

Beyond the Standard Embedding in M -Theory on S^1/Z_2

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ABSTRACT

In this paper we discuss compactifications of M-theory to four dimensions on $X \times S^1/Z_2$, in which nonstandard embeddings in the $E_8 \times E_8$ vacuum gauge bundle are considered. At the level of the effective field theory description of Horava and Witten, this provides a natural extension of well known results at weak coupling, to strongly coupled $E_8 \times E_8$ heterotic strings. As an application of our results, we discuss models which exhibit an anomalous $U(1)_A$ symmetry in four dimensions, and show how this emerges from the reduction of the $d = 11$ topological term $C \wedge G \wedge G$, and how it is consistent with $d = 4$ anomaly cancellation in M-theory. As a further application of nonstandard embeddings, we show how it is possible to obtain an inverse hierarchy of gauge couplings, where the observable sector is more strongly coupled than the hidden one. The basic construction and phenomenological viability of these scenarios is demonstrated.

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1 Introduction

The quest for the theory unifying all the known interactions has entered a qualitatively new and exciting phase with the realization of the existence of the web of dualities connecting apparently different string models and allowing for the exploration of nonperturbative phenomena in string theory. The underlying eleven dimensional quantum theory whose ten dimensional emanations are known superstring theories has been termed M -theory. One simple-minded way of viewing the relation of M -theory to strings is to say that in this framework the string coupling becomes a dynamical field, which may be interpreted as an extra spatial dimension. Many field theoretical limits corresponding to compactifications of M -theory on various manifolds and orbifolds are known. Among them special role is played by the chiral $N = 1$ supersymmetric model formulated by Horava and Witten, which is the low-energy limit of the M -theory compactified on the line segment S^1/Z_2 and further on a six dimensional Calabi-Yau manifold [1, 2, 3]. The field theoretical model is constructed as a consistent compactification of the eleven dimensional $N = 1$ supergravity theory on S^1/Z_2 . It has been demonstrated that the supergravity model obtained this way, living on the eleven dimensional manifold with two parts of its ten dimensional boundary inhabited by the ten dimensional E_8 gauge supermultiplets, forms the low-energy limit of the strongly coupled $E_8 \times E_8$ heterotic superstring [1]. One of the important results of this construction is that, in order to preserve supersymmetry and to assure the absence of anomalies, the usual stringy Bianchi identity has to be modified [2]. This results in a nonvanishing antisymmetric tensor field background. As a consequence, corrections to the Calabi-Yau metric are induced. Witten has shown that the volume of the C-Y space changes linearly with x^{11} along the eleventh dimension, with a coefficient which depends on the gauge and gravitational vacuum configurations. The important consequence for physics in four dimensions is that the gauge couplings in the hidden and visible sectors are split with the sign of $\delta(\frac{1}{\alpha}) = \alpha_h^{-1} - \alpha_v^{-1}$ and its magnitude depending on the particular embedding of the gauge vacuum bundle in the $E_8 \times E_8$ bundle. For the standard embedding, which consists in identifying the $SU(3)$ holonomy gauge field of the underlying Calabi-Yau manifold with some of the gauge fields from one of the two E_8 s - the one which is subsequently broken down to E_6 and called the visible gauge sector, $\alpha_v < \alpha_h$. Here indices v , h denote the visible and hidden sectors and $\alpha = g^2/(4\pi)$.

Moreover, additional consistency checks appear, namely to have $\alpha_h < \infty$, the physical radius of the eleventh dimension, R_{11} , must be smaller than certain critical value R_{crit} . On the other hand, to obtain the correct value of the Newton constant, G_N , we need $R_{11} \approx R_{crit}$. Therefore, α_h at M_{GUT} is relatively large, and this raises the question about the scale of the gaugino condensate, and about visible mass hierarchy which should follow (there seems to be no room left for running of the hidden gauge coupling below M_{GUT}).

This problem seems to be directly related to the fact that, although nonstandard embeddings were already discussed in [4], the examples of the specific M -theoretical model considered so far in the literature have been obtained precisely in the context of the standard embedding [5]. It is an obvious idea that relaxing the standard embedding assumption might help to obtain a more realistic scale of the gaugino condensation in M -theory. Moreover, as we shall see,

the standard embedding is consistent with the overall unification of couplings only with $\alpha_v < 4\pi^{3/2} \frac{1}{\epsilon^{3/4}} (M_{GUT}/M_{Pl})^{3/2}$ where ϵ is a parameter which cannot be too small (see Section 5)¹. Thus, typically, there is no room left for the possibility of unification at ‘strong’ coupling, which is characteristic for models with additional matter at intermediate scales [6]. Again, relaxing the standard embedding assumption opens up this avenue (see also [7]). One should stress that both problems are typical for M -theory. In the weakly coupled heterotic string, although there exist similar correlations between the gauge couplings of the two E_8 s, the difference $\delta(\frac{1}{\alpha})$ is generically small and phenomenologically irrelevant.

The above genuinely M -theoretical motivation for going beyond the standard embedding comes in addition to the reasons which are the same as for the weakly coupled string models [8, 9, 10]². As in the weakly coupled case, these constructions give a chance of constructing directly models similar to the MSSM, with non-semisimple and lower rank gauge group at the unification scale. Also, this seems to open up the possibility of gauge mediated supersymmetry transmission models [12] in the context of the Horava-Witten model. Another very important fact is that, as in the weakly coupled case, the anomalous $U(1)_A$ group can be present only in models which go beyond the standard embedding. The physics of models with the anomalous $U(1)_A$ has been studied extensively over the last years and it has been shown that it can be relevant for the fermion mass generation [13], and for the supersymmetry breaking issue [14, 15, 16]. Hence, it is of considerable importance to set up the framework for the discussion of the anomalous $U(1)_A$ symmetry in the context of the strongly coupled heterotic string models.

The purpose of this paper is to give a setup for the Horava-Witten model with non-standard embeddings and to discuss some of the phenomenological questions raised above.

Let us summarize here some basic facts worked out in the literature so far. First, we note that the calculation of Witten, which gives the picture of unification in M -theory, is valid for general embeddings. However, it gives only $\delta(\frac{1}{\alpha})$, not the normalization of the individual couplings separately, and it has been performed in the eleven dimensional setup, in particular using eleven dimensional metric and eleven dimensional equations of motion. Ultimately, we are interested in the four dimensional effective theory and would like, for instance, to express the gauge couplings in each gauge sector in terms of the four dimensional moduli chiral superfields S and T . In particular, supersymmetry breaking may be governed by the four dimensional physics. Moreover, the comparison to the weakly coupled heterotic string predictions is transparent in four dimensions.

The correct procedure for obtaining the effective four dimensional theory consists in integrating out physics in six dimensions compactified on the Calabi-Yau manifold and in the eleventh dimension compactified on the S^1/Z_2 [17, 18, 19, 20, 21]. This should be performed in this well defined order as, for phenomenological reasons, the eleventh dimension has to be about an order of magnitude larger than the Calabi-Yau radius. Integrating out the six C-Y dimensions introduces the Calabi-Yau modes into the (nonlinear sigma model) five dimensional supergravity Lagrangian, which in a sense is the fundamental Lagrangian of the M -theory. It

¹Models with $\epsilon = 0$ exist, but are phenomenologically uninteresting.

²A possibility of extending nonstandard stringy models to M -theory was considered in [11].

is in five dimensions where we should solve the equations of motion with the sources defined with the S^1/Z_2 compactification, and afterwards integrate out the fifth dimension [20, 21]. This approach is considerably complex and has not yet been fully completed. A simplification which has been taken in several papers (including the original paper [2]) and which we assume here is to neglect the effects of the Calabi-Yau physics on the dynamics in the fifth dimension. This way the equations of motion in the fifth (eleventh) dimension can be solved directly along the eleventh dimension. After dimensional reduction and truncation of the six C-Y dimensions, and Kaluza-Klein reduction from five to four dimensions, the effective four dimensional theory is obtained.

Returning now to the existing literature, it was the work of Banks and Dine [22] which has shown that with the standard embedding, Witten's result for $\delta(\frac{1}{\alpha})$ follows from the standard universal threshold corrections $\frac{1}{g_{(1),(2)}^2} = S_r \pm \epsilon T_r$ derived in the weakly coupled heterotic string from the Green-Schwarz terms in ten dimensions (for earlier work in this direction see Ibanez and Nilles [23], axionic couplings were also considered in [24]). The analysis of the relation between weakly coupled string one-loop threshold corrections and strongly coupled string threshold corrections has been performed also in [25], and recently threshold corrections for nonstandard embeddings have been computed in [26]. Then Lukas *et al.* [27] have shown that the Green-Schwarz terms used in [22] can be derived from the $11d \rightarrow 10d$ Kaluza-Klein reduction, in the spirit of the simplified approach discussed above. Hence, it has been shown that the overall normalization of the Witten's result follows from the axionic couplings, and actually for the standard embedding the effective four dimensional form $1/g_{v,h}^2 = S_r \pm \epsilon T_r$ is derived from the eleven dimensional theory, and it agrees with the explicit weak coupling calculation. This is very interesting since, in the context of the five dimensional physics, the origin of axionic corrections and scalar corrections is totally different (the former come through the topological interaction term in eleven or five dimensions, and the latter through the corrections to the C-Y metric). It confirms the fact that, at least for the gauge fields and charged matter sector, there should exist a reduced effective theory which has the form of the four dimensional supersymmetric gauge chiral model.

In this paper we generalize the standard embedding result and confirm that the general result $1/g_{(1),(2)}^2 = Re(S + (\pm\epsilon_i)T_i)$ (where $i = 1, \dots, h(1,1)$) is correct for a general embedding if one considers only zero-modes both on C-Y space and on the circle. In this derivation we follow the simplified approach outlined above to the construction of the effective four dimensional theory with, however, proper care taken of non-zero modes both on the Calabi-Yau space and on the circle in the eleventh dimension. The overall normalization of gauge couplings is fixed by the normalization of axionic corrections for general embeddings. Then, we consider the existence of an anomalous $U(1)_A$ in the non-standard embedding Horava-Witten model, Finally, we discuss the possibilities for changing the relative sign of threshold corrections between the different gauge group factors, a question which is crucial in understanding gaugino condensation hierarchy and 'strong coupling' unification in the context of the most general embeddings.

2 Effective Lagrangian of the M-Theory compactified on S^1/Z_2

To start with let us recall the form of the M-theory Lagrangian constructed by Horava and Witten [2, 3], which is given by $L_S + L_B$ where

$$\begin{aligned}
L_S &= \frac{1}{\kappa^2} \int_{M^{11}} d^{11}x \sqrt{g} \left\{ -\frac{1}{2}R - \frac{1}{2}\bar{\Psi}_I \Gamma^{IJK} D_J \Psi_K - \frac{1}{48} G_{IJKL} G^{IJKL} \right. \\
&\quad - \frac{\sqrt{2}}{384} (\bar{\Psi}_I \Gamma^{IJKLMN} \Psi_N + 12 \bar{\Psi}^J \Gamma^{KL} \Psi^M) (G_{JKLM} + \hat{G}_{JKLM}) \\
&\quad \left. - \frac{\sqrt{2}}{3456} \epsilon^{I_1 \dots I_{11}} C_{I_1 I_2 I_3} G_{I_4 \dots I_7} G_{I_8 \dots I_{11}} \right\} \tag{1}
\end{aligned}$$

$$\begin{aligned}
L_B &= \frac{1}{2\pi(4\pi\kappa^2)^{2/3}} \sum_{m=1}^2 \int_{M_m^{10}} d^{10}x \sqrt{g} \left\{ -\frac{1}{4} \text{Tr} F_{AB}^m F^{mAB} - \frac{1}{2} \text{Tr} \bar{\chi}^m \Gamma^A D_A(\hat{\Omega}) \chi^m \right. \\
&\quad - \frac{1}{8} \text{Tr} \bar{\Psi}_A \Gamma^{BC} \Gamma^A (F_{BC}^m + \hat{F}_{BC}^m) \chi^m + \frac{\sqrt{2}}{48} \text{Tr} \bar{\chi}^m \Gamma^{ABC} \chi^m \hat{G}_{ABC11} \left. \right\} \\
&\quad + O(\kappa^{4/3}) \text{ relative to } L_S \tag{2}
\end{aligned}$$

where in eqs(1,2), $I = 1, \dots, 11$ label coordinates on M_{11} ; $A = 1..10$ those on $M_4 \times X$, ($a, \bar{b} = 1..3$ will denote the holomorphic and antiholomorphic coordinates on X); $\kappa = m_{11}^{-9/2}$, with m_{11} the 11 dimensional Planck mass. The field strength $G_{IJKL} = [\partial_I C_{JKL} \pm 23]$ terms $+ O(\kappa^2)$ satisfy the modified Bianchi identities [2]

$$(dG)_{11ABCD} = -\frac{3\sqrt{2}\kappa^2}{\lambda^2} \sum_{m=1}^2 \delta^{(m)}(x^{11}) [\text{tr} F_{[AB}^m F_{CD]}^m - \frac{1}{2} \text{tr} R_{[AB} R_{CD]}] \tag{3}$$

where $\lambda^2 = 2\pi(4\pi\kappa^2)^{2/3}$ is the $d = 10$ gauge coupling constant. The delta functions $\delta^{(m)}(x^{11})$ have support on the two fixed point sets in $M_4 \times X \times S^1/Z_2$. The presence of these various source terms in eq(3) are an important difference with the corresponding Bianchi identities relevant to compactification of the perturbative $E_8 \times E_8$ heterotic string, where H_{ABC} plays the role of G_{11ABC} .

In the bulk $d = 11$ supergravity lagrangian L_S , $g = \det(g_{IJ})$ involves the $d = 11$ bulk metric. In the boundary lagrangian, the same quantity is understood as being the determinant of the $d = 10$ metric obtained as the restriction of the bulk metric to either of the two boundaries M_m^{10} , $m = 1, 2$. Similarly the two copies of the E_8 super Yang Mills fields defined on the boundaries are denoted by F^{mAB} , χ^m respectively. Ω_{ABC} are the usual $d = 10$ spin connections, with hatted quantities denoting the supercovariant generalizations, explicit definitions of which can be found in [2, 3].

2.1 Lagrangian at Order $O(\kappa^{2/3})$ and Unification

One of the reasons why the supergravity on a manifold with boundary [3] which is the field theoretical limit of the strongly coupled $E_8 \times E_8$ heterotic string is so interesting is that it is possible with its three independent low energy parameters to fit three fundamental low energy observables in an internally consistent way. Actually, fitting the fundamental parameters of the weakly coupled heterotic string (also three of them) to observed values of the Newton constant, G_N , g_{GUT}^2 and M_{GUT} is also possible, but the value of the dimensionless string coupling which corresponds to that fit is much larger than unity, signaling departure from the perturbative regime. When such a matching is performed within the scope of the Horava-Witten Lagrangian, the corresponding string coupling turns out to be large too, but here it is consistent with the assumptions under which the effective field theory limit has been obtained.

Below we shall briefly review the weakly and strongly coupled string unifications, forgetting in this section about the parts of the Horava-Witten Lagrangian which of the order $O(\kappa^{4/3})$ relative to the gravitational part. One of the reasons is to establish a clear correspondence between the weakly coupled string, and strongly coupled string degrees of freedom, which is necessary if one wants to use certain results, like specific values of threshold corrections, obtained in the weakly coupled region, in the Horava-Witten theory. The correspondence is also useful when one wants to realize why the matching conditions are different in both cases, and what is the physical interpretation of this difference.

Let us start with the weakly coupled string. The relevant terms are

$$-\frac{1}{2\kappa_{10}^2} \int \sqrt{g^{(10)}} R^{(10)} - \frac{1}{4\lambda^{(10)2}} \int \sqrt{g^{(10)}} \phi^{-3/4} F^2 \quad (4)$$

where the ten dimensional gauge coupling $\lambda^{(10)2}$ can be made equal to one through the rescaling of the ten dimensional dilaton field ϕ . It is instructive to recall, that the ten dimensional supergravity Lagrangian can be obtained as a reduction of the eleven dimensional supergravity Lagrangian. In terms of the underlying 11d metric g_{IJ} the dilaton is just $\phi^2 = g_{11\ 11}$. After truncation the canonical normalization of the 10d Einstein-Hilbert is achieved through the suitable Weyl rescaling of the 11d metric, $g_{MN} \rightarrow \phi^{-1/4} g_{MN}$. There is one more useful observation. When in the 10d Lagrangian one performs the Weyl rescaling $g_{MN} \rightarrow \phi^{-1} g_{MN}$, then the dilaton ϕ enters the Lagrangian only as an overall factor, multiplying all terms in the Lagrangian, namely as ϕ^{-3} . This agrees with the effective Lagrangian of the heterotic string if one identifies the dimensional string coupling as $g_s = \phi^{3/2}$. After these clarifications, let us go back to the canonical 10d Lagrangian, and let us perform further reduction down to $d=4$. Let us write the 10d metric in the form:

$$g_{\mu\nu}^{(10)} = e^{-3\sigma} g_{\mu\nu}^{(4)}, \quad g_{MN}^{(10)} = e^\sigma g_{MN}^{(0)} \quad (5)$$

where the $g^{(0)}$ is the reference metric on the C-Y space given in the context of the canonical metric in $d = 10$. The volume of the Calabi-Yau space is defined as $V_{Xw} = e^{3\sigma} \int_X \sqrt{\det g_{MN}^{(0)}}$ and w stands for weak coupling. Now, when we compactify to four dimensions, then it turns out that the 4d metric $g_{\mu\nu}^{(4)}$ defined above is canonical, and one can easily read off the 4d Planck

scale, and 4d gauge coupling:

$$\frac{M_{Pl}^{(4)2}}{16\pi} = \frac{M_{Pl}^{(10)8}}{16\pi} V_w^{(0)}, \quad \frac{1}{g_{GUT}^2} = \frac{V_w^{(0)}}{\lambda^{(10)2}} e^{3\sigma} \phi^{-3/4} \quad (6)$$

In the above there are three stringy parameters, $M_{Pl}^{(10)}$, ϕ , σ , and two reference constants which we are free to choose in any convenient way. The standard choice is $V_w^{(0)} = M_{Pl}^{(10)-6}$ which makes $M_{Pl}^{(10)} = M_{Pl}^{(4)}$, and $\frac{V_w^{(0)}}{\lambda^{(10)2}} = 1$. In addition, one has to express the four dimensional unification scale M_{GUT} in terms of the independent parameters. The natural choice is $M_{GUT}^{-6} = V_{Xw} = e^{3\sigma} M_{Pl}^{(10)-6}$. In the context of the weak string Lagrangian in four dimensions one introduces chiral moduli superfields S_w, T_w whose real parts are $S_{wr} = e^{3\sigma} \phi^{-3/4}$ and $T_{wr} = e^\sigma \phi^{3/4}$. If we take the values of observables to be $g_{GUT} = 0.7$, $M_{Pl} = 1.2 \times 10^{19} GeV$, $M_{GUT} = 2 \times 10^{16} GeV$, then we obtain from $S_{wr} = 1/g_{GUT}^2$, $S_{wr} T_{wr} = (M_{Pl}/M_{GUT})^8$ the values $S_{wr} = 2.04$ and $T_{wr} = 8 \times 10^{21}$ and the corresponding value of the string coupling $g_s^{-2} = S_{wr}/T_{wr}^3 = 4 \times 10^{-66}$ which is clearly inconsistent with the assumption of the perturbativity of the underlying string theory. Such huge values of T_{wr} are just symptomatic of the failure of isotropic C-Y spaces to 'decouple' the scales M_{Pl} from M_{GUT} . As has been pointed out in [2], a way of ameliorating this difficulty is to advocate anisotropic C-Y spaces where $V_w \sim \alpha'^{d/2} M_{GUT}^{d-6}$. In such a scheme it may be feasible to fit correctly the values of M_{Pl}, M_{GUT} and g_{GUT}^2 and obtain the more reasonable values of T_{wr} that have been used in the past literature, when discussing, for example the role of moduli dependent threshold corrections to the gauge couplings. Such a solution could hardly be described as natural, however.

The above picture of stringy unification in the weakly coupled regime can be supplemented by the additional requirement of numerical unification of the gauge couplings and the dimensionless gravitational coupling $g_{grav}^2(E) = 16\pi G_N E^2$ [28]. The observation which leads us to contemplate this possibility is that in ten dimensional *stringy* frame one has the relation $g_{GUT}^2 = 16\pi G_N/\alpha' = 16\pi G_N M_{string}^2$. A way to state that this universal unification does not work is to assume that all forces numerically unify at the observed scale of unification of gauge forces with the unified coupling at that scale equal to g_{GUT} . Then one easily obtains from the above relation the value of M_{pl} : $M_{pl} = \sqrt{16\pi} M_{GUT}/g_{GUT} \approx 2 \times 10^{17} GeV$ which is too small, by two orders of magnitude. If one goes to the four dimensional canonical frame, the equivalent statement is simply that $1/\alpha' = M_{Pl}^2/(16\pi)$ which is different from M_{GUT}^2 .

Let us move now to the case of the strongly coupled heterotic string. In the analysis presented in this section we shall neglect all the terms in the Horava-Witten Lagrangian which are beyond the order $O(\kappa^{2/3})$ relative to the gravitational Lagrangian.

In this case we start with the eleven dimensional Lagrangian, and in first step reduce it directly to five dimensions. As argued in [20, 21] it is five dimensions where the low energy Lagrangian of M-theory is naturally formulated, and where the presumed unification of gauge couplings with gravity might occur.

The reduction is made through the ansatz

$$g_{\alpha\beta}^{(11)} = e^{-2\beta} g_{\alpha\gamma}^{(5)}, \quad g_{MN}^{(11)} = e^\beta g_{MN}^{(0)} \quad (7)$$

where α, γ are five dimensional indices. We note that the volume of the Calabi-Yau space is defined now as $V_{X_s} = e^{3\beta} V_s^{(0)} = e^{3\beta} \int_X \sqrt{\det g_{MN}^{(0)}}$ in terms of the underlying eleven dimensional metric (s stands for strong coupling). In five dimensions we obtain with this choice the Lagrangian with canonically normalized Einstein-Hilbert term

$$-\frac{V_s^{(0)}}{2\kappa^2} \int_{M^5} \sqrt{g^{(5)}} R^{(5)} - \frac{V_s^{(0)}}{8\pi(4\pi\kappa^2)^{2/3}} \int_{M^4} \sqrt{g^{(4)}} e^{3\beta} F^2 \quad (8)$$

The interesting observation concerning this Lagrangian is that any further Weyl rotation of the 5d metric leaves the kinetic term of the gauge fields invariant. Hence, in principle, one could use some other frame in five dimensions, for instance the brane-frame. However, in this paper we stay within the spirit of [21] and choose 5d canonical metric. Then, to make the connection of the parameters of the Lagrangian with the four dimensional Planck scale one should reduce the gravitational action further to four dimensions. To this end let us define $g_{55}^{(5)} = e^{2\gamma}$. Then the gravitational action in four dimensions, in terms of the metric which is canonical in 5d, takes the form

$$-\frac{2V_s^{(0)}\pi\rho_0}{2\kappa^2} \int_{M^4} \sqrt{g^{(4)}} e^\gamma R^{(4)} \quad (9)$$

where $2\pi\rho_0$ is the coordinate length of S^1 . The physical distance between the walls, measured with respect to canonical five dimensional metric, is $\pi\rho_p = e^\gamma\pi\rho_0$. Similarly as in the weak string case we have here three physical parameters, which are κ, γ, β , and two reference numbers: $V_s^{(0)}, \rho_0$. The first of the two numbers, the fiducial volume, we choose in such a way that $e^{3\beta} = 1/g_{GUT}^2$, i.e. $V_s^{(0)} = 2\pi(4\pi\kappa^2)^{2/3}$. At this point one defines the four dimensional modulus S through the requirement analogous to that employed in the reduction of the weakly coupled string, namely that it becomes the dynamical gauge coupling, $S_r = 1/g^2 = e^{3\beta}$. The ρ_0 we choose in a more sophisticated way. We shall keep in mind that eventually we shall need to compare our results to the weakly coupled case. In the heterotic string theory in ten dimensions there is a single dimensionful parameter, which is the string tension, or its inverse called α' . Hence, it makes sense to measure all distances and mass scales in units which are suitable powers of the fundamental distance $\sqrt{\alpha'}$. Actually, this is the case in the reduction of the weakly coupled string action described earlier. There, in the stringy frame, $\sqrt{\alpha'} = \sqrt{2}\kappa_{10}/\lambda_{10} = 4\sqrt{\pi}/M_{Pl}$, and indeed, we have defined all the scales in terms of the Planck scale in that case. The correspondence between present M -theory parameters and the ten dimensional string parameters is easily obtained when one starts from the usual eleven dimensional Lagrangian and compactifies first down to ten dimensions. The resulting expression for α' , again - in the string frame, is

$$\alpha' = \frac{1}{2(4\pi)^{2/3}\pi^2} \frac{\kappa^{2/3}}{\rho_0} \quad (10)$$

Since ρ_0 is the fundamental length scale of our model, then in order to be in agreement with the weakly coupled case normalization we shall take $\rho_0 = \sqrt{\alpha'}$ which gives

$$\rho_0 = \left(\frac{\kappa}{4\pi}\right)^{2/9} \frac{1}{2^{1/3}\pi^{2/3}} \quad (11)$$

When one goes down to four dimensions, then exactly as in the effective Lagrangian for the weakly coupled case one tries to cast the supersymmetric part of the Lagrangian in the superfield language, and to this end defines two superfields whose kinetic terms do not mix (as in the weak case): $S_{sr} = e^{3\beta}$ and $T_{sr} = e^\gamma$. It is interesting to realize that, with the normalizations we have assumed above, there exists a simple correspondence between weak and strong S and T fields. The simplest way to find it out is to start with the canonical metric in eleven dimensional M -theory Lagrangian and reduce it down to four dimensions in two ways: one way is first to go to the canonical ten dimensional metric and then to 4d with the redefinitions of the metric chosen exactly as in the weak case, and the second way is to reduce the same 11d metric as in the strong case, first to 5d and then down to 4d. Then, remembering that one has started from the same canonical 11d metric, one can compare the same entries of the original metric in both final parametrizations - strong and weak ones. Comparing both forms of $g_{11\ 11}^{(11)}$ one obtains $\phi = e^{\gamma-\beta}$, and comparing the two expressions for $g_{MN}^{(11)}$ we obtain $e^\beta = \phi^{-1/4}e^\sigma$. The net conclusion one can draw from these manipulations is that one can identify $S_s = S_w$ and $T_s = T_w$. It is interesting to note the surprising fact, that the weak S has been chosen in such a way, that it corresponds to the Calabi-Yau physical volume measured in ‘strong’ eleven dimensional metric. Further to that, we can see that the ‘weak’ and ‘strong’ volumes of the same Calabi-Yau space are different. The fact that in each case we define the same physical M_{GUT}^6 to be equal to the inverse of the respective volume, which is not the same in ‘weak’ and ‘strong’ cases, amounts to the different relations between three fundamental physical parameters in both scenarios. To see what are the consequences of that difference, let us perform the fit to observables in the strongly coupled scenario. Now the fundamental relations are:

$$e^{3\beta} = S_{sr} = \frac{1}{g_{GUT}^2}, \quad M_{GUT}^{-6} = \frac{1}{g_{GUT}^2} 2\pi(4\pi\kappa^2)^{2/3}, \quad \frac{M_{Pl}^2}{16\pi} = \frac{g_{GUT}^2}{M_{GUT}^6} \frac{\pi\rho_0 T_{sr}}{\kappa^2} \quad (12)$$

Substituting the same numbers as before for four dimensional observables we obtain the physical distance between the hyperplanes or equivalently the mass scale at which the fifth dimension opens up $m_5 = (\pi\rho_p)^{-1} = 0.8 \times 10^{16} GeV$, and the values of the moduli $S_{sr} = 2.04$ and $T_{sr} = \frac{1}{\pi^{10/3}} \frac{1}{2^{19/3}} \left(\frac{M_{Pl}}{M_{GUT}}\right)^2 \frac{1}{S_{sr}^{1/3}} = 80$. This allows us to compute the dimensionless string coupling $g_s^{-2} = S_{sr}/T_{sr}^3 = 4 \times 10^{-6}$. The inverse of this number is still large (although much smaller than the one obtained from the fit to the weakly coupled case), but this time it is consistent with the initial assumption, which was that the underlying heterotic string *is* strongly coupled. To have more clear picture of the resulting pattern of scales let us quote the numerical values of the resulting eleven dimensional Planck scale $m_{11} = \kappa^{-2/9}$ and the ten dimensional string tension: $m_{11} = 0.2/\sqrt{\alpha'} = 4 \times 10^{16} GeV$.

It is interesting to discuss in the present context the definition of the dimensionless gravitational coupling obtained from the stringy frame and check how close is the numerical unification of gauge and dimensionless gravitational coupling in the strongly coupled scenario. First, let us note that among the physical scales the scale of the fifth dimension, m_5 , is the lowest, and about five times *smaller* than the unification scale M_{GUT} . Then the stringy scale $1/\sqrt{\alpha'}$ is about two times M_{GUT} , and the 4d Planck scale, which now plays the role of a low energy effective parameter, is the largest one. This means, that, as we have assumed, the unification with gravity takes place after the fifth dimension opens up. If we neglect, as we shall do here,

the threshold effects from heavy modes on the circle and on Calabi-Yau space, which are discussed in a forthcoming section, then the three gauge couplings run logarithmically without seeing the fifth dimension up to M_{GUT} , but the dimensionless gravitational coupling changes its power-law running at $E = m_5$ from $\propto E^2$ to $\propto E^3$ (where E is the energy scale). Just beyond M_{GUT} , the C-Y dimensions open up and so the dimensionless gauge coupling \tilde{g}^2 scales like $\propto E^6$ whilst the dimensionless gravitational coupling scales like E^9 (recall that in a space time with more than four dimensions the gauge coupling is no longer dimensionless, and it scales with energy. In d dimensions the dimensionless gauge coupling is $\tilde{g}^2 = g^2 E^{d-4}$, and the dimensionless gravitational coupling is $g_{grav}^2 = 16\pi G_N^{(d)} E^{d-2}$.)

To estimate the value of the numerical unification scale, $E^{(M)}$, we note that in terms of the low energy physical scales the dimensionless couplings at energies higher than M_{GUT} are

$$\tilde{g}^2 = \frac{g_{GUT}^2 E^6}{M_{GUT}^6}, \quad g_{grav}^2 = \frac{16\pi G_N}{m_5 M_{GUT}^6} E^9 \quad (13)$$

A quick computation shows that these couplings meet at $E^{(M)} \approx 1/\sqrt{\alpha'}$. This scale is essentially the string scale and lies in the region where the field theoretical description cannot be trusted. However, the result of this simple calculation shows that in a more complete formulation of M -theory the idea of grand unification, in terms of the numerical unification of the couplings we defined above, can work.

It is interesting to note, that if one performs such a naive dimensional running in the weakly coupled string case, this time taking $d = 10$ both in gauge and in the gravity sectors, one obtains numerical unification at $1/\sqrt{\alpha'}$. However, the common dimensionless coupling one obtains this way is orders of magnitude larger than one, signalling inconsistency of this approach in the weakly coupled case.

2.2 G as a Solution of the Bianchi Identity

As pointed out in [3] one can solve the Bianchi identity (3) by defining a modified field strength [2, 3]

$$G_{11ABC} = (\partial_{11} C_{ABC} \pm 23 \text{ terms} + \frac{\kappa^2}{\sqrt{2}\lambda^2} \sum_{m=1}^2 \delta^{(m)}(x^{11})(\omega_{ABC}^{(m)} - \frac{1}{2}\omega_{ABC}^{(L)}) \quad (14)$$

where $\omega^{(m)}$, and $\omega^{(L)}$ are (E_8) Yang Mills and Lorentz Chern-Simons 3 forms defined on the respective boundaries.

Another way of solving Bianchi identity (3), which does not lead to singularities in the field strength G_{IJKL} and in the equations of motion, is to assign the discontinuous limiting behaviour to the field strength components G_{ABCD} when they pass across the boundaries:

$$\lim_{x^{11} \rightarrow 0} G_{ABCD} = -\frac{3\kappa^2}{\sqrt{2}\lambda^2} \theta(x^{11}) (\text{tr} F_{[AB}^1 F_{CD]}^1 - \frac{1}{2} \text{tr} R_{[AB} R_{CD]}), \quad (15)$$

$$\lim_{x^{11} \rightarrow \pi\rho_0} G_{ABCD} = -\frac{3\kappa^2}{\sqrt{2}\lambda^2} \theta(x^{11} - \pi\rho_0) (\text{tr} F_{[AB}^2 F_{CD]}^2 - \frac{1}{2} \text{tr} R_{[AB} R_{CD]}) \quad (16)$$

With this choice, and with definite Z_2 -parity properties of all the bosonic fields, one can work on the half-circle $(0, \pi\rho_0)$ imposing the boundary conditions (15),(16), instead of working on the full circle with singular configurations of G . As pointed out in [27] algebraic manipulations allow then to convert Bianchi identity into eleven dimensional Laplace equation which has to be solved with boundary conditions (sources) following from (15),(16). It is not difficult to find out solutions for the components of the antisymmetric tensor field and its strength in terms of the sources located at the two fixed points. These solutions in turn determine the deviations from the underlying Calabi-Yau metric required to maintain the $N = 2$ supersymmetry in the five-dimensional bulk and $N = 1$ chiral supersymmetry at the fixed hyperplanes. They can be read off from the formulae given in [2, 27]. It is possible, using a momentum expansion [27, 29], to give explicit dependence of these solutions on the eleventh coordinate x^{11} . The easiest way to go is to define the form Σ which is Hodge dual to the field strength G , $\Sigma = d\Xi = \star G$, where Ξ is the potential for Σ . Substituting it into the Bianchi identity, and supplying with the Lorentz-gauge condition $d^\dagger \Xi = 0$, where d^\dagger is the Hermitian conjugate of d , leads to the equation:

$$\Delta_{11} \Xi_{J_1 J_2 J_3 J_4 J_5 J_6} = 0 \quad (17)$$

with boundary conditions

$$\lim_{x^{11} \rightarrow 0} \partial_{11} \Xi_{J_1 J_2 J_3 J_4 J_5 J_6} = \frac{1}{4\sqrt{2}\pi} \left(\frac{\kappa}{4\pi} \right)^{2/3} (\star I^1)_{J_1 J_2 J_3 J_4 J_5 J_6} \quad (18)$$

$$\lim_{x^{11} \rightarrow \pi\rho_0} \partial_{11} \Xi_{J_1 J_2 J_3 J_4 J_5 J_6} = \frac{1}{4\sqrt{2}\pi} \left(\frac{\kappa}{4\pi} \right)^{2/3} (\star I^2)_{J_1 J_2 J_3 J_4 J_5 J_6} \quad (19)$$

where

$$(\star I^i)_{J_1 J_2 J_3 J_4 J_5 J_6} = \frac{\sqrt{g}}{6!} \epsilon_{J_1 J_2 J_3 J_4 J_5 J_6 J_7 J_8 J_9 J_{10}} 30(\text{tr}(F^{(i)[J_7 J_8} F^{(i)J_9 J_{10}]}) - \frac{1}{2} \text{tr}(R^{[J_7 J_8} R^{J_9 J_{10}]})) \quad (20)$$

and $i = 1, 2$ counts the fixed hyperplanes. In specific cases these formulae simplify, for instance when one chooses all indices on F and R to be Calabi-Yau indices, one can factorize out the antisymmetric tensor of the visible four dimensions, and through the ansatz $\Xi_{\mu_1 \mu_2 \mu_3 \mu_4 J_5 J_6} = \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} \Xi_{J_5 J_6}$ one obtains a simplified problem for the $(1, 1)$ 2-form Ξ_{ab} . We shall return to this case in a moment. The other example consists in taking all indices on F and R to be noncompact indices. Then $\star I^i = \frac{1}{5!} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} 30(\text{tr}(F^{(i)[\mu_1 \mu_2} F^{(i)\mu_3 \mu_4]}) - \frac{1}{2} \text{tr}(R^{[\mu_1 \mu_2} R^{\mu_3 \mu_4}]}) V_6$ where V_6 is the six dimensional volume form. After defining $\Xi_{J_1 J_2 J_3 J_4 J_5 J_6} = \frac{1}{5!} \Xi V_6$ one obtains equation for the function Ξ . Similarly, one can solve for the antisymmetric tensor field in cases when sources with mixed, noncompact and compact, indices are excited. For the purpose of the present Section we shall be dealing with the pure cases given above. The solution for the antisymmetric field strength with purely non-compact indices is

$$\begin{aligned} G_{\mu\nu\rho\delta} &= -\frac{3}{2\sqrt{2}\pi} \left(\frac{\kappa}{4\pi} \right)^{2/3} \left((\text{tr}(F_{[\mu\nu}^{(1)} F_{\rho\delta]}^{(1)}) - \frac{1}{2} \text{tr}(R_{[\mu\nu} R_{\rho\delta]})) \left(1 - \frac{x^{11}}{\pi\rho_0}\right) \right. \\ &\quad \left. - \frac{x^{11}}{\pi\rho_0} (\text{tr}(F_{[\mu\nu}^{(2)} F_{\rho\delta]}^{(2)}) - \frac{1}{2} \text{tr}(R_{[\mu\nu} R_{\rho\delta]})) \right) \end{aligned} \quad (21)$$

and the solution for the antisymmetric field strength with purely compact indices is obtained by the replacement of the indices μ, ν, ρ, δ by the suitable Calabi-Yau indices I_1, I_2, I_3, I_4 . In fact, the solutions with mixed indices assume in the approximation we use here exactly the same form with corresponding indices on the sources. The result (21) implies through the relation $G = dC$ the form of the background part of C_{JKL} :

$$C_{JKL} = -\frac{1}{4\sqrt{2}\pi} \left(\frac{\kappa}{4\pi}\right)^{2/3} \left((\omega_3^{(1)YM} - \frac{1}{2}\omega_3^L) \left(1 - \frac{x^{11}}{\pi\rho_0}\right) - \frac{x^{11}}{\pi\rho_0} (\omega_3^{(2)YM} - \frac{1}{2}\omega_3^L) \right) \quad (22)$$

However, this also implies through the relation $G_{11JKL} = (\partial_{11}C_{JKL} \pm 23 \text{ perm})$ the specific form of G_{11JKL} :

$$G_{11JKL} = \frac{1}{4\sqrt{2}\pi^2\rho_0} \left(\frac{\kappa}{4\pi}\right)^{2/3} (\omega_3^{(1)YM} + \omega_3^{(2)YM} - \omega_3^L) \quad (23)$$

We want to point out that this is an interesting expression, as it looks gauge non-invariant since the Yang-Mills Chern-Simons form is noninvariant under the gauge transformation $\delta A_I = -D_I \epsilon$ with $\delta\omega_3^{(i)YM} = \partial_I(\text{tr}\epsilon F_{JK}^{(i)})$. To assure the gauge invariance of G_{11JKL} one needs to require that the ‘free’ part of the three-form C_{IJK} is gauge non-invariant and transforms as

$$\delta C_{11IJ} = -\frac{1}{18\sqrt{2}\pi^2\rho_0} \left(\frac{\kappa}{4\pi}\right)^{2/3} \text{tr}(\epsilon F_{IJ}) \quad (24)$$

It is an easy exercise to substitute this variation into the eleven-dimensional integral of the $C \wedge G \wedge G$ together with solutions for the G_{MNPQ} and to perform the integration over x^{11} in the resulting expression. This way one obtains the term which exactly cancels the usual gauge anomaly coming from ten-dimensional majorana-weyl gauginos. We discuss this point to show the slight difference in the ways the anomaly cancellation takes place depending on the way the Bianchi identity is solved. In the classic case discussed in [3] the required gauge variation is supported only on respective boundary, whereas here C is uniformly transformed in the whole space, and variation becomes ten-dimensional after integrating over the explicit dependence of the integrand on x^{11} obtained through solution of equations of motion.

Finally, to illustrate more explicitly the way non-zero modes of the six dimensional Laplacian enter solutions of the Bianchi identity, let us consider the first example mentioned below the formula (20). Let us assume that the sources on the fixed planes have the decomposition in the eigenfunctions of that Laplacian of the form

$$\begin{aligned} \sum_i h^{(1)i} \pi_{IJ}^i &= -\frac{3}{2\sqrt{2}\pi} \left(\frac{\kappa}{4\pi}\right)^{2/3} \epsilon_{IJJ_1I_2I_3I_4} \left(\text{tr}(F^{(1)[I_1I_2} F^{(1)I_3I_4]}) - \frac{1}{2} \text{tr}(R^{[I_1I_2} R^{I_3I_4]}) \right) \\ \sum_i h^{(2)i} \pi_{IJ}^i &= -\frac{3}{2\sqrt{2}\pi} \left(\frac{\kappa}{4\pi}\right)^{2/3} \epsilon_{IJJ_1I_2I_3I_4} \left(\text{tr}(F^{(2)[I_1I_2} F^{(2)I_3I_4]}) - \frac{1}{2} \text{tr}(R^{[I_1I_2} R^{I_3I_4]}) \right) \end{aligned} \quad (25)$$

In what follows, when we shall distinguish zero modes among $(1, 1)$ forms π^i then we shall call these zero modes ω^i . In terms of the mode expansion the solution of the relevant version of the

equation (15),(16) with boundary conditions (18), (19) is:

$$\begin{aligned}
\Xi_{a\bar{b}} &= \sum_{\text{heavy modes}} \pi_{a\bar{b}}^i \left(-\frac{1}{48\pi\rho_0} \frac{1}{\lambda_i^2} (h_{(2)i} + h_{(1)i}) + \frac{1}{48} \frac{1}{\lambda_i} (h_{(2)i} + h_{(1)i}) \frac{\cosh(\lambda_i x^{11})}{\sinh(\pi\rho_0\lambda_i)} \right. \\
&\quad \left. - \frac{1}{48} \frac{h_{(1)i}}{\lambda_i} \frac{\sinh(x^{11}\lambda_i - \frac{\pi\rho_0\lambda_i}{2})}{\cosh(\frac{\pi\rho_0\lambda_i}{2})} \right) \\
&\quad - \sum_{\text{zero modes}} \omega_{a\bar{b}}^i \left(-\frac{1}{48\pi\rho_0} (h_{(2)i}^{(0)} + h_{(1)i}^{(0)}) \left(\frac{(x^{11})^2}{2} - \frac{\pi^2\rho_0^2}{3!} \right) + \frac{1}{48} h_{(1)i}^{(0)} \left(x^{11} - \frac{\pi\rho_0}{2} \right) \right) + \chi
\end{aligned} \tag{26}$$

where χ is the solution of the equation $\Delta_X \chi_{a\bar{b}} = -1/(48\pi\rho_0) \pi_{a\bar{b}}^i (h_{(2)i} + h_{(1)i})$ with trivial boundary conditions. The eigenvalues λ_i are defined through the relation $\Delta_X \pi_{a\bar{b}}^i = -\lambda_i^2 \pi_{a\bar{b}}^i$. It is easy to see that with the above definitions we have $G_{IJKL} = -\epsilon_{IJKLMN} \partial_{11} \Xi_{MN}$. We note, that at the order linear in x^{11} , which is the order to which we solve in this paper the Horava-Witten model, the antisymmetric tensor field strength obtained from (26) gives (21). The (1, 1) form $\Xi_{a\bar{b}}$ which we have found here in the case of a general embedding is also interesting, as it gives directly corrections to the metric of the Calabi-Yau space, see [27]. One of the important findings of the next Section is that for a general embedding the coefficients of the zero modes in the decompositions of sources fulfil the equality

$$h_{(2)i}^{(0)} + h_{(1)i}^{(0)} = 0 \tag{27}$$

which is due to the requirement that the integrability condition for the Bianchi identity be fulfilled. The obvious consequence is that the solution given above simplifies in its zero-mode part, and that the zero-mode component of G_{MNPQ} (with all indices along C-Y space) is independent of x^{11} in the general case.

3 Axionic Thresholds

In this section we shall see that the solution of the Bianchi identity in the previous section, for general embeddings, implies that there will be axionic threshold corrections in the $d = 4$ theory. The condition that the Bianchi identity has a solution is that the source terms on its right-hand-side add up to zero in cohomology, i.e. that the cohomology class $[\sum_{i=1,2} \text{tr} F^{(i)} \wedge F^{(i)} - \text{tr} R \wedge R]$ is trivial. The discussion given above does not depend thus far in any way on the particular embedding one uses to solve this integrability condition. The standard embedding consists in identifying the spin connection of the compact six dimensional space with the $SU(3)$ subgroup of one of the two E_8 gauge groups present in the theory, $F_{IJ}^{(1)} = R_{IJ}$. This leads to the model which has one of the E_8 factors broken down to E_6 , with chiral matter in four dimensions. This is the solution which has been discussed so far in the context of the Horava-Witten model. However, this is by no means the unique possibility and, in fact, other embeddings, called non-standard, give models which have the group theoretical structure even more appealing than the simple E_6 . Hence it is interesting to see whether the basic features of the naive four-dimensional limit of the Horava-Witten model change in any significant way when one departs from the

standard embedding. As pointed out in [21, 20] the fundamental low energy theory coming from the compactification of that model on the deformed Calabi-Yau space lives in five dimensions. However, at least for the gauge sector which is basically four-dimensional after compactification, it makes sense to define the effective nonrenormalizable four-dimensional action which is globally supersymmetric at the lowest nontrivial order. We are primarily interested in the effective gauge kinetic terms. These terms, assuming four-dimensional supersymmetry, are

$$L_{gauge} = -\frac{1}{4} \int d^4x \operatorname{tr}(F_{\mu\nu}F^{\mu\nu}) \Re(f) + \frac{1}{8} \int d^4x \epsilon^{\mu\nu\rho\delta} \operatorname{tr}(F_{\mu\nu}F_{\rho\delta}) \Im(f) \quad (28)$$

We are interested in the form of the function f in the visible and hidden sectors. In terms of the moduli fields it should follow from the Kaluza-Klein reduction of the eleven-dimensional theory, or can be read off the effective Lagrangian of the weakly coupled heterotic string, as argued in [22], [25]. It is the first path that we shall follow, although at the end we turn back to the correspondence with the weakly coupled result. The easiest way to identify the particular combination of moduli which forms gauge kinetic functions of gauge groups on each of the fixed hyperplanes is to find out all the contributions to axionic couplings, i.e. the coefficients multiplying the four dimensional operators $\epsilon^{\mu\nu\rho\delta} \operatorname{tr}(F_{\mu\nu}^{(i)}F_{\rho\delta}^{(i)})$. One obvious contribution is the one coming from the bulk kinetic terms for the antisymmetric tensor field enhanced by the Chern-Simons forms implied by the sources in the Bianchi identity. However, it is easy to convince oneself that the massless axion which is the trivial zero-mode of $C_{11\mu\nu}$ couples in exactly the same way to both $\epsilon^{\mu\nu\rho\delta} \operatorname{tr}(F_{\mu\nu}^{(i)}F_{\rho\delta}^{(i)})$, $i = 1, 2$. Since the difference between the axionic couplings coming from the kinetic terms vanishes, we have to look for an additional source of axionic couplings which would give a nonzero difference which shall partner the difference of the Calabi-Yau volumes at both planes (the difference of the volumes has been computed in the embedding-independent way in [2]). Such terms are easily found through the Kaluza-Klein compactification of the eleven dimensional $C \wedge G \wedge G$ term. The relevant integral is that of the last term in formula (1). The part which gives axionic thresholds must be proportional to $G_{\mu\nu\rho\delta}G_{IJKL}C_{11MN}$. The components C_{11MN} have an expansion in terms of the eigenmodes of the six-dimensional Laplacian Δ_X . The harmonic forms which are zero-modes shall be denoted by ω_{IJ}^α and the nonzero modes by π_{IJ}^α with $\Delta_X \pi_{IJ}^\alpha = -(\lambda^\alpha)^2 \pi_{IJ}^\alpha$. Then we expand

$$C_{11MN} = \frac{2\sqrt{2}}{3} (\theta^\alpha(x) \omega_{MN}^\alpha + \bar{\theta}^\beta(x) \pi_{MN}^\beta) \quad (29)$$

where x denotes four noncompact coordinates plus x^{11} . One should note that we should also expand the θ s with respect to x^{11} . The lowest order terms are (all functions are periodic with the period $2\pi\rho_0$ and θ s are Z_2 -even):

$$\theta^\alpha(x^\mu; x^{11}) = \theta^\alpha(x^\mu) + \theta_1^\alpha(x^\mu) \frac{1}{\sqrt{\pi\rho_0}} \cos\left(\frac{x^{11}}{\rho_0}\right) + \dots \quad (30)$$

(and similarly for barred thetas). Let us consider the massless axions $\theta^\alpha(x^\mu)$. We substitute into the last term of eq.(1) the explicit solutions (21), with the linear dependence on the x^{11} . In principle we should perform the integrals over Calabi-Yau first, leaving the five-dimensional

coupling between the gauge fields and massless axions with the explicit x^{11} dependence. However, we are interested in four-dimensional expressions, hence we can integrate over the circle. After simple algebraic manipulations we obtain:

$$\begin{aligned}
\delta L^{(4)} &= \frac{1}{\kappa^2} \frac{\rho_0}{24\pi} \left(\frac{\kappa}{4\pi} \right)^{4/3} \text{tr}(F^{(1)} \tilde{F}^{(1)}) \left[\theta^\alpha \int_X \omega^\alpha \wedge (tr(F^{(1)} \wedge F^{(1)}) - \frac{1}{2} tr(F^{(2)} \wedge F^{(2)})) \right. \\
&- \left. \frac{1}{4} tr(R \wedge R) + \bar{\theta}^\beta \int_X \pi^\beta \wedge (tr(F^{(1)} \wedge F^{(1)}) - \frac{1}{2} tr(F^{(2)} \wedge F^{(2)}) - \frac{1}{4} tr(R \wedge R)) \right] \\
&+ \frac{1}{\kappa^2} \frac{\rho_0}{24\pi} \left(\frac{\kappa}{4\pi} \right)^{4/3} \text{tr}(F^{(2)} \tilde{F}^{(2)}) \left[\theta^\alpha \int_X \omega^\alpha \wedge (tr(F^{(2)} \wedge F^{(2)}) - \frac{1}{2} tr(F^{(1)} \wedge F^{(1)})) \right. \\
&- \left. \frac{1}{4} tr(R \wedge R) + \bar{\theta}^\beta \int_X \pi^\beta \wedge (tr(F^{(2)} \wedge F^{(2)}) - \frac{1}{2} tr(F^{(1)} \wedge F^{(1)}) - \frac{1}{4} tr(R \wedge R)) \right]
\end{aligned} \tag{31}$$

This formula holds for any embedding, standard or not. The question arises about the correlations between specific integrals over Calabi-Yau space entering the formula above. First, one has to note that in all these integrals one can take the same Calabi-Yau space, as taking the variation of C-Y would lead to corrections which are higher order in κ , beyond the order to which the Horava-Witten model is defined. Then one has to consider separately integrals with ω^α and with π^β . Let us consider the integrals containing harmonic forms. If one adds integrals with the same form ω^α which are coefficients of $F^{(1)} \tilde{F}^{(1)}$ and $F^{(2)} \tilde{F}^{(2)}$ respectively, let's call them γ_1^α and γ_2^α , one gets

$$\gamma_1^\alpha + \gamma_2^\alpha = \frac{1}{\kappa^2} \frac{\rho_0}{24\pi} \left(\frac{\kappa}{4\pi} \right)^{4/3} \frac{1}{2} \int_X \omega^\alpha \wedge (tr(F^{(1)} \wedge F^{(1)}) + tr(F^{(2)} \wedge F^{(2)}) - tr(R \wedge R)) \tag{32}$$

and we remember that $[F^{(1)} \wedge F^{(1)} + F^{(2)} \wedge F^{(2)} - R \wedge R] = 0$. If one restricts oneself to compact indices on the forms, then the reasoning from the weak string case can be adopted and one can write $dH = (tr(F^{(1)} \wedge F^{(1)}) + tr(F^{(2)} \wedge F^{(2)}) - tr(R \wedge R))$. The integral of dH over any closed four-dimensional surface must vanish, as H must be globally defined (it enters the energy density under the name of G_{11IJK}). If so, then for any embedding, the above integral is

$$\gamma_1^\alpha + \gamma_2^\alpha = -\frac{1}{\kappa^2} \frac{\rho_0}{24\pi} \left(\frac{\kappa}{4\pi} \right)^{4/3} \frac{1}{2} \int_X d\omega^\alpha \wedge H = 0 \tag{33}$$

because ω^α is a harmonic form. In exactly analogous way one proves the relation (27). Next, we also consider the contribution from the non-zero modes π^β . The sum of respective coefficients $\bar{\gamma}_1^\beta$ and $\bar{\gamma}_2^\beta$ is

$$\begin{aligned}
\bar{\gamma}_1^\beta + \bar{\gamma}_2^\beta &= \frac{1}{\kappa^2} \frac{\rho_0}{24\pi} \left(\frac{\kappa}{4\pi} \right)^{4/3} \frac{1}{2} \int_X \pi^\beta \wedge (tr(F^{(1)} \wedge F^{(1)}) + tr(F^{(2)} \wedge F^{(2)}) - tr(R \wedge R)) \\
&= -\frac{1}{\kappa^2} \frac{\rho_0}{24\pi} \left(\frac{\kappa}{4\pi} \right)^{4/3} \frac{1}{2} \int_X d\pi^\alpha \wedge H \neq 0
\end{aligned} \tag{34}$$

since in general $d\pi \neq 0$.

It should be stressed, that the threshold corrections we have given here are corrections to the *unified gauge couplings on the respective walls* for any embedding. After breaking $E_8 \times E_8$

down to a group containing a subgroup G , one should in principle change the normalization of generators T of G , using the relation $Tr_{Adj(E_8 \times E_8)} T^2 = f Tr_{Adj(G)} T^2$ where f turns out to be a small number (< 10). Once the detailed decomposition of both E_8 s is known, one can read-off corrections to the individual group factors. In the next Section we shall discuss a particularly interesting class of embeddings which mix the subgroups of the two E_8 s.

There are two important conclusion to be drawn from this reasoning. Firstly, if one restricts oneself to true four-dimensional pseudoscalar zero modes, which we call here θ^α , then the above calculation gives us canonical individual axionic threshold corrections to each separate gauge group. This is not so for the corrections to the coefficients of the F^2 terms, which come from the difference of the volumes of the C-Y spaces on both walls. There one computes directly the difference of the volumes, and in a general case we do not get any canonical reference point, which would tell us how the difference should be split between the planes.³ Secondly, the canonically chosen axionic threshold corrections (and, assuming underlying $N = 1$ 4d supersymmetry, full universal threshold corrections) are equal in magnitude, and of opposite sign if one restricts oneself to the zero-mode harmonic forms in the C-Y decomposition of the C_{11AB} . However, if one goes a step further, and considers the non-zero modes on Calabi-Yau, then the exact correlation of the threshold coefficients between walls is violated. This is actually not surprising, as the integration with the zero modes over C-Y space extracts only the averaged part of the integrands, whereas the higher modes are sensitive to the finer structure of these. Of course, the contributions of the higher modes are weighted by the expectation values of the four-dimensional fields $\bar{\theta}^\beta$. These fields have masses of the order of $V_{CY}^{-1/6}$, hence at low energies these expectation values are expected to be zero. However, when one reaches up in energies, considering for instance the details of the unification of gauge and gravitational couplings in the model, then these modes should be switched on around the scale where the largest of the six C-Y dimensions peels off. There, the expectation values should be put at $\langle \bar{\theta}^\beta \rangle = 1/R_{max}$ rather than at zero⁴.

Of course, once we consider the higher modes of the Laplacian on the C-Y space, we should consider also higher Fourier modes on the circle along the eleventh dimension. This is because one expects that the radius of the eleventh dimension is smaller than the largest radius of C-Y, $\pi\rho < R_{max}$. It is easy to compute the coefficients associated with the higher modes on the circle. The coefficients corresponding to the first nonhomogeneous modes θ_1^α are:

$$\begin{aligned} \delta_1 L^{(4)} &= \frac{1}{\kappa^2} \frac{\sqrt{\rho_0}}{18\pi^{7/2}} \left(\frac{\kappa}{4\pi}\right)^{4/3} tr(F^{(1)} \tilde{F}^{(1)}) \left[\theta_1^\alpha \int_X \omega^\alpha \wedge (tr(F^{(1)} \wedge F^{(1)}) - \frac{1}{2} tr(R \wedge R)) \right] \\ &- \frac{1}{\kappa^2} \frac{\sqrt{\rho_0}}{18\pi^{7/2}} \left(\frac{\kappa}{4\pi}\right)^{4/3} tr(F^{(2)} \tilde{F}^{(2)}) \left[\theta_1^\alpha \int_X \omega^\alpha \wedge (tr(F^{(2)} \wedge F^{(2)}) - \frac{1}{2} tr(R \wedge R)) \right] \end{aligned} \quad (35)$$

Hence, as one can easily check using the argument applied earlier in this section, at this level the respective threshold coefficients on the walls are equal and of the same sign

$$\gamma_{(1)1}^\alpha - \gamma_{(1)2}^\alpha = 0 \quad (36)$$

³In the case of the standard embedding one can argue that such a point emerges from the calculation of the correction to the background metric, and lies in the middle of the interval between walls, where the correction vanishes.

⁴Note that whenever necessary dimensions are supplied by suitable powers of $1/\sqrt{\alpha'}$.

Again, higher modes on the circle are actually expected to be excited when one reaches the energy scale at which the eleventh dimension becomes visible, $m_5 = 1/(\pi\rho)$. There the expected value of θ_1^α might be better approximated by its *rms* value $m_5/\sqrt{2}$ than by zero.

Identifying the axions θ_i as the imaginary parts of the $h_{(1,1)}$ moduli T_i (which generalize the situation with the single general modulus T), on the basis of the equations (31) and (33) and using the assumption of four dimensional supersymmetry and holomorphicity of 4d supersymmetric gauge couplings (28), we obtain the following result for the gauge couplings at the unification scale :

$$1/g_{(k)}^2 = \text{Re}(S + \epsilon_{i(k)}T_i) \quad (37)$$

($i = 1, \dots, h(1,1)$, $k = 1, 2$) where the chiral field S remains to be defined and the individual coefficients $\epsilon_{i(k)}$ are related through

$$\epsilon_{i(1)} = -\epsilon_{i(2)} = \gamma_{i(1)} \quad (38)$$

This result is valid for a general embedding, standard or not. This is due to the fact, that the integration over Calabi-Yau with all external four dimensional fields massless picks out only the massless contribution from the solution for G , which leads to the above property of $\epsilon_{i(1)}$ and $\epsilon_{i(2)}$ independently of the nature of the embedding. Using eq.(27), eq.(32) and eq.(33) the explicit form of the coefficients $\epsilon_{i(k)}$ reads

$$\epsilon_{i(1)} = -\frac{\pi\rho_0}{2(4\pi)^{4/3}\kappa^{2/3}} \frac{1}{8\pi^2} \int_X \omega_i \wedge (\text{tr}F^{(1)} \wedge F^{(1)} - \frac{1}{2} \text{tr}R \wedge R) \quad (39)$$

where ω_i are massless $(1,1)$ -forms, and the topological integral over C-Y space shall be parametrized in terms of instanton numbers in Section 5.

The difference of the couplings given above is of the relative order $o(\kappa^{4/3})$ with respect to the gravitational part of the Lagrangian, and represents the difference of volumes of the two C-Y spaces on the hidden and visible walls. When we incorporate these corrections into the effective four dimensional model, it constitutes a departure from the order $(\kappa^{2/3})$ situation discussed earlier in Section 2.1 . Hence, before we proceed with the discussion we need to make some comments on the interpretation of the ‘strong’ fields S and T in the present setup. As pointed out in [22] in the correct five dimensional theory there exists a scalar field belonging to the universal hypermultiplet, let us call it here \mathcal{S} , whose expectation value measures the C-Y volume along the fifth dimension. When one tries to solve the equation of motion for this field along the fifth dimension, the step which is necessary to obtain four dimensional model, then one encounters a zero mode of this field, which is homogeneous in the direction of x^5 . This is a good candidate for the effective 4d S . In this paper we work in a simplified setup, where to get the 4d model we solve only the 11d equations of motion, where one obtains a simple linear dependence of the volume on x^5 . The zero mode of such a linearly changing quantity can be defined as the arithmetic mean of the limiting values on the boundaries. When we define the effective S in such a way, as $S_r = (V(x^{11} = 0) + V(x^{11} = \pi\rho_0))/2V_s^{(0)}$ and choose the reference volume according to the normalization assumed in the Lagrangian containing only terms of relative order $o(\kappa^{2/3})$, then, indeed, in terms of this S the holomorphic gauge couplings look like $S \pm \Delta(\kappa^{4/3})$. Hence, what we described above is the five dimensional interpretation of the

‘strong’ field S we are going to use in the rest of the paper. As for the overall modulus T , the degree of freedom it contains does exist in five dimensions as the component e_{55} of the vielbein and belongs to the 5d gravitational multiplet (together with graviton and graviphoton, whose fifth component is the axial part of T). There is no trouble with defining the correct effective T : it is simply the $(\int_0^{\pi\rho_0} e_{55})/(\pi\rho_0)$. Analogous treatment applies to additional moduli T_i which from five dimensional point of view are parts of $h_{(1,1)} - 1$ vector supermultiplets. The definitions of ρ_0 , α' and the relation between T and four dimensional M_{Pl} we leave as defined in Section 2.1 since we do not want to go beyond the validity limit of the Horava-Witten construction. This summarizes the interpretation of S and T in the context of effective 4d Lagrangian containing terms of relative order $o(\kappa^{4/3})$.

4 Embeddings that Mix Gauge Sectors from Different Walls and Anomalous $U(1)$

The interesting aspect of non-standard embeddings in M-theory which deserves a separate discussion is the possibility of obtaining matter with charges that mix visible and hidden sector gauge groups. It should be stressed that this is the necessary condition for the presence of an anomalous $U(1)_A$ group, with cancellation of the $U(1)_A$ anomaly through the gauge transformation of the universal axion, as it happens in stringy models with ‘anomalous’ $U(1)_A$ factor. The standard embedding does not give mixed charges, hence does not give ‘anomalous’ $U(1)_A$, neither in smooth C-Y compactifications, or in orbifolds compactifications. Before we discuss how it can exist in nonstandard embeddings, let us specify first what we actually mean by nonstandard embeddings [8, 9].

Roughly speaking, let us take a Calabi-Yau manifold, and a vector bundle which is a direct sum of stable holomorphic vector bundles, $V = \bigoplus V_i$. Let us make the gauge fields of the gauge connections on that bundle satisfy the Einstein-Kähler equations⁵

$$F_{aa} = F_{\bar{a}\bar{a}} = 0, \quad g^{a\bar{a}} F_{a\bar{a}} = 0 \quad (40)$$

Suppose that, in addition, the second chern classes of the gauge bundles V_i sum up to the second chern class of the tangent bundle T ,

$$\sum c_2(V_i) = c_2(T) \quad (41)$$

(which is equivalent to $[\sum_i F^{(i)} \wedge F^{(i)} - R \wedge R] = [0]$ encountered earlier), and integrability conditions for the equations of motion [8] are satisfied. After compactification to four dimensions, these conditions lead to unbroken $N = 1$ supersymmetry. This abstract vacuum bundle is embedded into the actual $E_8 \times E_8$ gauge bundle, and $E_8 \times E_8$ is broken down to the subgroup G formed by the generators which commute with the structure group J of the vacuum bundle.

If one takes $V_1 = T$, $V_2 = 0$, where indices 1, 2 correspond to the two E_8 s, and one makes the natural identification of the $SU(3)$ holonomy group gauge fields with those of the $SU(3)$

⁵Supplied with integrability conditions $\int_X \omega \wedge \omega \wedge F = 0$.

subgroup of one of the E_8 s, then all the equations are fulfilled, and the standard embedding is realized. However, it is well known by now that many other choices both for the vacuum bundle V and for the actual embedding of the vacuum bundle into gauge bundle are possible, in manifold compactifications and in orbifold compactifications for instance. One extension of the above construction to the Horava-Witten type supergravity is when each item from the direct sum of vacuum bundles is embedded fully into one or the other E_8 bundle. However, the possibility which is in a sense most interesting is that a sub-bundle is partially embedded into both E_8 s. To be more specific let us take a line bundle with the structure group $U(1)$, and the corresponding vacuum gauge field $(1, 1)$ form $\mathcal{F}_{a\bar{a}}$. Then let us take a subgroup $U(1)_1$ from one E_8 and a $U(1)_2$ from the other. Finally, let us define the embedding with the condition

$$\left(\cos(\theta)F_{(1)a\bar{a}} + \sin(\theta)F_{(2)a\bar{a}} \right) = \mathcal{F}_{a\bar{a}} \quad (42)$$

where $F_{1a\bar{a}}^{(1)}$ and $F_{2a\bar{a}}^{(2)}$ are the corresponding $U(1)_1 \times U(1)_2$ field strengths. The mixing angle is given by $\cos(\theta) = \text{Tr}_{\text{Adj}(E_8 \times E_8)}(T^1 \mathcal{T})$ and $\sin(\theta) = \text{Tr}_{\text{Adj}(E_8 \times E_8)}(T^2 \mathcal{T})$, with T^1 , T^2 , \mathcal{T} being generators of the first and second $U(1)$ and of the structure group of the linear bundle.

Before we continue, it is worth pointing out that it is normal for the curvature of line bundles to lie in integral cohomology classes of the C-Y manifold X . If this is the case with the bundles associated with $U(1)_1$ and $U(1)_2$ then the question arises under what conditions can the curvature \mathcal{F} in (42) lie in integral cohomology classes? Using the constraint that the orthogonal combination $F = -\sin(\theta)F_{1a\bar{a}}^{(1)} + \cos(\theta)F_{2a\bar{a}}^{(2)} = 0$, we see that we must take $\sin(\theta) = p/n$, $\cos(\theta) = q/n$ where $n^2 = p^2 + q^2$, p, q, n being integers with q, p labeling the cohomology classes of the $U(1)_1$ and $U(1)_2$ bundles. If we substitute this back into the embedding (42) then it is clear that this equation now defines a new $U(1)$ bundle in a cohomology class labelled by the integer n , in terms of those bundles whose classes are labelled by q and p .

The kinetic terms for $F_1^{(1)}$, $F_2^{(2)}$ give

$$\begin{aligned} \frac{1}{4g_1^2} F_1^2 &\rightarrow \frac{1}{4g_1^2} \left(\cos^2(\theta) \mathcal{F}^2 - 2\sin(\theta)\cos(\theta) \mathcal{F}F + \sin^2(\theta) F^2 \right) \\ \frac{1}{4g_2^2} F_2^2 &\rightarrow \frac{1}{4g_2^2} \left(\sin^2(\theta) \mathcal{F}^2 + 2\sin(\theta)\cos(\theta) \mathcal{F}F + \cos^2(\theta) F^2 \right) \end{aligned} \quad (43)$$

where in the above we have extended the definitions of F and \mathcal{F} to include noncompact indices. The field \mathcal{F} is ‘Higgsed’ by one of the $h_{(1,1)}$ nonuniversal axions coming from the C-Y decomposition of the antisymmetric tensor field components $C_{11a\bar{a}}$ and decouples from the massless spectrum [30]. This happens in the following way. Since the field strengths $F_{1a\bar{a}}^{(1)}$, $F_{2a\bar{a}}^{(2)}$ define closed $(1, 1)$ forms on X ⁶, the decomposition of $C_{11a\bar{a}}$ will include massless $d = 4$ axionic fields a_1, a_2 , with $C_{11a\bar{a}} = a_1 F_{1a\bar{a}}^{(1)} + a_2 F_{2a\bar{a}}^{(2)}$. However from what we have said above, only the combination $a_1 \cos(\theta) + a_2 \sin(\theta)$ in fact appears in this expansion. If we look at the terms in the effective $d = 4$ action that contain $C_{11a\bar{a}}$, amongst these we find (using the form of the background part of G_{11JKL} as given in (23)) $\int_X d^6 y \sqrt{g^{(6)}} (\partial_\mu C_{11a\bar{a}} + \frac{1}{2\sqrt{2}\pi^2 \rho_0} (\frac{\kappa}{4\pi})^{2/3} (A_{1\mu}^{(1)} F_{1a\bar{a}}^{(1)} + A_{2\mu}^{(2)} F_{2a\bar{a}}^{(2)}))^2$. This

⁶We are considering manifolds with $h_{(1,1)} > 1$.

makes it clear, from the choice of our mixed embedding, that $a_1 \cos(\theta) + a_2 \sin(\theta)$ is eaten by gauge fields $\cos(\theta)A_{1\mu}^{(1)} + \sin(\theta)A_{2\mu}^{(2)}$ with field strength $\mathcal{F}_{\mu\nu}$.

The orthogonal combination of gauge fields with field strength F describes a massless $U(1)$ vector boson, that couples to charged particles on both walls. It can in principle correspond to an anomalous $U(1)_A$, as the trace of the associated generator over the massless spectrum does not have to be zero⁷. The puzzle comes from the fact that $U(1)$ gauge transformations on *different* and spatially separated hyperplanes are correlated. In fact, we have imposed the mixed embedding through the nonlocal constraint equations (42), which relates fields living on different fixed points of the underlying orbifold in the direction of x^{11} . It is the Bianchi identity which resolves the puzzle. First, let us remind the reader that the Bianchi identity holds not only at the level of vacuum configurations but, also, at the level of full fields, including fluctuations around vacuum. As we saw in Section 2. solving the Bianchi identity we obtain the configuration of the antisymmetric tensor field in the bulk which is determined in non-perturbative way by the gauge fields on both planes. The same mechanism works for our mixed $U(1)$. It is the field G which provides communication through the bulk between the two ‘components’ of the gauge field of this $U(1)$. From equation (43) one obtains the effective four dimensional kinetic term for the mixed $U(1)$ of the form

$$L_F = \frac{1}{4} \left(\frac{\sin^2(\theta)}{g_1^2} + \frac{\cos^2(\theta)}{g_2^2} \right) F^2 \quad (44)$$

Hence the effective four dimensional coupling is a harmonic average of the two couplings for groups coming exclusively from a single hyperplane:

$$g_{mixed}^2 = \frac{g_1^2 g_2^2}{\sin^2(\theta)g_1^2 + \cos^2(\theta)g_2^2} \quad (45)$$

Taking $1/g_1^2 = S_r + \sum_{i'} \epsilon_{i'} T_{r i'}$ and $1/g_2^2 = S_r - \sum_{i'} \epsilon_{i'} T_{r i'}$ one obtains

$$\frac{1}{g_{mixed}^2} = S_r + \sum_{i'} \epsilon_{i'} T_{r i'} (\sin^2(\theta) - \cos^2(\theta)) \quad (46)$$

where $i' = 1 \dots h_{(1,1)} - 1$. We have excluded the axion whose wave function on the C-Y space is proportional to $\mathcal{F}_{a\bar{a}}$ as it is eaten in the ‘Higgsing’ of $\mathcal{F}_{\mu\nu}$. One can check that the form of the moduli dependence in (46) implies, via supersymmetry, axionic couplings to $F\tilde{F}$ in $d = 4$ that can be explicitly verified by reduction of the CGG term in the eleven dimensional action, using the explicit solutions to the Bianchi identities for G . The effective Lagrangian for the anomalous $U(1)$ realizing the M-theoretical version of the four dimensional Green-Schwarz mechanism shall have the form

$$L_K = -\log(S + \bar{S} + c V_{U(1)_A})|_D \quad (47)$$

with $c \propto 1/S_r = g_{mixed}^2|_{\text{lowest order in } \kappa}$, (see discussion at the end of this Section.)

⁷Anomalous $U(1)_A$ in the Horava-Witten model has been discussed recently in [31] in different contexts.

The evaluation of the anomaly coefficient c can be achieved in the following way. The form of the Kähler potential L_K in (47) arises as a consequence of the supersymmetrization of the four dimensional terms $c \int d^4x B \wedge F$, where B is the two form potential, whose dualized 3-form field strength defines the pseudo-scalar field in S_r . Such a term is obtained from the reduction of the CGG Chern-Simons term in the M-theory Lagrangian, when we substitute the explicit solutions of the Bianchi identities, eq. (21), generalized to the case where we have mixed compact and non compact indices on G , and we identify $C_{11\mu\nu} = B_{\mu\nu}$. If the cancellation of $d = 4$ anomalies in compactified Horava-Witten theory mirrors what happens in the weak case, (as for example the cancellation of $d = 10$ anomalies does), then c should be proportional to the pure $U(1)^3$ anomaly and mixed $U(1)$ -gauge and $U(1)$ -gravitational anomalies of the compactified theory. In fact the coefficients of all three anomalies must be proportional to each other (if they are non zero), if one is to achieve complete cancellation by an appropriate $U(1)$ gauge transformation of the field S , as implied by the invariance of the Kähler potential L_K . Such 'universality' conditions have been described in the context of orbifold compactifications of the weakly coupled heterotic string in [32], and they provide quite a strong constraint on the kind of embeddings that can give rise to anomalous $U(1)$'s. Assuming an unbroken gauge group of the form $U(1)_A \times \Pi_a G_a$ with G_a semi-simple, the form of this universality relation is

$$\frac{1}{k_a} Tr_{G_a}(T(R)Q_A) = \frac{1}{3} Tr Q_A^3 = \frac{1}{24} Tr Q_A \quad (48)$$

where $2T(R)$ is the index of the representation R and k_a its level. Also, the $U(1)_A$ generator Q_A has been rescaled such that the level $k_A = 1$.

In studying non-standard embeddings and the possibility of anomalous $U(1)$ symmetries of the $d = 4$ theory, there are at least two basic situations to consider. The first (case I) is when the abelian parts of the background field strengths $F_{aa}^{(i)}$ on X are in a direction orthogonal to the $U(1)$'s under consideration, and the second (case II) is when the background values are taken to lie in the same directions as the $U(1)$'s. On top of this, in the second of these cases we have the possibility of mixing between $U(1)$'s (assuming there is more than one $U(1)$ factor), depending on the type of embedding chosen. The embedding defined in (42) is an example of the case II type discussed above, with mixing.

To have a better idea about anomaly cancellation in M -theory in four dimensions, before discussing potential $U(1)$ anomalies in case II, we shall first study case I, which is conceptually easier. Although we do not expect to find anomalous $U(1)$'s in this case since the embedding does not mix between hidden and observable sectors and we would not know how to cancel such an anomaly, it is instructive to show this formally, and moreover it turns out that the resulting formulae can easily be extended to case II type embeddings. Furthermore, in both cases I and II, for consistency, we should find that $d = 4$ triangle diagrams vanish precisely when the corresponding Green-Schwarz terms do, and if they are non-vanishing (which is the most interesting case), so must the Green -Schwarz terms. This is what we shall verify in the remainder of this section.

With this in mind, we start, as promised, by considering case I type embeddings where the E_8 and E'_8 gauge symmetries arising on each wall break via the subgroups $G \times U(1) \times J$ and

$G' \times U(1)' \times J'$ respectively. The background gauge fields $F_0^{(i)}$ on X are assumed to take values in the subgroups J and J' for $i = 1, 2$, which are not necessarily semi simple. This background will give rise to an unbroken gauge group $G \times U(1) \times G' \times U(1)'$, although whether the $U(1)$ factors are further broken depends on the anomaly analysis. We write the decomposition of the adjoint representations 248 and 248' of respective E_8 's as

$$248 = \sum_{a_1} L_{a_1} \otimes y_{a_1} \otimes Q_{a_1}, \quad 248' = \sum_{a_2} L_{a_2} \otimes y_{a_2} \otimes Q_{a_2} \quad (49)$$

where a_1, a_2 runs over the number of irreducible representations in the respective decompositions, with $L_{a_i}; Q_{a_i}$ defining irreducible representations of $G, G'; J, J'$ for $i = 1, 2$, while y_{a_i} define the corresponding $U(1), U(1)'$ charges.

Defining Y, Y' to be the generators of $U(1), U(1)'$ and following the reduction of the CGG term in $d = 11$ outlined above, one finds the terms $c_1 \int d^4x B \wedge F_1$ and $c_2 \int d^4x B \wedge F_2$ in four dimensions, where F_1, F_2 are the respective two form field strengths of $U(1)$ and $U(1)'$. The coefficients c_1, c_2 are given by

$$\begin{aligned} c_1 &= \frac{3\rho_0}{2\pi\kappa^2} \left(\frac{-\sqrt{2}}{3456}\right) \left(\frac{\kappa}{4\pi}\right)^{4/3} \int_X \frac{1}{30} Tr(YF_0^{(1)}) \wedge \left(\frac{1}{30} Tr(F_0^{(1)} \wedge F_0^{(1)})\right) \\ &\quad - \frac{1}{60} Tr(F_0^{(2)} \wedge F_0^{(2)}) - \frac{1}{4} tr(R \wedge R) \\ c_2 &= \frac{3\rho_0}{2\pi\kappa^2} \left(\frac{-\sqrt{2}}{3456}\right) \left(\frac{\kappa}{4\pi}\right)^{4/3} \int_X \frac{1}{30} Tr(Y'F_0^{(2)}) \wedge \left(\frac{1}{30} Tr(F_0^{(2)} \wedge F_0^{(2)})\right) \\ &\quad - \frac{1}{60} Tr(F_0^{(1)} \wedge F_0^{(1)}) - \frac{1}{4} tr(R \wedge R) \end{aligned} \quad (50)$$

where in (50) Tr means trace in the adjoint representation 248 or 248', and we recall that it is the coordinate radius ρ_0 that appears in these expressions. Both these coefficients vanish if we specialize to the standard embedding. The second one vanishes since in this case $F_0^{(2)} = 0$, whilst the first one vanishes because the $[Y, F_0^{(1)}] = 0$ allows the decomposition $Tr(YF^{(1)}) = \sum_{a_1} dim(L_{a_1}) tr_{y_{a_1}}(Y) tr_{Q_{a_1}}(F_0^{(1)})$, which is vanishing since $tr_{Q_{a_1}}(F_0^{(1)}) = 0$ for the standard embedding. Whether in the case I type embeddings we are considering, c_1, c_2 are always necessarily vanishing is not totally clear. Certainly, if we choose embeddings in such a way as the generator Y is orthogonal to $F^{(1)}$, i.e $Tr(YF^{(1)}) = 0$, and in a similar manner Y' orthogonal to $F^{(2)}$, then indeed these coefficients are vanishing. This was the case in the particular example, and for the particular choice of nonstandard embedding considered in [9], in (weakly coupled) $E_8 \times E_8$ heterotic (2,0) compactifications.

For consistency, it should be the case that c_1, c_2 must also be proportional to the coefficients of the various $U(1)$ anomalous triangle diagrams in $d = 4$. Checking this will provide a good test of the explicit form of the topological integrals over the C-Y space X in terms of which c_1, c_2 are defined. Consider, for example, the mixed $U(1) - GG$ or $U(1)' - G'G'$ anomalies. The

corresponding anomaly coefficients I_{UGG} and $I_{U'G'G'}$ may be written as

$$I_{UGG} = \sum_{a_1} n_{L_{a_1}, y_{a_1}} \text{tr}_{L_{a_1}, y_{a_1}}(YTT) \quad , \quad I_{U'G'G'} = \sum_{a_2} n_{L_{a_2}, y_{a_2}} \text{tr}_{L_{a_2}, y_{a_2}}(Y'T'T') \quad (51)$$

where $n_{L_{a_i}, y_{a_i}}$ represents the number of chiral fermions transforming in the corresponding irreducible representations labelled by (L_{a_i}, y_{a_i}) , and T, T' denote the generators of G, G' . These can be expressed, through the use of Index theorems, in terms of various topological integrals over X (e.g. see [8] suitably generalized to the present case):

$$n_{L_{a_i}, y_{a_i}} = \frac{1}{48(2\pi)^3} \int_X \text{tr}_{Q_{a_i}} \left(F_0^{(i)} \wedge F_0^{(i)} \wedge F_0^{(i)} \right) - \frac{1}{8} \text{tr}_{Q_{a_i}}(F_0^{(i)}) \wedge \text{tr}(R \wedge R) \quad (52)$$

If W, W' represents a generator of $G \times U(1)$, $G' \times U(1)'$ and Z, Z' that of J, J' , then it follows that $\sum_{a_1} \text{tr}_{L_{a_1}, y_{a_1}}(W) \text{tr}_{Q_{a_1}}(Z) = \text{Tr}(WZ)$ with a similar relation for W' and Z' . Using this fact, one can obtain

$$\begin{aligned} I_{UGG} &= \frac{1}{48(2\pi)^3} \int_X \left(\text{Tr}(YT^2(F_0^{(1)})^3) - \frac{1}{8} \text{Tr}(YT^2 F_0^{(1)}) \wedge \text{tr}(R^2) \right) \\ I_{U'G'G'} &= \frac{1}{48(2\pi)^3} \int_X \left(\text{Tr}(Y'T'^2(F_0^{(2)})^3) - \frac{1}{8} \text{Tr}(Y'T'^2 F_0^{(2)}) \wedge \text{tr}(R^2) \right) \end{aligned} \quad (53)$$

where in these equations we use the notation that $\text{Tr}(A^m) = \text{Tr}(A \wedge A \wedge A \dots \wedge A)$. To proceed we make use of the well known identities $\text{Tr}F^6 = \frac{1}{48} \text{Tr}F^2 \text{Tr}F^4 - \frac{1}{14400} (\text{Tr}F^2)^3$ and $\text{Tr}F^4 = \frac{1}{100} (\text{Tr}F^2)^2$. Although these identities involve the trace Tr in the adjoint of E_8 it is an important fact (which we shall return to later) that the first of these identities also holds for traces over the adjoint of $E_8 \times E_8$. Using these identities with the choice $F = \alpha Y + \beta T + \gamma F_0^{(1)}$ or $F = \alpha Y' + \beta T' + \gamma F_0^{(2)}$ and expanding the various monomials we obtain

$$\begin{aligned} I_{UGG} &= \frac{1}{48(2\pi)^3} \frac{1}{60 \cdot 1200} \int_X \left(\text{Tr}(T^2) \text{Tr}(Y F_0^{(1)}) \wedge (\text{Tr}(F_0^{(1)2}) - \text{Tr}(F_0^{(2)2})) \right) \\ I_{U'G'G'} &= \frac{1}{48(2\pi)^3} \frac{1}{60 \cdot 1200} \int_X \left(\text{Tr}(T'^2) \text{Tr}(Y' F_0^{(2)}) \wedge (\text{Tr}(F_0^{(2)2}) - \text{Tr}(F_0^{(1)2})) \right) \end{aligned} \quad (54)$$

In obtaining these equations we have also made use of the semi-simple properties of the generators T, T' , as well as the Bianchi identity $dH = 1/30(\text{Tr}F_0^{(1)2} + \text{Tr}F_0^{(2)2}) - \text{tr}R^2$. Note that the term dH does not contribute since an integration by parts gives zero. This follows from the fact that $\text{Tr}(Y F_0^{(1)})$, when decomposed into traces over the irreducible representations Q_{a_1} , gives an effective $U(1)$ valued 2-form, which satisfies an abelian Bianchi identity (the same holds for $\text{Tr}(Y' F_0^{(2)})$).

We can now compare the expressions for I_{UGG} and $I_{U'G'G'}$ and those for c_1 and c_2 in (50). First we notice that we can remove the ρ_0 dependence in the latter equations by using the equation (10) relating M-theory and string parameters. Furthermore, again using the Bianchi

identity involving dH and the integration by parts argument mentioned above, one can indeed verify that the forms of the expressions for c_1, c_2 are in agreement with those of $I_{UGG}, I_{U'G'G'}$.

Having established this connection for case I type embeddings, we now consider case II type. After some thought, it is apparent that the corresponding coefficients c_1, c_2 as well as their relation to the $U(1)-GG$ and $U(1)'-G'G'$ anomalies will be given by the same expressions, but now Y, Y' are identified with the $U(1)$ generators of J and J' . Then, in contrast to case I embeddings, it is clear that c_1, c_2 will not be vanishing in general, since $Tr(YF^{(1)}), Tr(Y'F^{(2)})$ will be non-zero, and we have the possibility of a surviving anomalous $U(1)$.

At this stage it remains only a possibility, because whilst it is a necessary condition that, for an anomalous $U(1)$ symmetry to exist, the coefficients of the corresponding $B \wedge F$ terms must be related to the various anomaly coefficients as we have seen above, it is not sufficient. This is because, as we discussed earlier, to some of these potentially anomalous $U(1)$'s the Higgs mechanism can be applied which involves the H field kinetic energy term improved by Chern-Simons terms, and this can happen precisely in the case II embeddings we are discussing. This is the case for the $U(1)$ s which belong to the structure group of the gauge vacuum bundle. These $U(1)$ s can be anomalous without being coupled to both walls. The reason is precisely that in this case the Higgs mechanism involves model-dependent axions through the improved kinetic term for the antisymmetric tensor field. In this paper, motivated by phenomenological applications, we are mainly interested in an anomalous $U(1)$ for which the Higgs mechanism involves the model-independent axion. As we have stressed earlier, (and which motivated the choice of mixed embedding in (42)) such an anomalous $U(1)$ must couple to both walls. If it does not then it should not be anomalous. Indeed it is a non-trivial check on the ideas presented in this section, that for example, if we imagine an embedding similar to the one in (42), but now defined purely in terms of line bundles on a single wall (say we are switching on the background which mixes two $U(1)$ s living on the first wall), with field strengths $F_{(1)a\bar{a}}, F'_{(1)a\bar{a}}$, then the surviving $U(1)$ should not be anomalous. This is an example of a case II type embedding, only we have mixing between two $U(1)$'s, both of which lies in the same E_8 factor. Whilst the formula presented earlier have not formally covered this possibility, it is easy to extend them to do so, simply reinterpreting formulae (50). For the resulting coefficients c_1 and c'_1 of the $B \wedge F$ terms one obtains

$$\begin{aligned}
c_1 &= \frac{3\rho_0}{2\pi\kappa^2} \left(\frac{-\sqrt{2}}{3456}\right) \left(\frac{\kappa}{4\pi}\right)^{4/3} \int_X \frac{1}{30} Tr(YF_0^{(1)}) \wedge \left(\frac{1}{30} Tr(F_0^{(1)} \wedge F_0^{(1)})\right) \\
&\quad - \frac{1}{60} Tr(F_0^{(2)} \wedge F_0^{(2)}) - \frac{1}{4} tr(R \wedge R) \\
c'_1 &= \frac{3\rho_0}{2\pi\kappa^2} \left(\frac{-\sqrt{2}}{3456}\right) \left(\frac{\kappa}{4\pi}\right)^{4/3} \int_X \frac{1}{30} Tr(Y'F_0'^{(1)}) \wedge \left(\frac{1}{30} Tr(F_0^{(1)} \wedge F_0^{(1)})\right) \\
&\quad - \frac{1}{60} Tr(F_0^{(2)} \wedge F_0^{(2)}) - \frac{1}{4} tr(R \wedge R)
\end{aligned} \tag{55}$$

For the embedding defined by

$$(\cos(\theta)F_{(1)a\bar{a}} + \sin(\theta)F'_{(1)a\bar{a}}) = \mathcal{F}'_{a\bar{a}} \tag{56}$$

it is easy to see that the corresponding coefficients of the surviving $B \wedge F$ indeed vanish. One may verify from (55) that they are proportional to the combination $-\sin(\theta)F_{(1)a\bar{a}} + \cos(\theta)F'_{(1)a\bar{a}}$, which by assumption is vanishing. Of course, the coefficient of the structure $U(1)$, whose gauge boson is eaten up by the Higgs mechanism, is nonzero. However, as we remarked before this is not the $U(1)$ of the type we are interested in. It decouples from the massless spectrum, hence its anomaly is harmless.

By the same reasoning, we should find that if we choose an embedding like (42) which mixes fields on different walls, then the coefficient should be non-vanishing in general. Following the same logic as before, we find that in $d = 4$ the term $c' \int B \wedge F$ arises from the CGG term reduction, where F is the orthogonal combination of F_1, F_2 that defines the unbroken $U(1)$, and $c' = \cos(\theta)c_2 - \sin(\theta)c_1$. Next, we have to use equations (50) (we stress that in these equations Y, Y' are taken to define the $U(1)$ generators in J, J'), appropriate for each wall. An important point is that using the dH Bianchi identity, we see that the 4-forms multiplying the $Tr(YF_1^{(1)})$ and $Tr(Y'F_2^{(2)})$ terms in (50) are equal but with opposite sign. From this we see that the combination $\cos(\theta)F_2^{(2)} + \sin(\theta)F_1^{(1)}$ occurs in the coefficient c' , which is equal to $2 \sin(\theta) F_1^{(1)}$ upon using the constraint discussed below (56). This is non-vanishing as implied by definition of the embedding (42). The crucial sign flip seen in c' , whose origin lies in the Bianchi identity, is responsible for providing (in general) an anomalous $U(1)$ in $d = 4$. This argument also illustrates the idea that such an anomalous $U(1)$ must couple to both walls. If we put $\theta = 0$ (or $\pi/2$), which confines embedding to a single wall, then c' vanishes.

We now make some comments concerning the mass scale entering the Green-Schwarz term of the anomalous $U(1)$ we obtained above. To do this it is convenient to solve the B -field equations of motion in the frame defined earlier in (7) (and recalling $g_{55} = e^{2\gamma}$), which allows one to make contact with the axionic pseudo-scalar field b

$$\star H_\mu = \frac{e^{\gamma-6\beta}}{\sqrt{g^{(6)}}} \partial_\mu b \quad (57)$$

where $\star H$ is the dual of the 3-form field strength of B . Substituting this solution back into the effective four dimensional action, one can verify that the Green-Schwarz term expressed in terms of the variable b is consistent with the expansion of the Kähler potential $K = -M_{Pl}^2 \ln(S + \bar{S} - cV)$. Thus, we find the Green-Schwarz term $\sim g^2 M_{Pl}^2 \partial_\mu b A^\mu$ where A^μ is the gauge field of the anomalous $U(1)$. Here $g^2 = g_1^2 = g_2^2$ is the lowest order term in the κ expansion of g_{mixed}^2 , where we note that it is only the threshold corrections (which are proportional to $\kappa^{-4/9}$) which distinguish g_1 from g_2 in the definition of g_{mixed}^2 . (Recall that the factor of M_{Pl}^2 already contains a factor of $\kappa^{-4/9}$)

Before ending this section, we make some comments concerning the connection between the above results and those of the weakly coupled case. We expect that in this case the generation of $B \wedge F$ terms in four dimensions comes from the reduction of the $d = 10$ Green-Schwarz terms [33]. The coefficient of such a term should once again be proportional to the various $U(1)$ triangle anomalies in four dimensions. We can derive expressions for the latter in terms of the index of the Dirac operator on X where now we are considering the full $E_8 \times E_8$ gauge group (as opposed to separate E_8 factors on each wall in the M-theory case). However it

turns out that the reduction of the GS terms gives expressions for the coefficient of $B \wedge F$ that mirror the expressions of c_1, c_2 calculated above, in the strongly coupled case. Therefore the $d = 4$ anomaly analysis must be mirrored likewise. This follows from the nontrivial property of the TrF^6 identity which we mentioned earlier, namely that it is satisfied for E_8 and $E_8 \times E_8$, separately. This is an important key to understanding why the anomaly analysis, in both $d = 10$ and as we have seen in this paper, $d = 4$, are mirrored in the weak and strong coupling cases.

5 Gauge Couplings in Nonstandard Embeddings

In this Section we shall discuss the issue which has direct phenomenological relevance: the correlation between the choice of embedding, hence of the structure of the unbroken low energy symmetry group, and the values of the unified gauge couplings on the fixed hyperplanes. Let us recall the result for the gauge coupling difference, originally given by Witten [2] and also following from Section 3:

$$\alpha_h^{-1} - \alpha_v^{-1} = \frac{2}{(4\pi\kappa^2)^{2/3}}(V_h - V_v) = \frac{\pi\rho_p s_i}{(4\pi)^{1/3}\kappa^{2/3}} \frac{1}{8\pi^2} \int_X \omega \wedge (tr F^{(i)} \wedge F^{(i)} - \frac{1}{2} tr R \wedge R) \quad (58)$$

where $s_i = +1, -1$ for $i = 1, 2$ respectively. We stress again, that this result is independent of the particular embedding. The split of the couplings given above is of the relative order $o(\kappa^{4/3})$ with respect to the gravitational part of the Lagrangian, and it has the interpretation of the difference between volumes of the two C-Y spaces on the hidden and visible walls (with the proportionality coefficient $2\pi(4\pi\kappa^2)^{2/3}$).

Let us rewrite (58) in terms of string units, i.e. in terms of α'

$$\alpha_h^{-1} - \alpha_v^{-1} = \frac{s_i}{8\pi^2\alpha'} \frac{1}{8\pi^2} T \int_X \omega \wedge (tr F^{(i)} \wedge F^{(i)} - \frac{1}{2} tr R \wedge R) \quad (59)$$

To better understand this expression one should write down the integrand in the explicit manner

$$\omega \wedge tr(F \wedge F) = g^{a\bar{a}} g^{b\bar{b}} tr(F_{a\bar{a}} F_{b\bar{b}}) - g^{a\bar{a}} g^{b\bar{b}} tr(F_{a\bar{b}} F_{\bar{a}b}) \quad (60)$$

Using the Einstein-Kähler equations given in the preceding section one easily finds that

$$\int_X \omega \wedge tr(F^{(i)} \wedge F^{(i)}) = -\frac{1}{2} \int_X tr F_{ij}^{(i)} F^{(i)ij} = -8\pi^2 n_{F_i} \leq 0 \quad (61)$$

The same steps can be repeated for the gravitational part of the integrand

$$\int_X \omega \wedge tr(R \wedge R) = -\frac{1}{2} \int_X tr(R_{ij} R^{ij}) = -8\pi^2 n_R \leq 0 \quad (62)$$

Hence the integral that gives the difference between the unified couplings is

$$\int_X \omega \wedge (tr(F^{(i)} \wedge F^{(i)}) - \frac{1}{2} tr(R \wedge R)) = -8\pi^2 (n_{F_i} - \frac{1}{2} n_R) \quad (63)$$

The integrability conditions for the equations of motion give constraint on the instanton numbers⁸ [8]

$$n_{F1} + n_{F2} = n_R \quad (64)$$

It is important to note that all the above instanton numbers are positive (some of the gauge ones may be zero). In fact, from the observation that on any Calabi-Yau manifold the standard embedding must be possible, it follows that Calabi-Yau spaces always have $n_R > 0$. Moreover, in the case of a general embedding n_{F_i} must be positive (or zero) for any bundle in the direct sum, since the Kähler-Einstein equations must be fulfilled for each V_i separately. In terms of the instanton numbers and in units in which $2\alpha' = 1$ the difference of inverse couplings is

$$\alpha_h^{-1} - \alpha_v^{-1} = -\frac{s_i}{4\pi^2} T_r (n_{F_i} - \frac{1}{2}n_R) \quad (65)$$

Let us take standard embedding first. There $n_{F_v} = n_R$, and

$$\alpha_v^{-1} - \alpha_h^{-1} = \frac{1}{8\pi^2} T_r n_R \quad (66)$$

This particular embedding gives the specific gauge group structure $E_{6(v)} \times E_{8(h)}$. We stress that since n_R has positive sign, there is no way of changing the sign of $\alpha_v^{-1} - \alpha_h^{-1}$. For individual gauge couplings we have

$$\alpha_v^{-1} = 4\pi (S_r + \epsilon T_r), \quad \alpha_h^{-1} = 4\pi (S_r - \epsilon T_r); \quad \epsilon = \frac{n_{F_v} - \frac{1}{2}n_R}{32\pi^3} \quad (67)$$

Let us look for another model, which has $\epsilon' = -\epsilon$. This means that $n_{F_v} - \frac{1}{2}n_R = -n'_{F_v} + \frac{1}{2}n'_R$. Of course, one of the solutions to this equation is the ‘complementary’ sector from the original standard embedding. This is the only solution if one demands that the CY metric does not change, i.e. if $n_R = n'_R$. If one allows for other metrics, then one can have more interesting solutions. Let us also note that it is possible to obtain the situation where $\epsilon = 0$, see [26] for $N = 2$ examples, however we shall not pursue in the present paper any specific model in detail.

As far as general layout is concerned, there are two broad classes of models: the ones where the visible sector belongs to the more weakly coupled, say weakest, part of the unbroken gauge group, and the ones where the visible sector is the most strongly coupled one. We shall always assume that none of the nonabelian groups from the visible sector is of the mixed type. These two distinct unification routes which open up in the general embedding case shall be discussed below.

5.1 Weaker Obervable Sector

We assume here that we live on the hyperplane where the C-Y volume is larger, hence the unified coupling is smaller than the one on the other hyperplane. The obvious constraint is

⁸The configurations we use here fulfill the Yang-Mills equations of motion and Einstein equations which justifies the term instanton.

that the hidden coupling is smaller than infinity, see (67). This requirement, formulated in the 11d supergravity frame used originally by Witten, leads to the notion of the critical radius. This is the physical distance between the planes that corresponds to vanishing volume of the C-Y space localized on the hidden plane. Using linear approximation for the variation of the volume along x^{11} , one can obtain the value of the physical critical radius $(\pi\rho_c)^{-1} = 0.8 \times 10^{15} \text{ GeV}$.

One should note that in the 5d supergravity frame the presence of the critical radius manifests itself in a different way than in the 11d canonical frame, since in the equation for M_{Pl} there is no modulus S . In the 5d canonical frame we see the critical radius once we allow the G_N to vary while g_{GUT} and M_{GUT} are fixed at their observed values, and one takes a specific, fixed by an embedding, value of ϵ . Then one can find that in order not to leave the field theoretical domain, the physical radius of the eleventh dimension must be smaller than $\rho'_{cr} = \rho_0/(2g_{GUT}^2\epsilon)$.

Another way to explore the constraint of finite g_h^2 is to allow the unified coupling constant to vary while keeping the other parameters fixed at realistic values. One obtains the condition

$$\alpha_v < 4\pi^{3/2} \frac{1}{\epsilon^{3/4}} (M_{GUT}/M_{Pl})^{3/2} \quad (68)$$

This puts some restrictions on model building. To be more specific, let us take the example of K3 fibration C-Y manifolds discussed in [34]. Guided by this example we see that a typical value of ϵ is $\epsilon = O(10)/(32\pi^3)$. This leads to the condition $\alpha_v < 0.047$. The upper limit for the standard embedding value $\epsilon_s = 3/(8\pi^3)$ is $\alpha_{(s)v} < 0.041$. This shows that while the observable unification coupling $\alpha_{GUT} = 0.04$ fits marginally within the limits, accommodating the scenarios of ‘strong’ unification [13], with $\alpha_{GUT} \approx 1$, within this branch of M -theory models is not possible. The problem is underlined by the fact that the hidden sector coupling is, by assumption, even stronger. Actually it grows with the growing visible coupling according to

$$\alpha_h = \left(\frac{1}{\alpha_v} - \epsilon(\alpha_v)^{1/3} \left(\frac{M_{Pl}}{M_{GUT}} \right)^2 \frac{1}{2^{2/3} 4\pi^2} \right)^{-1} \quad (69)$$

The value of α_h becomes larger than one at $\alpha_v = 0.045$ for a generic embedding and slightly above $\alpha_v = 0.04$ for the standard embedding. This means in particular that, generically, in this type of scenarios there is not much space for running of the gauge coupling in the hidden sector. In consequence, it is difficult to develop a condensation scale which would be hierarchically smaller than the unification scale, as needed for realistic supersymmetry breaking. The condensation scale is given by the formula:

$$\Lambda_c = M'_{GUT} e^{\frac{-1}{8\pi^{b_{(h)0}}\alpha_h}} \quad (70)$$

where $M'_{GUT} = M_{GUT}(\alpha_h/\alpha_v)^{1/6}$ is the unification scale in the hidden sector. For the standard embedding with the pure E_8 in the hidden sector ($\alpha_h = 0.97$, $b_0 = 0.57$) one obtains $\Lambda_c = 1.59 M_{GUT}$. Changing the unifying gauge group so that $0.1 < b_0 < 0.57$ and for embeddings with generic value of ϵ we get $0.18 M_{GUT} < \Lambda_c < 0.92 M_{GUT}$. Although nonstandard embeddings lower the value of the condensation scale by about an order of magnitude, this is still clearly above the border line of giving the proper gaugino condensation scale. Further decrease of the condensation scale would require additional matter fields, which would decrease the coefficient $b_{(h)0}$.

5.2 Stronger Observable Sector

A very interesting class of models arises, when one identifies the visible sector with the part of the unbroken gauge group which is most strongly coupled (this possibility has been independently considered in [7]). To obtain relations involving ϵ , S and T we solve again the strongly coupled string relations (12) from the Section 2.1, substituting there this time $1/\alpha_{GUT} = 4\pi(S_r - \epsilon T_r)$.

Obviously, there is no critical radius and no critical value of ϵ in the sense that the C-Y volume on the hidden hyperplane is always larger than that on the visible hyperplane (we remind that to orient our discussion we are assuming $\epsilon > 0$). Hence it never becomes smaller than $(\alpha')^3$.

However, there are limits on α_v or/and ϵ coming from two places. Firstly, the string coupling g_s^2 should remain larger than unity, to stay within the realm of the Horava-Witten model, and away from the transition limit between weakly and strongly coupled string. Thus, we demand $S/T^3 < 1^9$. If we treat α_v as a parameter, then this translates into the condition

$$\frac{1}{\alpha_v} + \epsilon \alpha_v^{1/3} \left(\frac{M_{Pl}}{M_{GUT}} \right)^2 \frac{1}{2^{11/3} \pi^2} < \alpha_v \left(\frac{M_{Pl}}{M_{GUT}} \right)^6 \frac{1}{2^{15} \pi^8} \quad (71)$$

which is fulfilled with a wide margin for typical values of $\epsilon \approx O(10)/(32\pi^3)$.

The other, more interesting limit comes from the requirement that it is hierarchical dynamical supersymmetry breaking on the hidden wall which is responsible for soft masses in the visible sector. In general, for this to work, one needs a dynamically generated condensation scale in the hidden sector to be $10^5 GeV < \Lambda_{cond} < 10^{13} GeV$ (a mass range which interpolates between gauge mediated and gravity mediated supersymmetry breaking models). Using the analog of eqs.(69) and (70) it is straightforward to work out the relation between the hidden condensation scale and ϵ . Taking a typical value of $\epsilon = 0.01$, one obtains $1.4 \cdot 10^{-8} M_{GUT} < \Lambda_c < 2.9 \cdot 10^{-7} M_{GUT}$ for $0.04 < \alpha_v < 0.11$, and then the value of the condensation scale becomes smaller again achieving $2.4 \cdot 10^{-11} M_{GUT}$ at $\alpha_v = 1$. In the above estimate we have put the value of one-loop beta-function coefficient to 0.1. In principle, it is possible to increase the condensation scale by increasing $b_{(h)0}$. For example, $b_{(h)0} = 0.25$ would give $\Lambda_c = 7 \cdot 10^{-3} M_{GUT}$ at $\alpha_v = 0.04$.

It is interesting to find out how large the hidden wall Calabi-Yau might be. It turns out it cannot be too large, or in other words the mass of the heavy Kaluza-Klein modes is as large as in the visible sector. The mass of the hidden Kaluza-Klein modes in terms of ϵ is given as: $M_{hKK} = \frac{M_{GUT}}{(4\pi\alpha_{GUT})^{1/6}} (S_r + \epsilon T_r)^{-1/6}$. For the typical choice of ϵ motivated by the $K3$ fibrations C-Y examples, the smallest value one can get is $M_{hKK} = 0.51 M_{GUT}$ corresponding to $\alpha_{GUT} = 1^{10}$.

In the presently discussed scenario where the coupling of the observable group is the

⁹This condition is always fulfilled if the observable sector is the weakest one.

¹⁰And $M_{hKK} = 0.91 M_{GUT}$ for $\alpha_{GUT} = 0.04$.

strongest, one can raise it up towards the nonperturbative region, without violating any bound, this way providing a realization of the ideas of strong unification advocated in [6].

It is also worth of pointing out, that the race-track gaugino condensation scenarios also need nonstandard hidden sector with hidden matter in order to work properly [35].

Finally, we observe that non-standard embedding M -theoretical models might naturally be good places for the realization of the scenario of supersymmetry breaking and moduli stabilization proposed in [36].

6 Summary

In this paper we have discussed theoretical and phenomenological aspects of general embeddings in M - theory compactified on S^1/Z_2 . Going beyond the standard embedding discussed so far in the literature is interesting for several reasons. We focused our attention on the existence of anomalous $U(1)_A$ and on the behaviour of the gauge couplings in the non-standard embedding scenarios.

As a necessary technical component of this discussion we first generalized the standard embedding result for the threshold corrections to the gauge kinetic couplings to non-standard embeddings. For general embeddings, we have formulated the effective four dimensional $N = 1$ supersymmetric theory by solving the Bianchi identity and equations of motion for the antisymmetric tensor background along the eleventh (fifth) dimension, and integrating out the explicit x^{11} dependence. The solution for that background, and implicitly also for the metric, is given in terms of massless and *massive* eigenmodes of the Laplacian on Calabi-Yau space. Using this result, we have shown that, when one considers only the effective couplings between the *massless* fields, the form of threshold corrections is the same as for the standard embedding in the sense that in the expression $1/g_{(1),(2)}^2 = Re(S + (\pm\epsilon_{i(1),(2)})T_i)$ (where $i = 1, \dots, h(1,1)$) the individual coefficients $\epsilon_{i(1),(2)}$ are related through $\epsilon_{i(1)} = -\epsilon_{i(2)}$. An easy way to obtain this result is to consider the axial part of the corrections which follow from the reduction of the $C \wedge G \wedge G$ term.

A particular class of embeddings which we have discussed in detail is the one in which gauge interactions ‘mix’ the two walls, that is under which fermions on *both* hyperplanes are charged. We explicitly give such a construction, and point out that this is the way the anomalous $U(1)_A$ gauge group can arise in M -theory models. We show how the four dimensional Green-Schwarz term $B \wedge F$, which serves to cancel the abelian anomaly, does arise in this case from the reduction of the eleven dimensional $C \wedge G \wedge G$ term. We also consider the issue of the cancellation of the four dimensional anomalies in M -theory with a general embedding, and discuss its relation to the weakly coupled case.

Finally, we have discussed several phenomenological aspects of the behaviour of the gauge couplings in non-standard embedding scenarios. The hierarchy of couplings in the visible and hidden sectors is well defined in the standard embedding. Changing it, by which we mean

having the coupling in the visible sector stronger than in the hidden sector (or sectors), can be achieved only by going to a non-standard embedding. Such an ‘inverse’ scenario has several advantageous features. One naturally goes around the problem of the existence of a critical radius. Moreover, the condensation scales become low enough compared to the hidden sector unification scale (since the hidden sector coupling g_h can be made sufficiently small) to have a suitable hierarchy of masses due to the associated supersymmetry breaking. Also, ‘strong’ unification becomes possible. The hidden sector may have several components and contains chiral matter, which helps to break dynamically supersymmetry in a satisfactory way. In addition, the mixed gauge embeddings with low condensation scale revive the scenarios where the transmission of the supersymmetry breaking is *not* gravitational - a situation impossible within the restricted framework of the standard embedding models.

We hope that the basic steps we took in this paper along the route towards nonstandard embeddings M -phenomenology justify further investigation of this matter.

Acknowledgements:

Z.L. and S.P. are supported in part by the Polish Committee for Scientific Research grant 2 P03B 040 12 1997/98, and by the M. Curie-Sklodowska Foundation grant F-115. The work of S.T. is funded by the Royal Society of Great Britain. The authors thank John Ellis and Graham G. Ross for interesting conversations, and Z.L. would like to thank Stephan Stieberger for stimulating discussions.

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