]². The following results for their singular parts have been found³:

$$\begin{aligned} J(a) &= N_2 \left[\frac{2}{\varepsilon^3} + \frac{23}{3} \frac{1}{\varepsilon^2} + \frac{35}{2} \frac{1}{\varepsilon} + O(1) \right] \\ J(b) &= N_2 \left[\frac{22}{27} \frac{1}{\varepsilon^3} + \frac{1}{\varepsilon^2} \left(\frac{83}{27} + \frac{7}{9} \log 3 \right) + \frac{1}{\varepsilon} \left(\frac{365}{54} + \frac{55}{18} \log 3 + \frac{1}{2} \log^2 3 \right) + O(1) \right] \\ J(c) &= N_2 \left[\frac{2}{9} \frac{1}{\varepsilon^3} + \frac{1}{\varepsilon^2} \left(\frac{23}{27} + \frac{2}{3} \log 3 \right) + \frac{1}{\varepsilon} \left(\frac{35}{18} + \frac{23}{9} \log 3 + \log^2 3 \right) + O(1) \right] \\ K(a) &= -N_1 \left[\frac{1}{\varepsilon^3} + \frac{17}{3} \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \left(\frac{67}{3} + 6A \right) + O(1) \right] \\ K(b) &= -N_1 \left[\frac{7}{9} \frac{1}{\varepsilon^3} + \frac{1}{\varepsilon^2} \left(\frac{13}{3} + \frac{1}{3} \log 3 \right) + \frac{1}{\varepsilon} \left(\frac{151}{9} + 3A + \frac{1}{3} B + 2 \log 3 \right) + O(1) \right] \\ K(c) &= -N_1 \left[\frac{5}{9} \frac{1}{\varepsilon^3} + \frac{1}{\varepsilon^2} \left(\frac{29}{9} + \frac{2}{3} \log 3 \right) + \frac{1}{\varepsilon} \left(\frac{37}{3} + \frac{2}{3} B + 4 \log 3 \right) + O(1) \right] \\ K(d) &= -N_1 \left[\frac{5}{9} \frac{1}{\varepsilon^3} + \frac{1}{\varepsilon^2} \left(3 + \frac{2}{3} \log 3 \right) + \frac{1}{\varepsilon} \left(\frac{101}{9} + \frac{2}{3} B + 4 \log 3 \right) + O(1) \right] \end{aligned} \quad (2)$$

¹The example of the direct application of the partial differential equation may be found in [8].

²The singular part of $M(a)$, $M(b)$ and $M(c)$ diagrams has been calculated in [10] using DEM [5]. The regular part of $M(a)$ has been found very recently by Broadhurst [15].

³Contrary to [12] we use the space $D = 4 - 2\varepsilon$ and sum the terms γ^n (γ - is Euler constant) and ζ_2^n to exponents.

$$\begin{aligned}
L(a) &= N_0 \left[\frac{1}{3} \frac{1}{\varepsilon^3} + \frac{2}{3} \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \left(\frac{2}{3} + 2A \right) + O(1) \right] \\
L(b) &= N_0 \left[\frac{1}{3} \frac{1}{\varepsilon^3} + \frac{2}{3} \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \left(\frac{2}{3} + A + C \right) + O(1) \right] \\
L(c) &= N_0 \left[\frac{1}{3} \frac{1}{\varepsilon^3} + \frac{1}{\varepsilon^2} \left(\frac{2}{3} + \log 3 \right) + \frac{1}{\varepsilon} \left(\frac{2}{3} + 2B + 2 \log 3 + \frac{3}{2} \log^2 3 \right) + O(1) \right] \\
L(d) &= N_0 \left[\frac{1}{3} \frac{1}{\varepsilon^3} + \frac{2}{3} \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \left(\frac{2}{3} + 2C \right) + O(1) \right] \\
M(a) &= M(b) = M(c) = N_0 \left[\frac{2}{\varepsilon} \zeta_3 + O(1) \right],
\end{aligned}$$

where the normaliation factor

$$N_k = \frac{m^{2k}}{(4\pi)^6} \left(\frac{m^2}{\mu^2} \right)^{-3\varepsilon} \cdot \exp(3/2\zeta_2\varepsilon^2) \quad \text{and} \quad \bar{\mu}^2 = (4\pi\mu^2)e^\gamma$$

and ζ_n are Euler ζ -functions.

The constants A , B and C have been represented [12] in the form:

$$A = f(1, 1), \quad B = f(1, 3) \quad \text{and} \quad C = f\left(\frac{1}{3}, \frac{1}{3}\right),$$

where

$$f(a, b) = \int_0^1 dx \left[\int_0^{1-z} dy \left(-\frac{\log(1-y)}{y} \right) - \frac{z \log z}{1-z} \right], \quad z = \frac{ax + b(1-x)}{x(1-x)} \quad (3)$$

The pupose of this short letter is to calculate analytically A , B and C constants and, thus, to obtain exact results for the singular parts of the FD presented in Fig. 1.

Before evaluations we would like to stress that the knowledge of the exact values for the singular parts of diagrams is very important, because they determine effective potentials and renormalization functions of the quantum field models. Moreover, for various physical processes, the regular parts of diagrams may be evaluated numerically (sometimes with rather good quality) but their singular parts should be known analytically because many types of them should be canceled in the end of calculations.

2. To evaluate $A = f(1, 1)$ we represent the r.h.s. of Eq.(3) as the sum of two terms A_1 and A_2 .

We introduce the new variable $s = (1-x)/2$ and represent A_2 as

$$A_2 \equiv - \int_0^1 dx \frac{z \log z}{1-z} = -4 \int_0^1 \frac{ds}{3+s^2} \log \left(\frac{1-s^2}{4} \right)$$

The term A_1 may be rewritten in the form ($y = 1-t$)

$$A_1 \equiv \int_0^1 dx \int_0^{1-z} dy \frac{\log(1-y)}{y} = \int_0^1 ds \int_1^{4/(1-s^2)} dt \frac{\log t}{1-t} \quad (4)$$

Changing the order of integration in the r.h.s. of (4), we have

$$A_1 = 6 \int_0^1 \frac{ds}{3+s^2} \log \left(\frac{1-s^2}{4} \right)$$

Thus, the evaluation of

$$A = 4 \int_0^1 \frac{ds}{3+s^2} \left[\log(1-s) + \log(1+s) - \log 4 \right]$$

is very simple. Integrating by part, we obtain the final result

$$A = -\frac{2}{\sqrt{3}} Cl_2\left(\frac{\pi}{3}\right), \quad (5)$$

where $Cl_2(\theta)$ is Clausen integral [16]

$$Cl_2(\theta) = \int_0^\theta \log(2\sin(\theta'/2)) d\theta'$$

Repeating above calculations, we have

$$B = -\frac{4}{\sqrt{3}} Cl_2\left(\frac{\pi}{3}\right) \quad \text{and} \quad C = -\frac{1}{2} \log^2 3 + \frac{4}{3\sqrt{3}} Cl_2\left(\frac{\pi}{3}\right) \quad (6)$$

3. We have obtained results for the singular parts of a FD class in closed analytical form. These analytical results are important in the calculations of physical processes of Standard Model, where many terms may be involved and the verification and the evaluation of the singularities is very important problem. Moreover, the constant A is only one numerical factor in the recent calculation [1] of three-loop correction to the effective potential of $O(N)$ φ^4 model:

$$\begin{aligned} -\frac{(4\pi)^6}{\lambda^2} \cdot V_{eff}^{(3)}(\varphi_c) &= \frac{\lambda^2 \varphi_c^4}{4} \left\{ \frac{1129}{192} + \frac{A}{8} \right\} + \frac{m^2 \lambda \varphi_c^2}{2} \left\{ -\frac{25}{96} - \frac{A}{4} \right\} + \left[\frac{\lambda^2 \varphi_c^4}{4} \left\{ -\frac{629}{96} - \frac{3A}{4} - \zeta_3 \right\} \right. \\ &+ \frac{m^2 \lambda \varphi_c^2}{2} \left\{ -\frac{287}{48} \right\} + m^4 \left\{ \frac{25}{96} + \frac{A}{4} \right\} \left. \right] \cdot \log \left(1 + \frac{\lambda \varphi_c^2}{2m^2} \right) + \left[\frac{\lambda^2 \varphi_c^4}{4} \cdot \frac{143}{48} \right. \\ &+ \frac{m^2 \lambda \varphi_c^2}{2} \cdot \frac{17}{6} + m^4 \cdot \frac{11}{48} \left. \right] \cdot \log^2 \left(1 + \frac{\lambda \varphi_c^2}{2m^2} \right) - \left[\frac{\lambda^2 \varphi_c^4}{4} \cdot \frac{9}{16} \right. \\ &+ \left. \frac{m^2 \lambda \varphi_c^2}{2} \cdot \frac{7}{12} + m^4 \cdot \frac{5}{48} \right] \cdot \log^3 \left(1 + \frac{\lambda \varphi_c^2}{2m^2} \right) \end{aligned}$$

After above-mentioned calculation of A (i.e. Eq.(5)), this three-loop correction becomes known fully analytically.

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References

- [1] J.-M. Chung and B.K. Chung, *Phys.Rev.* **D56**, 6508 (1997).

- [2] D. Broadhurst, *Z.Phys.* **C54**, 599 (1992); D. Broadhurst and A.G. Grozin, *Phys.Lett.* **B267**, 105 (1991); hep-ph/9504400; D. Broadhurst, J. Fleischer and O.V. Tarasov, *Z.Phys.* **C60**, 287 (1993).
- [3] J. Fleischer and O.V. Tarasov, *Z.Phys.* **C64**, 193 (1994); hep-ph/9502257.
- [4] L.V. Avdeev, *Comp.Phys.Commun.* **98**, 15 (1996); L.V. Avdeev and M.Yu. Kalmykov, *Nucl.Phys.* **B502**, 419 (1997).
- [5] A.V. Kotikov, *Phys.Lett.* **B254**, 185 (1991); **B259**, 314 (1991); *Mod.Phys.Lett.* **A6**, 677 (1991).
- [6] A.V. Kotikov, *Phys.Lett.* **B267**, 123 (1991) (E: **B295**, 409 (1992)); Proceeding of Third International Workshop on Software Engineering, Artificial Intelligence and Expert System for High Energy and Nuclear Physics (Oberammergau, Germany, October 4-8, 1993), p.453 (JINR preprint E2-93-414).
- [7] A.N. Vassiliev, Y.M. Pis'mak and Y.R. Honkonen, *Teor.Math.Phys.* **47**, 465 (1981); F.V. Tkachov, *Phys.Lett.* **B100**, 65 (1981) K.G. Chetyrkin and F.V. Tkachov, *Nucl.Phys.* **B192**, 159 (1981).
- [8] C. Ford, I. Jack and D.R.T. Jones, *Nucl.Phys.* **B387**, 373 (1992).
- [9] C. Ford and D.R.T. Jones, *Phys.Lett.* **B267**, 123 (1991).
- [10] A. Palitssetto and E. Vicari, *Nucl.Phys.* **B519**, 626 (1998).
- [11] J. Fleischer, A.V. Kotikov and O.L. Veretin, *Phys.Lett.* **B417**, 163 (1998).
- [12] J.-M. Chung and B.K. Chung, hep-ph/9805432.
- [13] K.G. Chetyrkin, M. Misiak and M. Muenz, *Nucl.Phys.* **B518**, 473 (1998).
- [14] B. Kastening, *Phys.Rev.* **D54**, 3965 (1996).
- [15] D. Broadhurst, hep-ph/9803091.
- [16] L. Lewin, *Polylogarithms and Associated Functions* (North Holland, New-York, 1981)

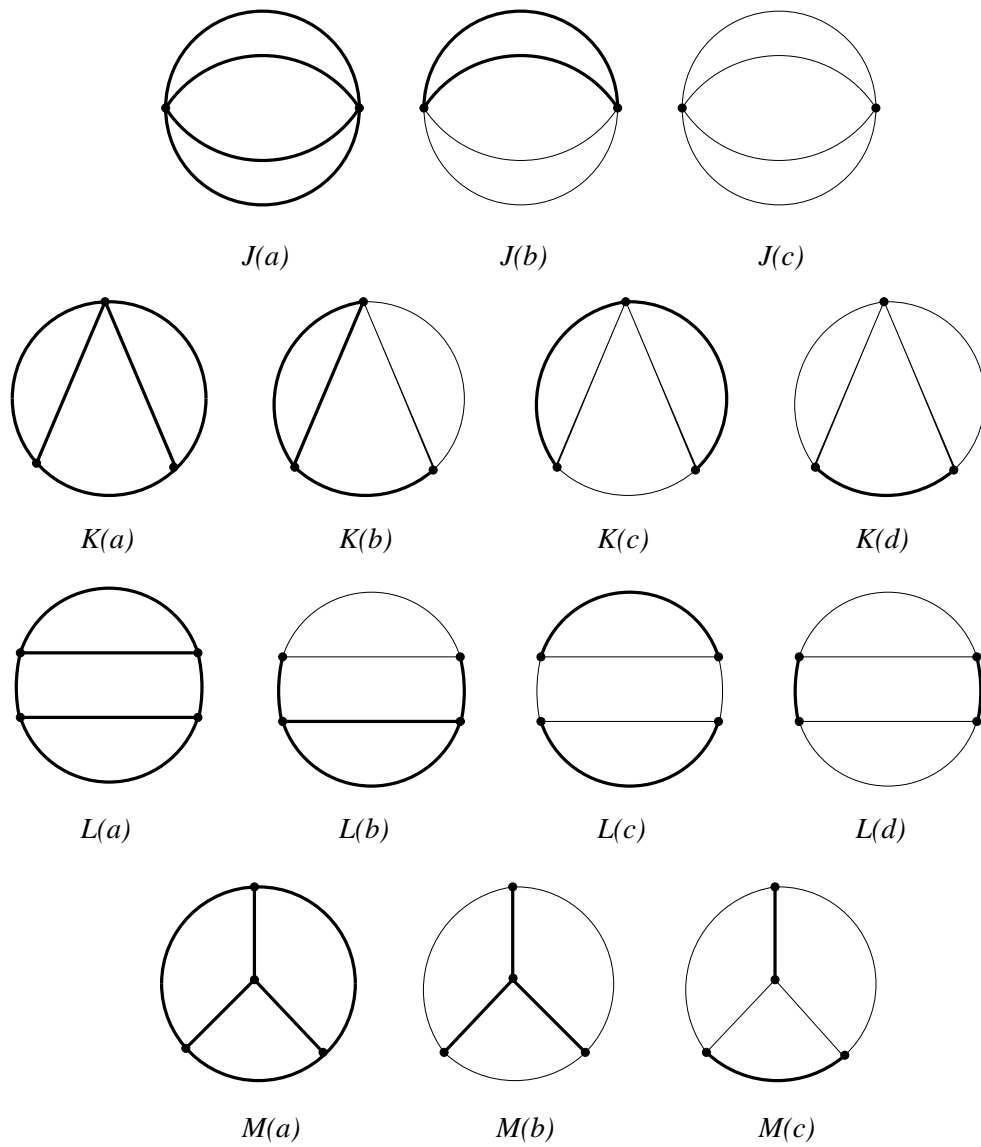


Figure 1: The class of three-loop bubbles. The masses of think lines are one-third of the masses of the bold lines.