

Special Symmetric Quark Mass Matrices

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Abstract

We give a procedure to construct a special class of symmetric quark mass matrices near the democratic limit of equal Yukawa couplings for each sector. It is shown that within appropriate weak-bases, the requirements of symmetry and $\arg[\det(M)] = 0$ are very strong conditions, that necessarily lead to a Cabibbo angle given by $|V_{us}| = \sqrt{m_d/m_s}$, and to $|V_{cb}| \sim m_s/m_b$, in first order. In addition, we prove that the recently classified ansätze, which also reproduce these mixing matrix relations, and which were based on the hypothesis of the Universal Strength for Yukawa couplings, where all Yukawa couplings have equal moduli while the flavour dependence is only in their phases, are, in fact, particular cases of the generalized symmetric quark mass matrix ansätze we construct here. In an excellent numerical example, the experimental values on all quark mixings and masses are accommodated, and the CP violation phase parameter is shown to be crucially dependent on the values of m_u and V_{us} .

1 Introduction

In the Standard Model (SM), the flavour structure of the Yukawa interactions is not constrained by any symmetry, and the charged currents depend only on the left handed quark fields. Thus, there is much freedom in defining a weak-basis for the quarks. This freedom is often used to construct ansätze with which one hopes to find relations between the quark masses and mixings, and perhaps some day, also find some deeper symmetry beyond the SM, to restrict the parameters in the Yukawa couplings [1].

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In the past there have been several attempts at relating the pattern of the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix elements to the quark mass ratios. Many of the ansätze, that one finds in the literature, are given in two distinct types of weak-bases. The well-known Fritzsche ansatz [2], the Hermitic ansätze classified by Ramond, Roberts and Ross, and others [3], use the so-called "heavy" weak-basis, where one of the quark mass matrix elements (by consensus usually the (3,3)-element) of both sectors is much larger than the others. Some of the later may even be zero, possibly as the result of some (unknown) symmetry.

A different approach is offered by the democratic weak-basis, where the quark Yukawa interactions become equal and indistinguishable in the limit of two zero lower masses (for each quark sector apart). Amongst others [4], one of the most interesting examples of this approach is given by the hypothesis of the Universal Strength for Yukawa couplings (USY) applied to the quark sector [5]. In USY all Yukawa couplings have equal moduli, and the flavour dependence is only in the phases of the Yukawa coupling matrix elements.

In this paper, we give, in section 2, a procedure to construct a special class of symmetric ansätze near to the democratic quark mass limit. We obtain this class by demanding that the quark mass matrices be symmetric and that, for both sectors, $\arg[\det(M)] = 0$. We then find that (within our framework) the mixing angles obey the phenomenological mass ratio relations where $|V_{us}| = (m_d/m_s)^{1/2}$ and $|V_{cb}| \sim m_s/m_b$. In section 3, we prove that an important group of recently classified ansätze, based on the USY idea, is, in fact, a particular case within this class. In section 4, a relation is found between m_u , V_{us} and the CP-violation phase. A numerical example is given, using the most successful of the constructed ansätze, and which accommodates all experimental results on quark masses and mixings. Finally, in section 5, we present our conclusions.

The purpose of this paper is to find specific ansätze by making use of the freedom that the SM model provides by choosing special weak-bases for the quarks. As a typical example, we mention the Nearest Neighbour Interaction (NNI) weak-basis [6]. It was proven that, without any constraint on the quark masses and mixings, the mass matrices could be written in the following way:

$$M_u = \begin{bmatrix} 0 & a_u & 0 \\ a'_u & 0 & b_u \\ 0 & b'_u & c_u \end{bmatrix}, \quad M_d = \begin{bmatrix} 0 & a_d & 0 \\ a'_d & 0 & b_d \\ 0 & b'_d & c_d \end{bmatrix} \quad (1)$$

The NNI weak-basis allows for different types of ansätze [7], [8]. By imposing the Hermiticity conditions, i.e., $a'_{u,d} = a_{u,d}^*$, $b'_{u,d} = b_{u,d}^*$, $c_{u,d}$ = real, one obtains the famous Fritzsche ansatz [2], now outruled because of the heavy top mass. Another ansatz, within the NNI weak-basis, can be constructed if one imposes the non-Hermiticity conditions $a'_{u,d} = a_{u,d}$, $b'_{u,d} = c_{u,d}$, which then allow for a heavy top mass [7].

2 Symmetric ansätze

In this section, we construct a class of complex quark mass matrix ansätze which lead to important mixing matrix relations, where $|V_{us}| = (m_d/m_s)^{1/2}$ and $|V_{cb}| \sim m_s/m_b$. The ansätze are symmetric and near to the democratic mass matrix limit. The procedure, we propose here, is analogous to the NNI-Fritzsh example. This means that we work in a specific weak-basis, but instead of requiring, e.g., the somewhat arbitrary Hermiticity conditions, we require that the quark mass matrices be,

- symmetric, and that,
- for each quark sector, $\arg[\det(M)] = 0$.

The importance of these two requirements, (which will also contribute to the calculability of the model), for the quark mass matrices is evident: symmetric fermion mass matrices are crucial in realistic unification schemes, such as in $SO(10)$ [9], and the requirement that $\arg[\det(M)] = 0$ is a *conditio sine qua non* for most solutions of the strong CP problem [10].

We begin with the most general, arbitrary, and complex quark mass matrix,

$$M_o = \begin{bmatrix} \alpha_o & \beta_o & \gamma_o \\ \delta_o & \epsilon_o & \zeta_o \\ \eta_o & \theta_o & \iota_o \end{bmatrix} \quad (2)$$

One can prove that there exists a weak-basis transformation of the right-handed quark fields, W , such that $M_o \rightarrow M = M_o \cdot W$, and where some of the mass matrix elements become equal:

$$M = \begin{bmatrix} u & u & \hat{z} \\ \hat{u} & v & w \\ z & \hat{w} & \hat{w} \end{bmatrix} \quad (3)$$

The proof is simple. Take the first line of the general complex quark mass matrix M_o in Eq.(2). This is the (line)vector $\mathbf{a}_o^+ = (\alpha_o, \beta_o, \gamma_o)$. Now construct three orthonormal (column)vectors $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, such that the ending points of \mathbf{v}_1 and \mathbf{v}_2 define a line which is perpendicular to \mathbf{a}_o^+ , then $\mathbf{a}_o^+ \cdot (\mathbf{v}_1 - \mathbf{v}_2) = 0$ or $\mathbf{a}_o^+ \cdot \mathbf{v}_1 = \mathbf{a}_o^+ \cdot \mathbf{v}_2$. By turning this line around \mathbf{a}_o^+ , we can assure that the third vector \mathbf{v}_3 lies in such a direction that another line, through the ending points of \mathbf{v}_3 and \mathbf{v}_2 , is perpendicular to the third (line)vector of M_o , $\mathbf{c}_o^+ = (\eta_o, \theta_o, \iota_o)$. Thus $\mathbf{c}_o^+ \cdot (\mathbf{v}_2 - \mathbf{v}_3) = 0$ or $\mathbf{c}_o^+ \cdot \mathbf{v}_2 = \mathbf{c}_o^+ \cdot \mathbf{v}_3$. Therefore, defining $W = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$, we get the result of Eq.(3). Furthermore, one can choose the phase of the complex number u in such a way that $u(z-w)^2 = \text{real}$. Up to this point, no conditions were imposed on M which constrain the mass spectrum or mixing angles. We stress that the form of M in Eq.(3) is just a choice of weak-basis.

Now, we demand the first of the two conditions: symmetry. Then $\hat{u} = u$, $\hat{z} = z$, $\hat{w} = w$, and M becomes of the form,

$$M = \begin{bmatrix} u & u & z \\ u & v & w \\ z & w & w \end{bmatrix} \quad (4)$$

Next, we implement the second condition, $\arg[\det(M)] = 0$. Computing the determinant of M ,

$$\det(M) = -u(z-w)^2 + (v-u)(uw - z^2) \quad (5)$$

and using $u(z-w)^2 = \text{real}$, we find that $v = u$ is a solution, because $\tan(\arg[\det(M)])$ is proportional to $|v-u|$. Therefore, requiring symmetry and $\arg[\det(M)] = 0$ reduces a general complex mass matrix in the weak-basis of Eq.(3) to a matrix of the form,

$$M = \begin{bmatrix} u & u & z \\ u & u & w \\ z & w & w \end{bmatrix} = u \begin{bmatrix} 1 & 1 & z' \\ 1 & 1 & w' \\ z' & w' & w' \end{bmatrix} \quad (6)$$

where $z' = z/u$ and $w' = w/u$.

In the following, we prove that the correct quark mass hierarchy implies that the ansatz of Eq.(6) is near to the democratic limit, thus $z', w' \approx 1$, and that it reproduces the crucial mixing matrix relations, where $|V_{us}| = (m_d/m_s)^{1/2}$ and $|V_{cb}| \sim m_s/m_b$. In order to do this, we introduce the dimensionless square mass matrix $H = 3(M \cdot M^\dagger)/\text{tr}(M \cdot M^\dagger)$. It is clear that by construction $\text{tr}(H) = 3$. Taking the right-handed side of Eq.(6), (we drop the primes on z' and w' to simplify the notation), we get,

$$H = \frac{1}{t_{zw}} \cdot \begin{bmatrix} 2 + |z|^2 & 2 + zw^* & z^* + w^* + zw^* \\ 2 + z^*w & 2 + |w|^2 & z^* + w^* + |w|^2 \\ z + w + z^*w & z + w + |w|^2 & |z|^2 + 2|w|^2 \end{bmatrix} \quad (7)$$

where $t_{zw} = (4 + 2|z|^2 + 3|w|^2)/3$. The determinant δ , the second invariant χ , and the trace t , which are the three invariants of H , can be expressed in its dimensionless eigenvalues λ_i ,

$$\begin{aligned} \delta &= \det(H) = \lambda_1 \lambda_2 \lambda_3 \\ \chi &= \chi(H) = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 \quad ; \quad \lambda_i = 3 \frac{(m_i/m_3)^2}{[1 + (m_2/m_3)^2 + (m_1/m_3)^2]} \\ t &= \text{tr}(H) = \lambda_1 + \lambda_2 + \lambda_3 = 3 \end{aligned} \quad (8)$$

In order to find the predictions of the ansatz of Eq.(6), it is very useful to introduce the following parameterization for z and w ,

$$z = 1 + \rho e^{i\alpha} \quad , \quad w = 1 + \rho e^{i\alpha} - \rho_\circ e^{i\beta} \quad (9)$$

This parameterization is general, but we shall now see that the mass hierarchy imposes constraints on ρ_\circ and ρ , such that these are small. Therefore $z, w \approx 1$ and M is almost democratic.

Because of the parametrization of Eq.(9) one can give a simple expression for ρ_\circ in terms of the determinant δ of H and t_{zw} ,

$$\rho_\circ^2 = 3\sqrt{3\delta} \cdot \left(\frac{t_{zw}}{3}\right)^{3/2} \quad (10)$$

where t_{zw} can also be expressed as a function of the variables defined in Eq.(9),

$$t_{zw} = 3 + \frac{10}{3}\rho \cos(\alpha) - 2\rho_o \cos(\beta) + \frac{5}{3}\rho^2 - 2\rho\rho_o \cos(\alpha - \beta) + \rho_o^2 \quad (11)$$

Using δ in Eq.(8) in terms of mass ratios, we find that the parameter ρ_o is proportional to $(m_1 m_2)^{1/2}/m_3$. From H in Eq.(7), one computes also the second invariant as a function of ρ , ρ_o , $\cos(\alpha)$ and $\cos(\beta)$. From this, one deduces an expression for the ρ written in terms of quark mass ratios, a rest-term, and t_{zw} ,

$$\rho^2 = \frac{9}{4} \cdot \chi \cdot \left(\frac{t_{zw}}{3}\right)^2 \cdot [1 + o_{zw}]^{-1} \quad (12)$$

where the rest-term o_{zw} is dependent on powers of ρ , ρ_o , ρ_o/ρ , $\cos(\alpha)$ and $\cos(\beta)$. Using the expression for χ in Eq.(8) in mass ratios, we find that ρ is proportional to m_2/m_3 . We conclude that ρ_o and ρ are indeed small. Then from Eq.(11), we find that $t_{zw} \approx 3$. Consequently, it is possible to give a leading order approximation for ρ_o and ρ :

$$\rho_o = 3\sqrt{3} \frac{\sqrt{m_1 m_2}}{m_3}, \quad \rho = \frac{9}{2} \frac{m_2}{m_3}. \quad (13)$$

A more complete expression for ρ_o and ρ as a power series in the mass ratios can be derived from the Eqs.(10, 11, 12) using the method of iteration. Starting with the leading order approximations, one obtains after a couple of iterations,

$$\begin{aligned} \rho_o &= 3\frac{\sqrt{3m_1 m_2}}{m_3} \cdot \left[1 + \frac{15}{4} \left(\frac{m_2}{m_3}\right) \cos(\alpha) - \frac{3\sqrt{3}}{2} \frac{\sqrt{m_1 m_2}}{m_3} \cos(\beta) + \dots\right] \\ \rho &= \frac{9}{2} \frac{m_2}{m_3} \cdot \left[1 - \frac{m_1}{m_2} + \frac{1}{2} \left(\frac{m_2}{m_3}\right) \cos(\alpha) + \sqrt{3} \frac{\sqrt{m_1 m_2}}{m_3} \cos(\beta) + \dots\right] \end{aligned} \quad (14)$$

Resuming: we have extracted ρ_o and ρ from the two (δ and χ) mass ratio relations of H . The two remaining phase parameters α , β will be free. However, from Eq.(14), we see that the contribution of α and β to ρ_o and ρ are small.

The next step is to compute the unitary matrix which diagonalizes H . The way to do this, is to introduce the power series of ρ_o , ρ in the parametrization of z , w of Eq.(9), and in the matrix H of Eq.(7). Thus, we obtain the square matrix H as a power series in the mass ratios, and it is then easy to calculate the eigenvectors as a series in these ratios, because the eigenvalues of H are also expressed as functions of ratios of masses. We prefer, however, to calculate the diagonalization matrix V in an appropriate "heavy" weak-basis for H . In this weak-basis all matrix elements of H are small, except $H_{33} \approx 3$, and only the relevant contributions of H_u and H_d to V_{CKM} are present. Thus the irrelevant contributions to the Cabibbo Kobayashi Maskawa mixing matrix V_{CKM} , where,

$$V_{CKM} = V_u^\dagger \cdot V_d \quad (15)$$

and which would cancel out in the matrix product, are absent. In this way, V_u and V_d are both near to $\mathbb{1}$. The "heavy-basis" is defined in the following way:

$$\begin{aligned}
H_u &\rightarrow H_u^{\text{Heavy}} = F^\dagger \cdot H_u \cdot F \\
H_d &\rightarrow H_d^{\text{Heavy}} = F^\dagger \cdot H_d \cdot F
\end{aligned}
\quad ; \quad
F = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \quad (16)$$

We find, in leading and next leading order,

$$\begin{aligned}
|V_{12}| &= \sqrt{\frac{m_1}{m_2}} \left[1 - \frac{m_1}{2m_2} + \frac{m_2}{4m_3} \cos(\alpha) \right]; & |V_{23}| &= \sqrt{2} \frac{m_2}{m_3} \left[1 - \sqrt{\frac{3m_1}{4m_2}} \cos(\alpha - \beta) \right] \\
|V_{13}| &= \frac{1}{\sqrt{2}} \frac{\sqrt{m_1 m_2}}{m_3} \left[1 + \frac{13m_2}{4m_3} \cos(\alpha) \right]; & |V_{31}| &= \frac{3}{\sqrt{2}} \frac{\sqrt{m_1 m_2}}{m_3} \left[1 - \sqrt{\frac{m_1}{3m_2}} \cos(\alpha - \beta) \right]
\end{aligned} \quad (17)$$

where we stress again that the phases α, β are free parameters, in the sense that they are not constrained by mass relations. This freedom will be used in section 4, where we shall define a specific ansatz by fixing these two free phases to accommodate the CP violation phase together with the quark masses and mixings.

To complete the description of our special symmetric ansätze near the democratic limit, we construct another explicit example. As before, we take the most general complex quark mass matrix of Eq.(2), but choose a slightly different weak-basis from the previous one in Eq.(3). We choose,

$$M = \begin{bmatrix} u & u & \hat{z} \\ \hat{u} & v & v \\ z & \hat{v} & w \end{bmatrix} \quad (18)$$

where $u(v - z)^2$ real. We follow the prescribed scheme, i.e., require symmetry and $\arg[\det(M)] = 0$, and obtain the ansatz,

$$M = \begin{bmatrix} u & u & z \\ u & u & u \\ z & u & w \end{bmatrix} = u \begin{bmatrix} 1 & 1 & z' \\ 1 & 1 & 1 \\ z' & 1 & w' \end{bmatrix} \quad (19)$$

where again $z' = z/u$ and $w' = w/u$. The procedure to solve this ansatz is as in the first example. However, here a different but general parameterization for z and w is better suited, (we mean in fact the z' and w' on the right-handed side of Eq.(19) but as before, in the following, leave out the prime). In order to obtain similar relations as in Eqs.(10, 12), we propose,

$$z = 1 + \rho_\circ e^{i\beta} \quad , \quad w = 1 + \rho_\circ e^{i\beta} + \rho e^{i\alpha} \quad (20)$$

and find that the diagonalization matrix elements V_{23}, V_{13} and V_{31} , (again in the "heavy weak-basis" of Eq.(16)), are somewhat different from the previous ansatz. We obtain in leading and next leading order,

$$\begin{aligned}
|V_{12}| &= \sqrt{\frac{m_1}{m_2}} \left[1 - \frac{m_1}{2m_2} - \frac{m_2}{4m_3} \cos(\alpha) \right]; & |V_{23}| &= \frac{1}{\sqrt{2}} \frac{m_2}{m_3} \left[1 + \sqrt{\frac{3m_1}{m_2}} \cos(\alpha - \beta) \right] \\
|V_{13}| &= \frac{1}{\sqrt{2}} \frac{\sqrt{m_1 m_2}}{m_3} \left[1 + \frac{5m_2}{4m_3} \cos(\alpha) \right]; & |V_{31}| &= \sqrt{\frac{3}{2}} \frac{m_1}{m_3} \left[1 - \sqrt{\frac{m_1}{3m_2}} \cos(\alpha - \beta) \right]
\end{aligned} \quad (21)$$

By now it should be clear how to obtain similiar symmetric ansätze near the democratic limit.

3 The particular case of USY

Next, we shall apply the procedure of the previous section to an important case, where the quark mass matrices are based on the hypothesis of a Universal Strength for Yukawa couplings. In particular, in this section we prove that ansätze of USY, thus obtained, coincide with other special USY-ansätze, which were classified recently, in Ref.[11], using different arguments of calculability, and for which all parameters are given by the quark mass ratios. These also predicted $|V_{us}| = (m_d/m_s)^{1/2}$ and $|V_{cb}| \sim m_s/m_b$.

In USY, it is assumed that there is only one universal Yukawa strength λ for all quarks. Two different Higgs doublets Φ_u, Φ_d give mass to the up and down quarks respectively, and all flavour dependence is in the phases of the Yukawa couplings. The quark mass matrices have the following form:

$$M_u = c_u [e^{i\phi_{ij}^u}] \quad , \quad M_d = c_d [e^{i\phi_{ij}^d}] \quad (22)$$

with $c_u = \lambda v_u, c_d = \lambda v_d$, where $v_u = \langle \Phi_u \rangle, v_d = \langle \Phi_d \rangle$ ¹.

We apply our scheme to the USY mass matrix. As can be seen from Eq.(22), the general USY matrix has 9 parameter-phases for each sector. However, one can choose some of the phases in Eq.(22) to be equal, in the same way as it was done for the matrix elements of the general case of Eq.(3). Then with the symmetry and $\arg[\det(M)] = 0$ procedure, we obtain a symmetric mass matrix of the following form:

$$M = c_\circ \begin{bmatrix} e^{ia} & e^{ia} & e^{-i(a+c)} \\ e^{ia} & e^{ia} & e^{ic} \\ e^{-i(a+c)} & e^{ic} & e^{ic} \end{bmatrix} = c'_\circ \begin{bmatrix} 1 & 1 & e^{iq} \\ 1 & 1 & e^{i(q-r)} \\ e^{iq} & e^{i(q-r)} & e^{i(q-r)} \end{bmatrix} \quad (23)$$

where $c'_\circ = c_\circ e^{ia}, q = -2a - c, r = -a - 2c$. Comparing Eq.(23) with Eq.(6), we see that this USY ansatz is a special case, where general complex numbers have been replaced by complex numbers of modulus one. With regard to the parameter space, we have, applying the parameterization of the general case, given in Eq.(9) for z and w in terms of ρ, ρ_\circ, α and β , to this USY ansatz,

$$\begin{aligned} z = 1 + \rho e^{i\alpha} = e^{iq} & \implies \rho = 2 |\sin(\frac{q}{2})| \quad , \quad \alpha = \pm \frac{\pi}{2} + \frac{q}{2} \\ w = 1 + \rho e^{i\alpha} - \rho_\circ e^{i\beta} = e^{i(q-r)} & \implies \rho_\circ = 2 |\sin(\frac{r}{2})| \quad , \quad \beta = \pm \frac{\pi}{2} - \frac{r-2q}{2} \end{aligned} \quad (24)$$

where the sign for the phases α and β depend on the sign of q and r respectively.

¹In our original paper [5], we did not discuss how a USY mass matrix could be obtained. This work was done recently by Fishbane and Hung with a minimum of six Higgs fields [14]

In principle, one can write the mass power series for the parameters q and r , derived from the series of ρ and ρ_o as given in Eq.(14). However, for this USY-ansatz there exist exact formulæ for q and r in term of δ and χ , the mass ratio invariants of Eq.(8):

$$\sin^2\left(\frac{r}{2}\right) = \frac{3}{4}\sqrt{3\delta} \quad , \quad \sin^2\left(\frac{q}{2}\right) = \frac{\frac{9}{16} \chi - \frac{9}{8}\sqrt{3\delta}}{1 - \frac{3}{4}\sqrt{3\delta}} \quad (25)$$

These exact relations are only possible because for the USY case, e.g., the function t_{zw} , related to the trace of H in Eq.(11), becomes very simple, and is equal to 3. One obtains, of course, the same leading order approximation relations as in Eq.(13):

$$|r| = 3\sqrt{3} \frac{\sqrt{m_1 m_2}}{m_3} \quad , \quad |q| = \frac{9}{2} \frac{m_2}{m_3}. \quad (26)$$

Next, we show that the USY ansatz of Eq.(23) is equivalent to one of the ansätze of Ref.[11]. In fact, the expressions of Eq.(25), for the phases r and q , were given in Ref.[11] with regard to a different USY-ansatz:

$$M = c_o \begin{bmatrix} 1 & e^{ir} & 1 \\ e^{iq} & 1 & e^{i(q-r)} \\ 1 & 1 & 1 \end{bmatrix} \quad (27)$$

which apparently does not correspond to our USY ansatz in Eq.(23), obtained with the symmetry and $\arg[\det(M)] = 0$ argument. What is the connection? Writing,

$$M = c_o \begin{bmatrix} 1 & e^{ir} & 1 \\ e^{iq} & 1 & e^{i(q-r)} \\ 1 & 1 & 1 \end{bmatrix} = c_o \cdot P_{23} \cdot \begin{bmatrix} 1 & 1 & e^{iq} \\ 1 & 1 & e^{i(q-r)} \\ e^{iq} & e^{i(q-r)} & e^{i(q-r)} \end{bmatrix} \cdot K_R \cdot P_{23} \quad (28)$$

where $K_R = \text{diag}(1, 1, e^{-i(q-r)})$, and P_{23} is the permutation of the second with the third quark field, it becomes obvious that our USY-ansatz in Eq.(23), and the USY ansatz of Ref.[11] in Eq.(27), are, in fact, equivalent. This is because these ansätze are related by a weak-basis transformation as in Eq.(28).

The diagonalization matrix elements for this USY ansatz can be read off from the matrix elements for the corresponding general case in Eq.(17), using the specific USY phases $\alpha = \pm\pi/2 + q/2$ and $\beta = \pm\pi/2 - (r - 2q)/2$. The diagonalization matrix elements, that one obtains in this way are, of course, the same as in Ref.[11].

To complete our discussion of the particular hypothesis of USY, we give a second USY example, derived in a similar way as the previous one, and which is related to the second general case in Eq.(19):

$$M = c_o \begin{bmatrix} e^{ia} & e^{ia} & e^{-i2a} \\ e^{ia} & e^{ia} & e^{ia} \\ e^{-i2a} & e^{ia} & e^{ic} \end{bmatrix} = c'_o \begin{bmatrix} 1 & 1 & e^{ip} \\ 1 & 1 & 1 \\ e^{ip} & 1 & e^{iq} \end{bmatrix} \quad (29)$$

where $c'_o = c_o e^{ia}$, $p = -3a$, $q = c - a$. Again, in Ref.[11] this ansatz was given in a different weak-basis. One can show the equivalence between the two, by explicitly writing the weak-basis relation:

$$M = c_o \begin{bmatrix} 1 & 1 & e^{ip} \\ 1 & 1 & 1 \\ e^{ip} & 1 & e^{iq} \end{bmatrix} = c_o \cdot P_{13} \cdot \begin{bmatrix} e^{ip'} & e^{-iq'} & 1 \\ e^{iq'} & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{\text{Ref.}[11]} \cdot P_{321} \cdot K_R \quad (30)$$

where $p' = q - p$, $q' = -p$, the P 's are self-evident permutations, and the phase unitary matrix $K_R = \text{diag}(1, 1, e^{ip})$. The diagonalization matrix elements are deduced in the same way as in the previous USY example.

Finally, one can prove that for each USY case in Ref.[11] there exists a corresponding general ansatz, which is obtained with our symmetry and $\arg[\det(M)] = 0$ scheme.

4 CP-violation and a numerical example

In this section, we analyse the CP-violation for a typical case of the ansätze that we constructed, and give a numerical example. From the Refs.[11], [12] we already know that the particular case of USY can accommodate the quark masses and mixings.

There are, however, two difficulties with the USY cases of Ref.[11]. The first is related to V_{us} . For these USY cases, one can choose the phases in V_{CKM} in such a way that, in leading order, V_{cs} , V_{ub} and V_{cb} are real, while,

$$V_{us} = \sqrt{\frac{m_d}{m_s}} \pm \sqrt{\frac{m_u}{m_c}} \quad (31)$$

where the sign is a matter of choice, dependent on the specific USY sign of the phase-parameters as explained for Eq.(24). If one combines the experimental limits on m_d/m_s , m_s , m_c , then the experimental value for $|V_{us}| = 0.2205(18)$ can only be accommodated if one takes a very small value for $m_u \leq 1 \text{ MeV}$ or even $m_u = 0$.

The second problem has to do with $J_{CP} = \text{Im}(V_{us}V_{cb}V_{cs}^*V_{ub}^*)$, which measures the CP violation [13]. In the USY ansätze of the Ref.[11], with the phase convention given above, only second and higher order terms of V_{cs} , V_{ub} , V_{cb} and V_{us} are complex and contribute to J_{CP} . The reason for this shall be made explicitly clear, but essentially this is because, in the USY cases mentioned, all phase parameters, which enter in the mass matrix elements, are small, thus, a mass matrix element like e^{iq} in the ansatz of Eq.(23), is in leading order equal to $1 + iq$. The mass matrix M is, therefore, in this order, equal to the democratic mass matrix plus a small imaginary matrix. The diagonalization matrix elements V_{12} , V_{22} , V_{23} and V_{13} , of both quark sectors get the same phase factor which cancels out in the CKM matrix product, and only higher order terms give a contribution to J_{CP} . One finds $|J_{CP}| = o(10^{-6} - 10^{-7})$, and it is impossible to obtain a large value for $|J_{CP}|$.

In fact, the two problems are related². If it were possible to exchange the \pm sign in the expression for V_{us} of Eq.(31) for a phase factor $e^{i\delta}$, while at the same time keeping V_{cs} , V_{ub} and V_{cb} real, then one would solve both problems. The new relation,

$$|V_{us}| = \left| \sqrt{\frac{m_d}{m_s}} + e^{i\delta} \cdot \sqrt{\frac{m_u}{m_c}} \right| \quad (32)$$

would support a larger value for m_u , and one would find in leading order for $|J_{CP}|$, supposing that $|\sin(\delta)|$ is large,

$$|J_{CP}| = |V_{us}V_{cb}V_{cs}V_{ub}| \cdot |\sin(\delta)| \quad (33)$$

The solution to both problems lies, therefore, in choosing a different complex phase content for the mass matrices of the two quark sectors. Let us specify more precisely how to obtain this. First we have to choose a specific ansatz. We find it appealing to have similar ansätze for the down as well as for the up quarks, and propose (the already discussed) mass matrices of the form:

$$M_u = c_u \begin{bmatrix} 1 & 1 & z_u \\ 1 & 1 & w_u \\ z_u & w_u & w_u \end{bmatrix}, \quad M_d = c_d \begin{bmatrix} 1 & 1 & z_d \\ 1 & 1 & w_d \\ z_d & w_d & w_d \end{bmatrix} \quad (34)$$

In order to find out what the contribution of the CP-phase is to V_{us} , we write the diagonalization equation in the following form: $V \cdot D \cdot V^\dagger = H$, (the indices $\{u,d\}$ have been dropped to simplify), where H is given in the "heavy-basis" of Eq.(16), and $D = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$; the λ_i are the eigenvalues of H as given in Eq.(8). One obtains, thus, (also using the unitarity of V):

$$\begin{aligned} (\lambda_2 - \lambda_1) \cdot V_{12}V_{32}^* + (\lambda_3 - \lambda_1) \cdot V_{13}V_{33}^* &= H_{13} \\ (\lambda_2 - \lambda_1) \cdot V_{22}V_{32}^* + (\lambda_3 - \lambda_1) \cdot V_{23}V_{33}^* &= H_{23} \\ (\lambda_2 - \lambda_1) \cdot V_{12}V_{22}^* + (\lambda_3 - \lambda_1) \cdot V_{13}V_{23}^* &= H_{12} \end{aligned} \quad (35)$$

With the general parametrization $z = 1 + \rho e^{i\alpha}$ and $w = 1 + \rho e^{i\alpha} - \rho_\circ e^{i\beta}$, for H in Eq.(7), one finds that H_{12} is an exact real number, while, in first order, $H_{13} = (1/\sqrt{6}) \rho_\circ e^{i\beta}$ and $H_{23} = (2\sqrt{2}/3) \rho e^{i\alpha}$. Therefore, in leading order, the following expressions, relating the phases α and β of the parameterization of z and w with the phases of V_{13} and V_{23} , hold:

$$\begin{aligned} V_{13} &= |V_{13}| e^{i\beta} \\ V_{23} &= |V_{23}| e^{i\alpha} \end{aligned} \quad (36)$$

where we have used the possibility to choose the phase of some of the elements of V ; in this case $V_{33} = \text{real}$. Furthermore, combining the expressions for V_{12} , V_{23} and V_{13} in Eq.(17), one gets the leading order relation: $|\lambda_2 \cdot V_{12}V_{22}^*| = |\lambda_3 \cdot V_{13}V_{23}^*|$. Then, using $H_{12} = \text{real}$, we find with Eq.(35), also in this order,

$$\underline{\arg(V_{12}V_{22}^*) = -\arg(V_{13}V_{23}^*)}, \quad \arg(V_{11}V_{21}^*) = -\arg(V_{12}V_{22}^*) \quad (37)$$

²Fritzsch pointed this out in the context of a different ansatz [15]

where the second relation follows from unitarity.

Finally combining the results for the phases of the matrix elements in Eq.(36) and Eq.(37) for both sectors V_d and V_u , and ignoring higher order contributions, we obtain for the CP violating phase in V_{us} :

$$\delta = \pi + (\alpha_d - \beta_d) - (\alpha_u - \beta_u) \quad (38)$$

This equation explains exactly why, for the USY cases of Ref.[11], the CP violation is so small: for these USY cases, the α 's and the β 's are all equal to $\pm\pi/2$, in leading order (see e.g. Eq.(24)).

With regard to our numerical example, we specify the ansatz in Eq.(34) further, by explicitly giving all the phases of the mass matrix elements $z_{u,d}$, $w_{u,d}$. We keep the USY ansatz of Eq.(23) for the down sector, but introduce for the up sector, matrix elements with moduli that are not anymore equal to 1. We propose the following ansatz:

$$M_u = c_u \begin{bmatrix} 1 & 1 & z_u \\ 1 & 1 & w_u \\ z_u & w_u & w_u \end{bmatrix} \quad ; \quad M_d = c_d \begin{bmatrix} 1 & 1 & e^{iq_d} \\ 1 & 1 & e^{i(q_d-r_d)} \\ e^{iq_d} & e^{i(q_d-r_d)} & e^{i(q_d-r_d)} \end{bmatrix} \quad (39)$$

where for the up sector:

$$z_u = 1 + q_u e^{i\pi/3} \quad ; \quad w_u = 1 + q_u e^{i\pi/3} - r_u \quad (40)$$

The difference in the phases of the matrix elements of the up and down sectors is evident:

$$\begin{aligned} \alpha_u &= \frac{\pi}{3} & \alpha_d &= \pm\frac{\pi}{2} + \frac{q_d}{2} \\ & & & ; \\ \beta_u &= 0 & \beta_d &= \pm\frac{\pi}{2} - \frac{r_d-2q_d}{2} \end{aligned} \quad (41)$$

where the \pm sign for the down sector depends, as was explained, on the sign of q_d and r_d . Remember also that r_d and q_d are small. Taking different signs for r_d and q_d , we get with Eq.(38) in leading order (mod 2π),

$$\delta = -\frac{\pi}{3} \quad (42)$$

From the Eqs.(32,33) one would expect that a choice for $\delta = \pm\pi/2$ is better suited to support a large m_u value and a large $|J_{CP}|$. However, numerically one must also adjust the other CKM matrix elements, and J_{CP} depends not only on the CP-phase δ , but also on the moduli of V_{us}, V_{cb}, V_{cs} and V_{ub} . We have found the best fit for all experimental results with a CP-phase $\delta = -\pi/3$, the following parameters and corresponding quark masses:

Input:

$$\begin{aligned} q_u &= 1.891 \times 10^{-2} & r_u &= 1.074 \times 10^{-3} \\ q_d &= -9.264 \times 10^{-2} & r_d &= 2.205 \times 10^{-2} \end{aligned} \quad (43)$$

which correspond to mass ratios given by the masses, at 1 *GeV* [16]:

$$\begin{aligned} m_u &= 3.4 \text{ MeV} & m_c &= 1.43 \text{ GeV} & m_t &= 340 \text{ GeV} \\ m_d &= 5.3 \text{ MeV} & m_s &= 135 \text{ MeV} & m_b &= 6.3 \text{ GeV} \end{aligned} \quad (44)$$

Output:

$$|V_{CKM}| = \begin{bmatrix} 0.9754 & 0.2206 & 0.0029 \\ 0.2204 & 0.9746 & 0.0395 \\ 0.0106 & 0.0382 & 0.9992 \end{bmatrix} \quad (45)$$

In this numerical example no approximations were made. The ratio $|V_{ub}/V_{cb}| = 0.074$ is perfectly well within its experimental limit, $|V_{ub}/V_{cb}|^{\text{Exp}} = 0.08 \pm 0.02$. We get for the CP violation $|J_{CP}| = 2.02 \cdot 10^{-5}$. This value corresponds to a CP-phase $|\sin(\delta)| = 0.816$, which is somewhat smaller (8%) than the first order prediction $\sin(\pi/3) = 0.886$, because of the influence the higher order contributions to δ .

5 Concluding remarks

We have shown how to construct a class of symmetric ansätze near the democratic limit, which reproduce the important phenomenological mixing matrix relations where $|V_{us}| = (m_d/m_s)^{1/2}$ and $|V_{cb}| \sim m_s/m_b$. We have proven that the recently classified USY-ansätze of Ref.[11], which also reproduce these mixing matrix expressions, are particular examples within this class. In addition, we have also shown, for an ansatz-example of the constructed class, how the CP-violation phase can be computed. For this example, all the experimental values of the quark masses and mixings, including the CP violation phase, can be accommodated with great success.

We find it very surprising that, with this class, the important issues of symmetry and $\arg[\det(M)] = 0$, for the quark mass matrices, become suggestively linked with the expected phenomenological mixing matrix relations in terms of quark mass ratios.

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