HD-THEP-97-30

# The Barbieri–Remiddi solution of the bound state problem in QED

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**Abstract**

We derive the so-called Barbieri–Remiddi solution of the Bethe–Salpeter equation in QED in its general form and discuss its application to the bound state energy spectrum.

PACS: 12.20.-m, 11.10.St, 36.10.Dr

## 1 Introduction

The Bethe–Salpeter (BS) equation [1] is usually considered the rigorous framework in which to approach the bound state problem in QED. The increasing precision of the experimental data concerning QED bound states (e. g. decay rate, energy levels, etc. for some recent reviews we refer the reader to [2]) makes more and more urgent to effort the evaluation of physical quantities by handling the BS equation with a systematic and unified formalism.

In this paper we will focus our attention to the bound state energy levels in QED and will discuss the so-called Barbieri–Remiddi (BR) formalism. This formalism was first suggested for positronium [3, 4] (for a clarifying quanto-mechanical example see also [5]), but has been

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used in recent years also for hydrogenic atoms [6], QCD bound states [7] and scalar-scalar bound states [8]. The main idea is to solve exactly the BS equation for a suitable zeroth-order kernel containing the relevant binding interaction (i. e. the Coulomb potential) and then to perform a perturbative expansion in terms of the difference between the complete two-body kernel and the zeroth-order one. What is appealing in this approach is that the zeroth-order solution is completely known in analytic closed form. Therefore the perturbative expansion obtained in this way is completely self-contained and does not need to be improved for higher correction in the fine structure constant $\alpha$.

In the following the BR formalism will be derived in the general case of muonium (i. e. different masses). This result is new and contains the positronium and hydrogenic case as a particular one. Moreover it furnishes a way to treat radiative and recoil corrections in the same theoretical framework, which seems to be very promising.

The paper contains two main sections. In section 2 we derive the perturbative expansion of the energy levels from the BS equation in the so-called Kato formalism. In section 3 we derive in some detail the BR solution for muonium. Section 4 is devoted to some comments and conclusions.

# 2 The Bethe–Salpeter equation

In this section we review some basics concerning the Bethe–Salpeter equation in QED and set up the theoretical background for the next section. The main result will be the perturbative expansion of the energy levels of the two fermion bound state given by Eq. (12).

Let us consider a system of two fermions (of mass $m_1$ and $m_2$ and electric charge $-e$ and $Ze$ respectively) like muonium. The four point Green function $G$ is the sum of the Feynman graphs shown in Fig. 1 (notice that for a particle-antiparticle system, like positronium, one has to add the annihilation graphs). Let us define $G_0$ the two fermions free propagator:

$$G_0(E;p) = S_F^{(1)}(E_1;p) S_F^{(2)}(-E_2;p)^T, \qquad S_F(E_j;p) = \frac{i}{E_j\gamma^0 + \not{p} - m + i\epsilon}, \tag{1}$$

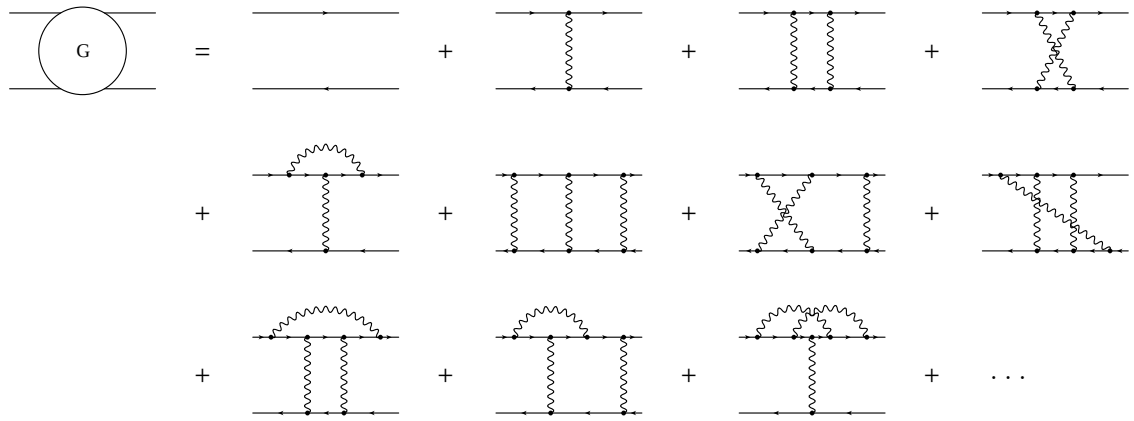

Figure 1: *Four point Green function $G$: graphs up to two loops.*

where $^{(1)}$ and $^{(2)}$ refer to the two fermion lines, $E_j \equiv E\ m_j/(m_1+m_2)$, $E$ is the bound state energy and $p$, $q$ are the relative momenta of the outcoming and incoming particles in the centre-of-mass reference frame [1]. It is found that the Green function $G$ satisfies the equation:

$$G(E;p,q) = G_0(E;p)\left[(2\pi)^4\delta^4(p-q) + \int \frac{d^4k}{(2\pi)^4} K(E;p,k)G(E;k,q)\right]. \tag{2}$$

This equation (for simplicity we have neglected the spinor indices) is known as the Bethe–Salpeter equation [1]. The kernel $K$, describing the interaction between the two fermions,

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[1] Let $p^{(1)}$ and $p^{(2)}$ be the momenta of the outcoming particles, and

$$P \equiv p^{(1)} + p^{(2)}, \qquad p \equiv \frac{\mu}{m_1}p^{(1)} - \frac{\mu}{m_2}p^{(2)},$$

with $\mu = m_1m_2/(m_1+m_2)$ the reduced mass of the two particles. In the centre-of-mass frame $\vec{p}^{\ (1)} = -\vec{p}^{\ (2)}$ implies $P = (p_0^{(1)} + p_0^{(2)}, \vec{0}\ ) = (E, \vec{0}\ )$. From the previous equations we obtain

$$\begin{aligned} p^{(1)} &= p + \frac{m_1}{m_1+m_2}P = (p_0 + E_1, \vec{p}\ ), \\ p^{(2)} &= -p + \frac{m_2}{m_1+m_2}P = (-p_0 + E_2, -\vec{p}\ ), \end{aligned}$$

where $E_j = E\ m_j/(m_1+m_2)$ and therefore

$$E = E_1 + E_2.$$

Finally, we note that in the static limit

$$E \to m_1 + m_2, \qquad E_j \to m_j.$$

In the same way we can treat the incoming momenta.

is not known in analytic closed form and is given by all the two particle irreducible graphs without external legs shown up to two loop in Fig. 2. Graphically the BS equation can be represented as in Fig. 3.

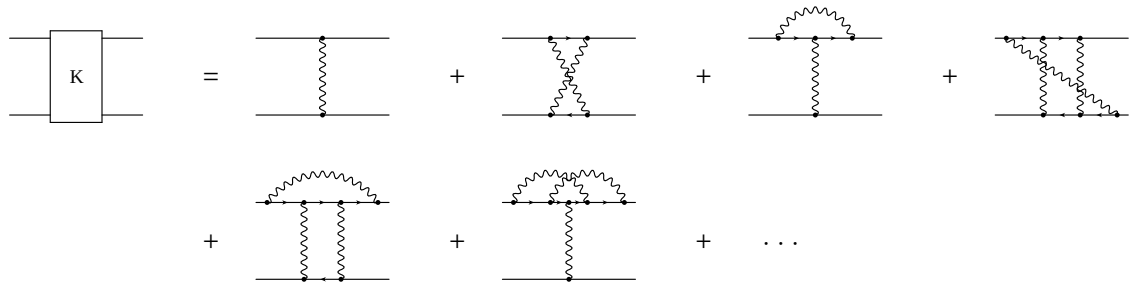


Figure 2: *The interaction kernel $K$.*

$G$, as a function of $E$, has simple poles in the bound state energy levels $E_{n...}$ [9] ($n...$ is a convenient set of quantum numbers which classifies the levels). Since the Coulomb interaction is contained in $K$, $E_{n...}$ (without mass terms) has to coincide with the Bohr levels at the leading order in $\alpha$. Therefore for any $n...$, we can write

$$G(E) = \frac{R_{n...}}{E - E_{n...}} + \hat{G}_{n...}(E)\,, \qquad E_{n...} = m_1 + m_2 - \frac{\mu}{2}\left(\frac{Z\alpha}{n}\right)^2 \; + \; \cdots\,, \tag{3}$$

where $R_{n...}$ is the residuum at the pole, $\hat{G}_{n...}$ is non singular in the limit $E \to E_{n...}$ and $\mu$ is the reduced mass of the two particle system. From now on we will neglect the explicit indication of the momenta in the argument of the functions, where considered not strictly necessary.

Inserting (3) into the BS equation and comparing the residua, we obtain:

$$R_{n...} = G_0(E_{n...})K(E_{n...})R_{n...}, \tag{4}$$

which is known as the homogeneous Bethe–Salpeter equation. Moreover, from the comparison of the non singular parts in $E - E_{n...}$ we obtain the normalization condition [10]:

$$R_{n...} = R_{n...}\,\frac{\partial}{\partial E}\left(G_0^{-1} - K\right)(E_{n...})\; R_{n...}\;. \tag{5}$$

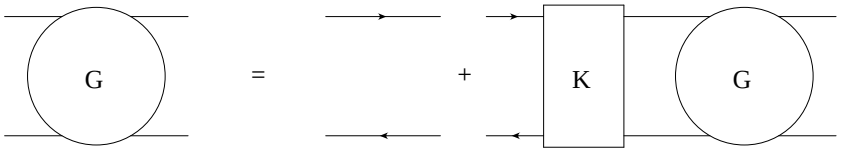


Figure 3: *Inhomogeneous BS equation:* $G(E) = G_0(E) + G_0(E)K(E)G(E)$.

The BS equation (2) is, up to now, not solvable in analytic closed form. Let $K_c$ be an interaction kernel satisfying the following two properties:

*i)* $K_c$ reproduces the same non relativistic limit of $K$, i.e. the Coulomb potential $V = -4\pi Z\alpha/p^2$ times some spinorial factors;

*ii)* the BS equation for $K_c$:

$$G_c(E) = G_0(E) + G_0(E)K_c(E)G_c(E), \tag{6}$$

is analytically solvable in closed form.

With these assumptions it is possible to solve the BS equation for $G$ at least perturbatively in terms of $\delta K \equiv K - K_c$ and to give a perturbative expansion for the bound state energy levels (the poles of $G$).

From the property *i)* it follows that $G_c$ has simple poles for $E = E_n^c$. These poles are surely more degenerate than the poles of the complete Green function $G$ and give back, at the leading order in $\alpha$, the Bohr levels:

$$G_c(E) = \frac{\sum R^c_{n\ldots}}{E - E_n^c} + \hat{G}_n^c(E)\,, \qquad E_n^c = m_1 + m_2 - \frac{\mu}{2}\left(\frac{Z\alpha}{n}\right)^2 \; + \; \cdots\,, \tag{7}$$

where $R^c_{n\ldots}$ is the residuum at the pole and $\hat{G}_n^c$ is non singular in the limit $E \to E_n^c$. The sum is extended over all the degenerate states for each $n \in \mathrm{N}$. The residuum $R^c_{n\ldots}$ satisfies the analogous of equations (4) and (5):

$$R^c_{n\ldots} \;=\; G_0(E_n^c)K_c(E_n^c)R^c_{n\ldots}\,, \tag{8}$$

$$R^c_{n\ldots} \;=\; R^c_{n\ldots}\,\frac{\partial}{\partial E}\left(G_0^{-1} - K_c\right)(E_n^c)\; R^c_{n\ldots}\,. \tag{9}$$

From the definition of $\delta K$ and from (2) and (6) we obtain the perturbative expansion of $G$ in terms of $\delta K$:

$$G(E) = G_c(E) + G_c(E)\delta K(E)G_c(E) + G_c(E)\delta K(E)G_c(E)\delta K(E)G_c(E) + \cdots . \qquad (10)$$

In order to obtain from (10) the perturbative expansion of the poles $E_{n\ldots}$ we will use the so-called Kato perturbation theory [11]. Since the energy levels are the poles of $G$ we can write:

$$E_{n\ldots} = \frac{\oint_{\Gamma_n} dz\ z\ \mathrm{Tr}\left\{G(z)O_{n\ldots}\right\}}{\oint_{\Gamma_n} dz\ \mathrm{Tr}\left\{G(z)O_{n\ldots}\right\}}, \qquad (11)$$

where $\Gamma_n$ is a closed curve in $\mathbb{C}$ which contains only the poles $E_{n\ldots}$ and $E^c_n$ of $G$ and $G_c$ respectively, $O_{n\ldots}$ is an operator which does not vanish on $R_{n\ldots}$ and $R^c_{n\ldots}$ and Tr means the trace over the spinor indices. A convenient choice is

$$O_{n\ldots} \equiv \frac{\partial}{\partial E}\left(G_0^{-1} - K_c\right)(E^c_n)\ R^c_{n\ldots}\ \frac{\partial}{\partial E}\left(G_0^{-1} - K_c\right)(E^c_n).$$

Inserting (10) in (11) integrating in $z$ and taking into account (9), we obtain (up to order $\delta K^4$):

$$\begin{aligned}
E_{n\ldots} &= E^c_n + \frac{1}{D_{n\ldots}}\left\langle \delta K(E^c_n)\right\rangle_{n\ldots} + \frac{1}{D^2_{n\ldots}}\left\langle \delta K(E^c_n)\hat{G}_c(E^c_n)\delta K(E^c_n)\right\rangle_{n\ldots} \\
&+ \frac{1}{D^2_{n\ldots}}\left\langle \delta K(E^c_n)\right\rangle_{n\ldots}\left\langle \frac{\partial}{\partial E}\delta K(E^c_n)\right\rangle_{n\ldots} \\
&+ \frac{1}{D^3_{n\ldots}}\left\langle \delta K(E^c_n)\hat{G}_c(E^c_n)\delta K(E^c_n)\hat{G}_c(E^c_n)\delta K(E^c_n)\right\rangle_{n\ldots} \\
&+ \frac{1}{D^3_{n\ldots}}\left\langle \delta K(E^c_n)\hat{G}_c(E^c_n)\delta K(E^c_n)\right\rangle_{n\ldots}\left\langle \frac{\partial}{\partial E}\delta K(E^c_n)\right\rangle_{n\ldots} \\
&+ \frac{1}{D^3_{n\ldots}}\left\langle \frac{\partial}{\partial E}\left(\delta K\hat{G}_c\delta K\right)(E^c_n)\right\rangle_{n\ldots}\left\langle \delta K(E^c_n)\right\rangle_{n\ldots} \\
&+ \frac{1}{D^3_{n\ldots}}\left\langle \delta K(E^c_n)\right\rangle_{n\ldots}\left[\left\langle \frac{\partial}{\partial E}\delta K(E^c_n)\right\rangle_{n\ldots}\right]^2 \\
&+ \frac{1}{2}\frac{1}{D^3_{n\ldots}}\left[\left\langle \delta K(E^c_n)\right\rangle_{n\ldots}\right]^2\left\langle \frac{\partial^2}{\partial E^2}\delta K(E^c_n)\right\rangle_{n\ldots} + O(\delta K^4), \qquad (12)
\end{aligned}$$

where

$$\left\langle A\right\rangle_{n\ldots} \equiv \mathrm{Tr}\left\{A\ R^c_{n\ldots}\right\} = \int \frac{d^4p}{(2\pi)^4}\int \frac{d^4q}{(2\pi)^4}\mathrm{Tr}\left\{A(p,q)\ R^c_{n\ldots}(p,q)\right\}.$$

$D_{n...}$ is the degeneracy of the $E = E_{n...}$ level and is defined to be

$$D_{n...} \equiv \left\langle \frac{\partial}{\partial E} \left( G_0^{-1} - K_c \right) (E_n^c) \right\rangle_{n...} . \tag{13}$$

Equation (12) expresses the bound state energy $E_{n...}$ as an expansion in $\delta K$. Since $\delta K$ is the difference between the sum of the infinit series of Feynman graphs drawn in Fig. 2 and the kernel $K_c$, $\delta K$ is not known in closed form. Each graph of Fig. 2 contributes to (12) with a series of powers of $\alpha$, because the dependence of the residuum $R_n^c$ on the fine structure constant (like in the well-known non-relativistic case where the hydrogen wave-functions depend on $\alpha$). For consistency with *i*) the explicit calculation must exhibit that to an increasing order in $\delta K$ it corresponds an increasing leading order in $\alpha$ in the contributions to the energy levels. In this sense expansion (12) can be interpreted as a perturbative expansion in the fine structure constant. Once $K_c$ is explicitly given and the corresponding BS equation is solved (this means we have an analytic expression for $E_n^c$, $R_{n...}^c$ and $\hat{G}_n^c$) the expansion (12) allows to obtain without any ambiguity the energy levels up to a given order in $\alpha$ for all the two-fermions bound states in QED. We emphasize that, in absence of an exact solution of Eq. (6), expression (12) could be evaluated only for an approximate choice of $G_c$ to improve at any increasing of the requested precision.

# 3 The Barbieri–Remiddi solution

In this section we work out with some detail the so-called Barbieri–Remiddi solution of equation (6) (for and exhaustive description see [12]). With this name we mean a zeroth order kernel $K_c$ which satisfies the previous given properties *i*) and *ii*) (in other words $K_c$ should describe correctly at the leading order in $\alpha$ the bound state and make solvable the corresponding BS equation (6)) as well as the solution of the corresponding BS equation. The BR solution was first given for the positronium [3]. In the following we will give the generalization of that solution for a bound state of two fermions with different masses, i. e. muonium. Once $K_c$ is given, we solve the equation for $G_c$ and work out the poles $E_n^c$ and

the residua $R^c_{n\ldots}$. At that point the perturbative expansion of the energy levels (12) will be completely defined.

Let us define the energy projectors

$$\Lambda_\pm(\vec{p}, m_j) \equiv \frac{E_{pj} \pm (m_j - \vec{p}\cdot\vec{\gamma})\gamma^0}{2E_{pj}}; \tag{14}$$

with $E_{pj} \equiv \sqrt{\vec{p}^{\,2} + m_j^2}$. In terms of $\Lambda_\pm$ the free fermion propagator $S_F$ can be written as

$$S_F(E_j; p) = i\left(\frac{\Lambda_+(\vec{p}, m_j)\gamma^0}{p^0 + E_j - E_{pj} + i\epsilon} + \frac{\Lambda_-(\vec{p}, m_j)\gamma^0}{p^0 + E_j + E_{pj} - i\epsilon}\right).$$

Moreover, we define

$$\begin{aligned}\Lambda(\vec{p}, \vec{q}\,) \;&\equiv\; \left(\frac{16E_{p1}E_{p2}E_{q1}E_{q2}}{(E_{p1}+m_1)(E_{p2}+m_2)(E_{q1}+m_1)(E_{q2}+m_2)}\right)^{\frac{1}{2}} \\ &\times\; \left(\gamma^0\Lambda_+(\vec{p}, m_1)\frac{1+\gamma^0}{2}\Lambda_+(\vec{q}, m_1)\right)^{(1)}\left(\gamma^0\Lambda_-(\vec{q}, m_2)\frac{1-\gamma^0}{2}\Lambda_-(\vec{p}, m_2)\right)^{(2)T}.\end{aligned} \tag{15}$$

The zeroth order BR interaction kernel for the muonium is

$$K_c(E; \vec{p}, \vec{q}\,) \equiv iR(E; \vec{p}\,)R(E; \vec{q}\,)V(\vec{p} - \vec{q}\,)\Lambda(\vec{p}, \vec{q}\,), \tag{16}$$

with

$$R(E; \vec{p}\,) = \left(\frac{8\mu E^2}{(E + E_{p1} + E_{p2})(E - E_{p1} + E_{p2})(E + E_{p1} - E_{p2})}\right)^{\frac{1}{2}}. \tag{17}$$

We assume (16) as a definition. In the following we will verify that this choice satisfies the properties $i$) and $ii$) given in the previous section.

In the static limit ($E \to m_1 + m_2$, $E_j \to m_j$ and $\vec{p}, \vec{q} \to 0$),

$$K_c(E; \vec{p}, \vec{q}\,) \to -i\; V(\vec{p} - \vec{q}\,)\left(\frac{1+\gamma^0}{2}\right)^{(1)}\left(\frac{1-\gamma^0}{2}\right)^{(2)T},$$

i. e. $K_c$ reproduces the Coulomb potential times some spinorial factors.

In order to verify that the choice (16) makes solvable the BS equation (6) it is useful to express the Green function $G_c$ in terms of a new function $H_c$ [2]:

$$G_c(E; p, q) \equiv G_0(E; p) + iR(E; \vec{p}\,)R(E; \vec{q}\,)H_c(E; \vec{p}, \vec{q}\,)G_0(E; p)\Lambda(\vec{p}, \vec{q}\,)G_0(E; q). \tag{18}$$

[2] In general $H_c$ could depend on each component of the momenta $p$ and $q$. The explicit calculation, however, will show that $H_c$ does not depend on $p_0$ and $q_0$ (see Eq. (20)).

Including (18) and (16) in (6), we have

$$H_c(E;\vec{p},\vec{q}\,) = V(\vec{p}-\vec{q}\,) \;-\; i\int \frac{d^4k}{(2\pi)^4}\frac{R(E;\vec{k}\,)^2}{(k^0+E_1-E_{k1}+i\epsilon)(k^0-E_2+E_{k2}-i\epsilon)} \times\; V(\vec{p}-\vec{k}\,)H_c(E;\vec{k},\vec{q}\,).$$

Integrating on $k^0$ we obtain

$$H_c(E;\vec{p},\vec{q}\,) = V(\vec{p}-\vec{q}\,) + \int \frac{d^3k}{(2\pi)^3}\frac{1}{E^*-k^2/2\mu}V(\vec{p}-\vec{k}\,)H_c(E;\vec{k},\vec{q}\,), \tag{19}$$

where

$$E^* \equiv \frac{(E-m_1-m_2)(E-m_1+m_2)(E+m_1-m_2)(E+m_1+m_2)}{8\mu E^2}.$$

Equation (19) is nothing else than the Schrödinger equation for the propagator of a non relativistic particle in an external Coulomb field. Therefore its solution is known. A way to express it is by means of the Gegenbauer polynomia $C_j^\lambda$ (for the definition and some properties see [13]) [14]:

$$H_c(E;\vec{p},\vec{q}\,) = V(\vec{p}-\vec{q}\,) - \frac{16\pi\mu(Z\alpha)^2\gamma}{(p^2+\gamma^2)(q^2+\gamma^2)}\sum_{j=0}^{\infty}\frac{1}{j+1-\mu Z\alpha/\gamma}C_j^1\left(\hat{\xi}(\vec{p}\,)\cdot\hat{\eta}(\vec{q}\,)\right), \tag{20}$$

where $\gamma \equiv \sqrt{-2\mu E^*}$. Substituting Eq. (20) in (18) we obtain the explicit analytic expression of the Green function corresponding to the kernel $K_c$ given by (16). As we will see, once $G_c$ is given, it is straightforward to work out the poles $E_n^c$, the residua $R_{n\ldots}^c$ and $\hat{G}_n^c$.

From (20) we have immediately that $G_c$ has poles in

$$E^* = -\frac{\mu}{2}\left(\frac{Z\alpha}{n}\right)^2 \Rightarrow E = E_n^c = \sqrt{m_1^2-(\mu Z\alpha/n)^2}+\sqrt{m_2^2-(\mu Z\alpha/n)^2} \quad n\in\mathbb{N}. \tag{21}$$

Notice that up to order $\alpha^2$

$$E_n^c \approx m_1+m_2-\frac{\mu}{2}\left(\frac{Z\alpha}{n}\right)^2.$$

The poles of $G_c$ give back the mass terms plus the Bohr levels, i. e. the physically correct levels up to order $\alpha^2$. Moreover we have

$$E_j(E_n^c) = E_n^c\frac{m_j}{m_1+m_2} \approx m_j - \frac{m_j}{m_1+m_2}\frac{\mu}{2}\left(\frac{Z\alpha}{n}\right)^2.$$

The residuum at the pole $E = E_n^c$, as defined in (7), is

$$\begin{aligned}\sum R^c_{n\ldots}(p,q) &= \frac{i}{4\mu}\frac{E_n^c}{\sqrt{m_1^2-(\mu Z\alpha/n)^2}\sqrt{m_2^2-(\mu Z\alpha/n)^2}}R(E_n^c;\vec{p}\,)R(E_n^c;\vec{q}\,)\\ &\times \left(p^2+(\mu Z\alpha/n)^2\right)\left(q^2+(\mu Z\alpha/n)^2\right)r_n(\vec{p},\vec{q}\,)\\ &\times G_0(E_n^c;p)\Lambda(\vec{p},\vec{q}\,)G_0(E_n^c;q),\end{aligned} \tag{22}$$

where

$$r_n(\vec{p},\vec{q}\,)\equiv 64\pi n\left(\frac{\mu Z\alpha}{n}\right)^5\frac{C^1_{n-1}\left(\hat{\xi}(\vec{p}\,)\cdot\hat{\eta}(\vec{q}\,)\right)}{\left[p^2+(\mu Z\alpha/n)^2\right]^2\left[q^2+(\mu Z\alpha/n)^2\right]^2}=\sum_{l=0}^{n-1}\sum_{m=-l}^{l}\varphi_{nlm}(\vec{p}\,)\varphi^*_{nlm}(\vec{q}\,). \tag{23}$$

$\varphi_{nlm}(\vec{p}\,)\equiv R_{nl}(p)Y_{lm}(\hat{p})$ are the well-known hydrogen atom wave functions [15]. Moreover, we can write

$$\begin{aligned}&\left(\Lambda_+(\vec{p},m_1)\frac{1+\gamma^0}{2}\Lambda_+(\vec{q},m_1)\gamma^0\right)^{(1)}\left(\Lambda_-(\vec{q},m_2)\frac{1-\gamma^0}{2}\Lambda_-(\vec{p},m_2)\gamma^0\right)^{(2)T}=\\ &\left(\Lambda_+(\vec{p},m_1)\frac{1+\gamma^0}{2}\Lambda_+(\vec{q},m_1)\gamma^0\right)_{\alpha\beta}\left(\Lambda_-(\vec{q},m_2)\frac{1-\gamma^0}{2}\Lambda_-(\vec{p},m_2)\gamma^0\right)_{\gamma\delta}=\\ &\left(\Lambda_+(\vec{p},m_1)\right)_{\alpha\rho}\left(\frac{1+\gamma^0}{2}\right)_{\rho\sigma}\left(\Lambda_+(\vec{q},m_1)\gamma^0\right)_{\sigma\beta}\left(\Lambda_-(\vec{q},m_2)\right)_{\gamma\nu}\left(\frac{1-\gamma^0}{2}\right)_{\nu\tau}\left(\Lambda_-(\vec{p},m_2)\gamma^0\right)_{\tau\delta}\\ &=\frac{1}{2}\sum_{Ss}\left(\Lambda_+(\vec{p},m_1)\Gamma_{Ss}\Lambda_-(\vec{p},m_2)\gamma^0\right)_{\alpha\delta}\left(\Lambda_-(\vec{q},m_2)\Gamma^\dagger_{Ss}\Lambda_+(\vec{q},m_1)\gamma^0\right)_{\gamma\beta}\ ,\end{aligned} \tag{24}$$

where we have used the Fiertz identity:

$$\left(\frac{1+\gamma^0}{2}\right)_{\rho\sigma}\left(\frac{1-\gamma^0}{2}\right)_{\nu\tau}=\frac{1}{2}\sum_{Ss}\left(\Gamma^\dagger_{Ss}\right)_{\nu\sigma}\left(\Gamma_{Ss}\right)_{\rho\tau}\quad S\in 0,1,\quad s\in -S,...,S\ ;$$

with the definitions:

$$\begin{aligned}\Gamma_{00} &= \frac{1+\gamma^0}{2}\gamma^5,\quad \Gamma_{1s}=i\frac{1+\gamma^0}{2}\vec{v}_s\cdot\vec{\gamma}\ ,\\ \Gamma^\dagger_{00} &= \frac{1-\gamma^0}{2}\gamma^5,\quad \Gamma^\dagger_{1s}=i\frac{1-\gamma^0}{2}\vec{v}_s{}^*\cdot\vec{\gamma}\ ,\\ \vec{v}_0 &= (0,0,1)\qquad \vec{v}_{\pm1}=-\frac{1}{\sqrt{2}}(\pm1,i,0)\ .\end{aligned}$$

Eqs. (23) and (24) allow to identify the quantum numbers $n...$ with the principal quantum number $n$, with the numbers $S, s$ describing the spin of the bound state and with the numbers

$l, m$ describing the angular momentum. The corresponding states $R^c_{nlmSs}$ are

$$\begin{aligned}
\left(R^c_{nlmSs}(p,q)\right)_{\alpha\beta\gamma\delta} &= \frac{i}{2\mu}\frac{E^c_n}{\sqrt{m_1^2-(\mu Z\alpha/n)^2}\sqrt{m_2^2-(\mu Z\alpha/n)^2}} R(E^c_n;\vec{p}\,) R(E^c_n;\vec{q}\,) \\
&\times \left(\frac{E_{p1}E_{p2}E_{q1}E_{q2}}{(E_{p1}+m_1)(E_{p2}+m_2)(E_{q1}+m_1)(E_{q2}+m_2)}\right)^{\frac{1}{2}} \\
&\times \frac{(p^2+(\mu Z\alpha/n)^2)}{(p^0+E_1(E^c_n)-E_{p1}+i\epsilon)(p^0-E_2(E^c_n)+E_{p2}-i\epsilon)} \\
&\times \frac{(q^2+(\mu Z\alpha/n)^2)}{(q^0+E_1(E^c_n)-E_{q1}+i\epsilon)(q^0-E_2(E^c_n)+E_{q2}-i\epsilon)} \\
&\times \varphi_{nlm}(\vec{p}\,)\varphi^*_{nlm}(\vec{q}\,) \\
&\times \left(\Lambda_+(\vec{p},m_1)\Gamma_{Ss}\Lambda_-(\vec{p},m_2)\gamma^0\right)_{\alpha\delta}\left(\Lambda_-(\vec{q},m_2)\Gamma^\dagger_{Ss}\Lambda_+(\vec{q},m_1)\gamma^0\right)_{\gamma\beta}. \qquad (25)
\end{aligned}$$

These states are not degenerate (how it is possible to verify directly by calculating $D_{nlmSs}$). Sometimes in the literature the residua at the poles of the Green function are written as

$$\left(R^c_{nlmSs}(p,q)\right)_{\alpha\beta\gamma\delta} = \left(\psi^c_{nlmSs}(p)\right)_{\alpha\delta}\left(\overline{\psi}^c_{nlmSs}(q)\right)_{\gamma\beta};$$

the functions $\psi^c_{nlmSs}$ and $\overline{\psi}^c_{nlmSs}$ are called the BR wave functions of the bound state:

$$\begin{aligned}
\left(\psi^c_{nlmSs}(p)\right)_{\alpha\delta} &= \left(\frac{i}{2\mu}\frac{E^c_n}{\sqrt{m_1^2-(\mu Z\alpha/n)^2}\sqrt{m_2^2-(\mu Z\alpha/n)^2}}\right)^{\frac{1}{2}} R(E^c_n;\vec{p}\,) \\
&\times \left(\frac{E_{p1}E_{p2}}{(E_{p1}+m_1)(E_{p2}+m_2)}\right)^{\frac{1}{2}} \\
&\times \frac{(p^2+(\mu Z\alpha/n)^2)}{(p^0+E_1(E^c_n)-E_{p1}+i\epsilon)(p^0-E_2(E^c_n)+E_{p2}-i\epsilon)} \\
&\times \varphi_{nlm}(\vec{p}\,)\left(\Lambda_+(\vec{p},m_1)\Gamma_{Ss}\Lambda_-(\vec{p},m_2)\gamma^0\right)_{\alpha\delta}, \qquad (26) \\
\overline{\psi}^c_{nlmSs} &= \gamma^0\left(\psi^c_{nlmSs}\right)^\dagger\gamma^0. \qquad (27)
\end{aligned}$$

Actually there are no reasons to introduce the bound state wave functions: from the formalism developed in the previous section it is clear that all the physical quantities can be expressed in terms of residua.

Finally, we obtain $\hat{G}^c_n$ subtracting from $G_c$ the singular part:

$$\begin{aligned}
\hat{G}^c_n(E^c_n;p,q) &= (2\pi)^4\delta^4(p-q)G_0(E^c_n;p) + G_0(E^c_n;p)K_c(E^c_n;\vec{p},\vec{q}\,)G_0(E^c_n;q) \\
&+ \hat{H}_c(E^c_n;p,q), \qquad (28)
\end{aligned}$$

where $\hat{H}_c$ takes into account all the contributions non singular in $E = E_n^c$ which come from the second term in (20). For an explicit expression of $\hat{H}_c$ in the positronium case we refer the reader to [16].

# 4 Conclusions

From the above given expressions we can recover some interesting limiting cases.

Putting $Z = 1$ and $m_1 = m_2 \equiv m$ ($\mu = m/2$ and $E_1 = E_2 = E/2$) the above given zeroth-order solution of the BS equation reduces to the original BR solution given in [3] for positronium. The main difference with the muonium case is that for positronium also annihilation graphs contribute to the interaction kernel $K$. In the literature the singlet state ($S = 0$) is usually referred as *parapositronium* and the triplet state ($S = 1$) as *orthopositronium*. Some applications can be found in [12, 16, 17, 18, 19].

Taking the limit of one particle mass to infinity the case of a particle in an external Coulomb field is recovered. In this case (e. g. $m_2 \to \infty$ ) $\mu = m_1 \equiv m$, the difference $E_2 - m_2$ is finite and the bound state energy $E$ is given by

$$E = \lim_{m_2 \to \infty} (E_1 + E_2 - m_2).$$

This case has been extensively studied in [6, 12, 20] for the evaluation of the pure radiative corrections to the energy levels of hydrogenic atoms.

As a conclusive remark, we stress that all these systems and muonium can be studied now in the same framework. In particular the formalism provides a powerful tool in dealing simultaneously with radiative, recoil and radiative-recoil corrections. Extremely interesting seems also to be the study of the infinit mass limit of one particle in the energy expansion (12) evaluated on muonium states. This should eventually clarify how this limit works in a purely off mass-shell context.

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