## Quantum Groups as Flavor Symmetries: Account of Nonpolynomial SU(3)-Breaking Effects in Baryon Masses

A.M. Gavrilik<sup>1</sup>, N.Z. Iorgov

Bogolyubov Institute for Theoretical Physics, Metrologichna str. 14b, Kiev-143, Ukraine

## Abstract

The use, for flavor symmetries, of the quantum (or  $q$ -) analogs of unitary groups  $SU(n_f)$  yields new, very accurate, baryon mass sum rules. We show, in the 3-flavor case, that such approach accounts for nonpolynomial  $SU(3)$ -breaking effects in the octet and decuplet baryon masses. A version of this approach with manifestly qcovariant mass operator is given. The obtained new version of the q-deformed mass relation is simpler than those derived before, but, for its empirical validity, the parameter q is to be fixed by fitting. As shown, the well-known Gell-Mann–Okubo octet mass sum rule results, besides usual  $SU(3)$ , also from an exotic "symmetry" encoded in the singular case  $q = -1$  of the q-algebra  $U_q(su_3)$ .

<sup>1</sup>E-mail: omgavr@bitp.kiev.ua

1. Application of quantum groups and quantum, or q-deformed, algebras [1] to diverse physical problems remains the subject of intensive study (see, e.g., overviews [2]). In recent years, some attempts were made to exploit various deformed algebras in the context of hadron phenomenology, both in the scattering sector [3] and in the sector of such static properties as (ground state) hadron masses [4-8]. It was found, in particular, that  $U_q(su_n)$ based approach enables one to obtain mass sum rules of an accuracy essentially improved not only with respect to the equal spacing rule (ESR)  $M_{\Omega}-M_{\Xi^*}=M_{\Xi^*}-M_{\Sigma^*}-M_{\Delta^*}-M_{\Delta^*}$ for decuplet baryons, but also in comparison to the slightly better decuplet formula [9,10]  $M_{\Omega} - M_{\Delta} = 3(M_{\Xi^*} - M_{\Sigma^*})$ , and even in comparison to the famous Gell-Mann–Okubo (GMO) octet sum rule [11]  $M_N + M_{\Xi} = \frac{3}{2}M_{\Lambda} + \frac{1}{2}M_{\Sigma}$  which holds with 0.58% accuracy.

Our main purpose in this note is to demonstrate transparently that the application of the q-algebras  $U_q(su_n)$  taken for flavor symmetries of hadron dynamics efficiently takes into account the  $SU(3)$ -breaking effects which are of highly nonlinear nature, namely, nonpolynomial effects. As another result, we obtain, utilizing properties of q-tensor operators and the important ingredients of the Hopf algebra structure of  $U_q(su_n)$  (comultiplication, antipode), a different version of the  $q$ -deformed mass relation, see (29) below. The new interesting feature of this relation is that it produces the classical GMO sum rule not only in the classical limit  $q = 1$  but also in the nonclassical situation of  $q = -1$ . A description of the mathematical structure (operator algebras) corresponding to  $q = -1$  is also given.<sup>2</sup>

2. In order to describe mass splittings for particles from isomultiplets within the octet of baryons  $J^P = \frac{1}{2}$ 2 <sup>+</sup>, we adopt, like in [4-7], that the algebra of hadron flavor symmetry in the 3-flavor case is not su<sub>3</sub>, but its q-analogue  $U_q(su_3)$  (broken down to the isospin algebra  $U_q(su_2)$ . We use a correspondence between baryons of the octet and basis vectors of the carrier space of irreducible representation (irrep) 8 of the highest weight  $\{p+2, p+1, p\}$ ,  $p \in \mathbf{Z}$ , of  $U_q(su_3)$ .. Here and below, we will take the highest weight of irrep of  $U_q(su_n)$  as that of  $U_q(u_n)$ , thus identifying two weights of the latter if their difference is a weight of the form  $\{s, s, \ldots, s\}$ ,  $s \in \mathbb{Z}$  (our final results will be insensitive w.r.t. such a difference). Let us remark that necessary details concerning the "compact"  $U_q(u_n)$  and "noncompact"  $U_q(u_{n,1})$  algebras as well as their irreps can be found, e.g., in [6].

Since the algebra  $U_q(u_2)$  will be always unbroken within our treatment (exact isospin– hypercharge symmetry of strong interaction), we have mass degeneracy within each isomultiplet of octet and, thus, 4 different masses within the octet. Isomultiplets from the octet are put into correspondence with the following representations  ${m_{12}, m_{22}}$  of  $U_q(u_2)$ :

$$
N \leftrightarrow \{p+2, p+1\} , \qquad \Sigma \leftrightarrow \{p+2, p\} ,
$$
  

$$
\Lambda \leftrightarrow \{p+1, p+1\} , \qquad \Xi \leftrightarrow \{p+1, p\} ,
$$

<sup>&</sup>lt;sup>2</sup>The quantum algebra  $U_q(su_n)$  is known to be undefined in this case [1,12].

and will be denoted as  $|B_i\rangle$ , where i runs over isoplets of the octet, i.e.  $i = 1, ..., 4$ . The masses  $M_{B_i}$  of particles are given as diagonal matrix elements

$$
M_{B_i} = \langle B_i | \hat{M} | B_i \rangle \tag{1}
$$

of the mass operator [6,5]

$$
\hat{M} = \hat{M}_0 + \alpha \hat{L} + \beta \hat{R},
$$
\n
$$
\hat{M}_0 = \widetilde{M}_0 \mathbf{1} + \gamma A_{45} A_{54} + \delta A_{54} A_{45},
$$
\n
$$
\hat{L} = A_{35} \tilde{A}_{53} + \tilde{A}_{35} A_{53} , \qquad \hat{R} = A_{53} \tilde{A}_{35} + \tilde{A}_{53} A_{35}
$$
\n(3)

which is composed from operators representing basis elements of the  $U_q(su_5)$  algebra <sup>3</sup> and acting in the space of "dynamical" representation  ${m_{15}, p+1, p, m_{55}}$  of this algebra.

In accordance with the chain of embeddings of  $q$ -algebras

$$
U_q(su_3) \subset U_q(su_4) \subset U_q(su_5)
$$

there is the corresponding embedding of the representations under consideration:

$$
\{p+2, p+1, p\} \subset \{p+2, p+1, p, p\} \subset \{m_{15}, p+1, p, p, m_{55}\}.
$$
 (4)

It is clear from the action formulas of  $A_{45}$  and  $A_{54}$  (see Jimbo in [1], and [12]) that the operators  $A_{45}A_{54}$  and  $A_{54}A_{45}$  do not give splitting between the isoplet masses  $M_{B_i}$ , but only shifting the common background mass  $\widetilde{M}_0$ . For that reason, we put  $\langle B_i | \hat{M}_0 | B_i \rangle = M_0$ .

Diagonal matrix elements of the operators  $\hat{L}$  and  $\hat{R}$  in (3) can be rewritten as <sup>4</sup>

$$
\langle B_i|(A_{35}\tilde{A}_{53} + \tilde{A}_{35}A_{53})|B_i\rangle
$$
  
\n
$$
= \langle B_i|([2]A_{34}A_{45}A_{54}A_{43} + [2]A_{45}A_{34}A_{43}A_{54} - 4A_{34}A_{45}A_{43}A_{54})|B_i\rangle,
$$
\n
$$
\langle B_i|(A_{53}\tilde{A}_{35} + \tilde{A}_{53}A_{35})|B_i\rangle
$$
  
\n
$$
= \langle B_i|([2]A_{54}A_{43}A_{34}A_{45} + [2]A_{43}A_{54}A_{45}A_{34} - 4A_{54}A_{43}A_{45}A_{34})|B_i\rangle
$$
  
\n
$$
= [2]\langle B_i|A_{54}A_{43}A_{34}A_{45}|B_i\rangle.
$$
\n(6)

The last equality in (6) is a consequence of the particular choice of representations in (4). Matrix elements (5) and (6) are evaluated [6] in the framework of the Gel'fand- Tsetlin formalism. Here, we take into account the identifications (we put  $p = 0$ )

<sup>&</sup>lt;sup>3</sup> We embed the  $U_q(su_3)$  algebra into  $U_q(su_4)$  and further into  $U_q(su_5)$ , in order to lift the  $\Lambda-\Sigma$  mass degeneracy.

<sup>&</sup>lt;sup>4</sup> Symbol [x] denotes the q-number  $[x] \equiv [x]_q \equiv (q^x - q^{-x})/(q - q^{-1})$  corresponding to the number x; the latter, conversely, results as the 'classical' limit of [x], that is,  $[x] \rightarrow x$  if  $q \rightarrow 1$ .

$$
m_{12} = \frac{Y}{2} + I + 1
$$
,  $m_{22} = \frac{Y}{2} - I + 1$ ,

where  $Y$  and  $I$  are hypercharge and isospin, respectively (their values label each isoplet unambiguously), and obtain for the summands in (5) and (6) the following expressions:

$$
\langle B_i | A_{45} A_{34} A_{43} A_{54} | B_i \rangle = -[m_{15} + 4][m_{55}][6]^{-1}[2 - Y], \tag{7}
$$

$$
\langle B_i | A_{34} A_{45} A_{43} A_{54} | B_i \rangle = -[m_{15} + 4][m_{55}][6]^{-1}[1 - Y], \tag{8}
$$

$$
\langle B_i | A_{34} A_{45} A_{54} A_{43} | B_i \rangle
$$
  
= 
$$
\frac{[Y/2][Y/2+1] - [I][I+1]}{[2][3]} \left( \frac{[m_{15} + 1][m_{55} - 3]}{[3]} - \frac{[2][m_{15} + 4][m_{55}]}{[6]} \right)
$$
  
- 
$$
\frac{[Y/2 - 1][Y/2 - 2] - [I][I+1]}{[2][5]} \left( [m_{15} - 1][m_{55} - 5] + \frac{[4]}{[6]}[m_{15} + 4][m_{55}] \right), (9)
$$
  
[2] $\langle B_i | A_{54} A_{43} A_{34} A_{45} | B_i \rangle \sim [m_{15} - 3][m_{55} - 7] = [3 - Y/2][2 - Y/2] - [I][I + 1].(10)$ 

Exact coefficient of proportionality in the matrix element (10) is unimportant, since it can be absorbed by redefining of  $\beta$ .

It is clearly seen from the definition of  $q$ -quantities (see footnote 3) that baryon masses which follow from (7)-(10) depend on hypercharge Y and isospin I (and, hence, on  $SU(3)$ breaking effects) in highly nonlinear – nonpolynomial – fashion.

Substitution of (7)-(10) in (5),(6) and then in (1) gives final expressions for  $M_N$ ,  $M_{\Xi}$ ,  $M_{\Lambda}$ ,  $M_{\Sigma}$ . Excluding from these the unknown constants  $M_0$ ,  $\alpha$  and  $\beta$ , we obtain the  $q$ -deformed mass relations of the form [6,7,13]

$$
[2]M_N + \frac{[2]}{[2]-1}M_{\Xi} = [3]M_{\Lambda} + \left(\frac{[2]^2}{[2]-1} - [3]\right)M_{\Sigma} + \frac{A_q}{B_q}(M_{\Xi} + [2]M_N - [2]M_{\Sigma} - M_{\Lambda}),
$$
\n(11)

where  $A_q$  and  $B_q$  are certain polynomials of  $[2]_q$  with non-coinciding sets of zeros. It should be emphasized that different dynamical representations, see the dependence on  $m_{15}$  and  $m_{55}$  in (7)-(10), produce different pairs  $A_q$ ,  $B_q$ . Any  $A_q$  (rewritten in factorized form) possesses the factor ( $[2]_q - 2$ ) and, thus, the "classical" zero  $q = 1$ . In the limit  $q = 1$ , each q-deformed mass relation reduces to the standard GMO sum rule for octet baryons. At some value(s) of q which are zeros (other than  $q = 1$ ) of particular  $A<sub>q</sub>$ , we obtain mass sum rules which hold with better accuracy than the GMO one. The two mass sum rules

$$
M_N + \frac{1 + \sqrt{3}}{2} M_{\Xi} = \frac{2}{\sqrt{3}} M_\Lambda + \frac{9 - \sqrt{3}}{6} M_\Sigma , \qquad (12)
$$

$$
M_N + \frac{1}{[2]_{q_7} - 1} M_{\Xi} = \frac{1}{[2]_{q_7} - 1} M_{\Lambda} + M_{\Sigma} ,
$$
\n(13)

where  $[2]_{q7} = 2 \cos(\pi/7)$ , are obtained [6,7,13] from two different dynamical representations  $D^{(1)}$  and  $D^{(2)}$  with corresponding polynomials  $A_q^{(1)}$  and  $A_q^{(2)}$ , by fixing zeros  $q = \exp(i\pi/6)$  and  $q = \exp(i\pi/7)$ , respectively. These sum rules show the precision of, resp., 0.22% and 0.07%, which is essentially better than the precision 0.58% of GMO. The case corresponding to  $q = \exp(i\pi/7)$  turns out [13] to be the best possible one. It was supposed in [7] that this value of q may be interpreted as follows:  $\pi/7 = 2\theta_C$  (Cabibbo angle).

3. Since our goal is to analyze a high nonlinearity in  $SU(3)$  breaking effectively accounted by the model, let us first check the classical limit  $q \to 1$  for the expressions (7)-(10). These can be summarized to give the formula

$$
M_{B_i} = M_{B(Y,I)} = M_0 + \alpha Y + \beta (Y^2/4 - I(I+1))
$$
\n(14)

for masses from the octet, by suitable redefinition of  $M_0, \alpha, \beta$ . This relation coincides with the mass formula of Gell-Mann and Okubo [9,11], as it should be.

Now consider, at q which is close to 1 and taken to be pure phase:  $q = e^{ih}$ , first few terms of the Tailor expansion for the expressions  $(7)-(10)$  which enter baryon octet masses. Using the formulas

$$
[n] = \frac{\sin(nh)}{\sin(h)} = n - \frac{n(n^2 - 1)}{6}h^2 + O(h^4),
$$
  
\n
$$
[n][n+1] = n(n+1) - h^2 \left(\frac{(n(n+1))^2}{3} - \frac{n(n+1)}{6}\right) + O(h^4)
$$
\n(15)

valid for small h, from  $(7)-(10)$  we get

$$
M_{B_i} = M_0 + \alpha \Big( \frac{-((Y/2)(Y/2+1) - I(I+1))(10/3 - 161h^2 + 5h^2/9)}{+((Y/2-1)(Y/2-2) - I(I+1))(8 - 84h^2 + 4h^2/3)} + \frac{((Y/2)^2(Y/2+1)^2 - (I(I+1))^2)(10h^2/9)}{-(Y/2-1)^2(Y/2-2)^2 - (I(I+1))^2)(8h^2/3)} \Big)
$$
  
+  $\beta \Big( \frac{((Y/2-3)(Y/2-2) - I(I+1))(1 + h^2/6)}{-(Y/2-3)^2(Y/2-2)^2 - (I(I+1))^2)(h^2/3)} \Big)$ , (16)

where, for simplicity, the choice  $m_{15} = 9$ ,  $m_{55} = 0$  has been fixed.

It is instructive to compare this result with the expansion in [9] in terms of the  $SU(3)$ breaking interaction, whose lowest orders involve summands with the following dependences on hypercharge and isospin:

$$
Y, \quad (Y^{2}/4 - I(I+1)) - 1st \text{ order terms in } SU(3) \text{ breaking}
$$
  
\n
$$
Y^{2}, \quad Y(Y^{2}/4 - I(I+1)), \quad (Y^{2}/4 - I(I+1))^{2} - 2nd \quad \text{'''}.
$$
  
\n
$$
Y^{3}, \quad Y^{2}(Y^{2}/4 - I(I+1)), \quad Y(Y^{2}/4 - I(I+1))^{2}, \quad (Y^{2}/4 - I(I+1))^{3} - 3rd \quad \text{''''}
$$
  
\n
$$
Y^{4}, \quad Y^{3}(Y^{2}/4 - I(I+1)), \quad \dots - 4th \quad \text{''''}.
$$

Let us remark that the 1st order (the top row above) is the highest possible one that allows getting a mass sum rule for octet baryons in the traditional approach [9] which treats the constants assigned to different terms as independent. On the other hand, dependence of baryon masses on q through the q-quantities like  $(15)$  determines unambiguously the coefficients of expansions in h and, thus, there appear no new parameters. This, together with the fact that the deformation parameter  $q$  appears in (expressions for) baryon masses through representation matrix elements (due to deformation of symmetry), shows a sharp distinction of the dimensionless parameter q from the dimensionful constants  $M_0, \alpha, \beta$ , introduced explicitly in the mass operator (2) as symmetry breaking parameters.

It is seen that the first order terms in  $h^2$  in the expression (16) correspond to terms up to the 4th order of Okubo's expansion, some of which are  $(Y^2/4 - I(I+1))^2$ ,  $Y^2(Y^2/4 - I(I+1))^2$  $I(I + 1)$ ,  $Y<sup>4</sup>$ . On the other hand, there is the 2nd order term in Okubo's expansion, namely  $Y(Y^2/4-I(I+1))$ , which does not appear in the expression (16). This means that the expansion in terms of small  $h^2$  is consistent with, but not the same as, the expansion in terms of  $SU(3)$  breaking.

4. Formula (14) is valid not only for octet baryons but also for the  $J^P = \frac{3}{2}$ 2 + baryons from decuplet 10 of  $U_q(su_3)$ . Taking into account the specific property that, for the decuplet, hypercharge and isospin obey the relation

$$
I = 1 + Y/2 \tag{17}
$$

one can rewrite (14) in the form  $M = M_0 + \alpha Y$ , which produces the equal spacing rule for decuplet baryon masses.

It is easy to see that the expressions (7)-(10) which lead in the q-deformed ( $q \neq 1$ ) case to octet baryon masses are equally well applicable in the decuplet case. Indeed, using (17) and the easily verifiable identity

$$
[x - Y/2][x + 1 - Y/2] - [I][I + 1] = -[Y - x + 1][x + 2]
$$

valid for all  $x$ , we arrive at the  $q$ -average formula for masses of decuplet baryons:

$$
\frac{M_{\Omega} - M_{\Xi^*} + M_{\Sigma^*} - M_{\Delta}}{[2]} = M_{\Xi^*} - M_{\Sigma^*} \tag{18}
$$

This formula previously was obtained in [5] within somewhat another context and shown to possess the important property of universality (independence on the choice of a dynamical representation of  $U_q(su_5)$  or  $U_q(su_{4,1})$  [5], under the only condition that such a dynamical representation contains the 20-plet  $\{p+3, p, p, p\}$  of  $U_q(su_4)$  in which the  $U_q(su_3)$ -decuplet is embedded). For example, within the dynamical irrep  $\{p+4, p, p, p, p\}$ of  $U_q(su_5)$ , the masses are  $M_{\Delta} = M_{10} + \beta$ ,  $M_{\Sigma^*} = M_{10} + \alpha + [2]\beta$ ,  $M_{\Xi^*} = M_{10} + [2]\alpha + [3]\beta$ ,  $M_{\Omega} = M_{10} + [3]\alpha + [4]\beta$ , and these obviously satisfy (18). Since each isoplet from the baryon decuplet is uniquely fixed by its strangeness (or hypercharge) value, all these expressions can be comprised by the single mass formula

$$
M_{B_i^*} = M_{10} + \alpha [1 - Y] + \beta [2 - Y], \qquad (19)
$$

where  $B_i^*$  runs over four different isoplets in 10-plet. From definition of q-numbers, it follows that the dependence of both quantities:  $[1 - Y]$ ,  $[2 - Y]$  on hypercharge Y is essentially nonlinear and becomes linear only in the classical (non-deformed) limit  $q = 1$ .

Comparison of the relation (18) with empirical data for baryon  $J^P = \frac{3}{2}$  $\frac{3}{2}^+$  masses [14] is successful if q is fixed as  $q = exp(i\theta_{10}), \theta_{10} \simeq \pi/14$ . The latter angle, as argued in [7], can be juxtaposed with the Cabibbo angle  $\theta_C$ .

5. Up to now we used only a representation-theoretic part of the structure of the quantum algebra  $U_q(su_n)$ . In this section, we follow somewhat different approach and treat the mass operator on the base of  $q$ -tensor operators. To this end, below we will need such ingredients of the Hopf algebra structure of  $U_q(su_n)$  as comultiplication  $\Delta$  and antipode S operations. These are defined on the generators  $E_i^+ \equiv A_{i,i+1}, E_i^- \equiv A_{i+1,i}$  and  $H_i \equiv A_{ii} - A_{i+1,i+1}$  according to the formulas [1]:

$$
\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i \quad , \quad S(H_i) = -H_i \quad , \quad S(q^{H_i/2}) = q^{-H_i/2} \quad , \quad S(1) = 1,
$$

$$
\Delta(E_i^{\pm}) = E_i^{\pm} \otimes q^{H_i/2} + q^{-H_i/2} \otimes E_i^{\pm} , \quad S(E_i^{\pm}) = -q^{H_i/2} E_i^{\pm} q^{-H_i/2} . \tag{20}
$$

Consider the adjoint action of  $U_q(su_n)$  defined as [12]

$$
ad_AB = \sum A_{(1)}BS(A_{(2)}) ,
$$

where  $A, B \in U_q(su_n)$ ,  $A_{(1)}$  and  $A_{(2)}$  are defined by comultiplication  $\Delta(A) = \sum A_{(1)} \otimes A_{(2)}$ . With the account of (20) this yields

$$
ad_{H_i}B = H_iB1 + 1BS(H_i) = H_iB - BH_i,
$$

$$
ad_{E_i^{\pm}}B = E_i^{\pm}Bq^{-H_i/2} - q^{-H_i/2}Bq^{H_i/2}E_i^{\pm}q^{-H_i/2}.
$$

Below we will need the *q*-tensor operators [15]  $(V_1, V_2, V_3)$  and  $(V_1, V_2, V_3)$ , which transform under  $U_q(su_3)$  as 3 and 3<sup>\*</sup>, respectively. Let us denote  $[X, Y]_q \equiv XY - qYX$ . Direct calculations show that the triple of elements from  $U_q(su_4)$ ,

$$
V_1 = [E_1^+, [E_2^+, E_3^+]_q]_q q^{-H_1/3 - H_2/6} , \quad V_2 = [E_2^+, E_3^+]_q q^{H_1/6 - H_2/6} ,
$$
  
\n
$$
V_3 = E_3^+ q^{H_1/6 + H_2/3} ,
$$
\n(21)

transforms as 3 under the adjoint action of  $U_q(su_3)$ . Moreover,  $V_1$  corresponds to the highest weight vector, the pair  $(V_1, V_2)$  is an isodoublet and  $V_3$  isosinglet under  $U_q(su_2)$ .

Likewise, by direct calculation it can be shown that the triple of elements from  $U_q(su_4)$ ,

$$
V_{\bar{1}} = q^{-H_1/3 - H_2/6} [E_1^-, [E_2^-, E_3^-]_{q^{-1}}]_{q^{-1}} , \quad V_{\bar{2}} = q^{H_1/6 - H_2/6} [E_2^-, E_3^-]_{q^{-1}} ,
$$
  
\n
$$
V_{\bar{3}} = q^{H_1/6 + H_2/3} E_3^-,
$$
\n(22)

transforms as  $3^*$  under the adjoint action of  $U_q(su_3)$ . Moreover,  $V_3$  corresponds to the highest weight vector, the pair  $(V_{\bar{1}}, V_{\bar{2}})$  is an isodoublet and  $V_{\bar{3}}$  isosinglet under  $U_q(su_2)$ .

As before, we take  $U_q(su_3)$  broken down to  $U_q(su_2)$  as the algebras of global internal symmetry of hadrons and make use of the correspondence between baryons  $J^P = \frac{1}{2}$ 2 + and basis vectors in the representation space of 8 as well as between baryons  $J^P = \frac{3}{2}$ 2 + and basis vectors in the representation space of 10. Like in the case of the usual nondeformed flavor symmetry algebra  $su(3)$  broken down to its subalgebra  $su(2)$ , we take the mass operator in the form

$$
\hat{M} = \hat{M}_0 + \hat{M}_8 \tag{23}
$$

where  $\hat{M}_0$  is a scalar of  $U_q(su_3)$ ,  $\hat{M}_8$  is the operator which transforms as the  $I=0, Y=0$ component of the tensor operator of 8-irrep of  $U_q(su_3)$ .

If  $|B_i\rangle$  is a basis vector of the representation 8 (or 10) space which corresponds to some baryon with spin  $J = 1/2$  (or  $J = 3/2$ ), respectively, then  $M_{B_i} = \langle B_i | \hat{M} | B_i \rangle$  is the mass of this baryon, see (1).

Consider first the case of octet baryons. The irrep 8 occurs twice in the decomposition

$$
8 \otimes 8 = 1 \oplus 8^{(1)} \oplus 8^{(2)} \oplus 10^* \oplus 10 \oplus 27 . \qquad (24)
$$

This fact, the Wigner-Eckart theorem for  $U_q(su_n)$  quantum algebras [15] applied to qtensor operators transforming as irrep 8 of  $U_q(su_3)$ , and symmetry properties [16] of q-Clebsch– Gordan coefficients lead us to the conclusion that the mass operator is of the form

$$
\hat{M} = M_0 \mathbf{1} + \alpha V_8^{(1)} + \beta V_8^{(2)} \tag{23'}
$$

and the baryon masses are calculated as

$$
M_{B_i} = \langle B_i | (\hat{M}_0 + \hat{M}_8) | B_i \rangle = \langle B_i | (M_0 \mathbf{1} + \alpha V_8^{(1)} + \beta V_8^{(2)}) | B_i \rangle . \tag{25}
$$

Here 1 is the identity operator,  $V_8^{(1)}$  and  $V_8^{(2)}$  are two fixed tensor operators with nonproportional matrix elements, which both have the same transformation property as the  $I = 0, Y = 0$  component of irrep 8 of  $U_q(su_3)$  (i.e., the same as that of  $\hat{M}_8$ );  $M_0$ ,  $\alpha$  and  $\beta$ are some unknown constants depending on details (dynamics) of the model.

In the decuplet case, operator  $(23')$  is equivalent to the operator

$$
\hat{M} = M_0 \mathbf{1} + \tilde{\alpha} V_8 \ .
$$

This follows from the fact that the irrep 8 occurs only once in the decomposition

$$
{\bf 10}^*\otimes{\bf 10}={\bf 1}\oplus{\bf 8}\oplus{\bf 27}\oplus{\bf 64}\ .
$$

Formally, we may use the operator  $(23')$  in the case of decuplet baryons, too. But in this case, the matrix elements of  $V_8^{(1)}$  and  $V_8^{(2)}$  become proportional to each other, and that effectively leads to a single constant  $\tilde{\alpha}$  instead of the two  $\alpha$  and  $\beta$  in (23').

From the decompositions

$$
3 \otimes 3^* = 1 \oplus 8, \qquad 3^* \otimes 3 = 1 \oplus 8 , \qquad (26)
$$

it is seen that the operators  $V_3$   $V_3$  and  $V_3$   $V_3$  from  $(21),(22)$  are just the two isosinglets needed in the context of equation (23′ ). It follows from decompositions (26) that each of them transforms as the sum of a singlet (scalar) and the  $I = 0, Y = 0$  component of the octet. Hence, the mass operator (23') can be rewritten (after redefinition of  $M_0, \alpha, \beta$ ) in the equivalent form

$$
\hat{M} = M_0 \mathbf{1} + \alpha V_3 V_{\bar{3}} + \beta V_{\bar{3}} V_3
$$

or

$$
\hat{M} = M_0 \mathbf{1} + \alpha E_3^+ E_3^- q^Y + \beta E_3^- E_3^+ q^Y , \qquad (27)
$$

where the formula  $Y = (H_1 + 2H_2)/3$  for hypercharge is used.

To obtain matrix elements (25), we use an embedding of 8 or 10 into some concrete representation of  $U_q(su_4)$ . Embedding the octet 8 of  $U_q(su_3)$ , for instance, into 15 (adjoint representation) of  $U_q(su_4)$ , on the base of  $(25),(27)$  we obtain the following expressions for the octet baryon masses:

$$
M_N = M_0 + \beta q
$$
,  $M_\Sigma = M_0$ ,  $M_\Lambda = M_0 + \frac{[2]}{[3]}(\alpha + \beta)$ ,  $M_\Xi = M_0 + \alpha q^{-1}$ . (28)

Let us emphasize that the expressions for  $M_N, M_\Xi$  are not invariant under  $q \to q^{-1}$  and no transformation of  $M_0, \alpha, \beta$  exists which makes them invariant.

Excluding  $M_0$ ,  $\alpha$  and  $\beta$  from (28), we obtain the following q-analogue of the GMO formula for octet baryons:

$$
[3]M_{\Lambda} + M_{\Sigma} = [2](q^{-1}M_N + qM_{\Xi}). \tag{29}
$$

Observe apparent simplicity of (29) as compared with  $q$ -MR (11). This same formula (29) is obtained by embedding 8 into other dynamical representations of  $U_q(su_4)$ .

However, what concerns validity of (29) with empirical baryon masses [14], there is no other way to fix the deformation parameter  $q$  than to apply a fitting procedure. One can check that for each of the values  $q_{1,2} = \pm 1.035$ ,  $q_{3,4} = \pm 0.903\sqrt{-1}$ , the left hand side of  $q\text{-MR}$  (29) coincides with its r.h.s within experimental uncertainty (note that for  $q_3$ ,  $q_4$ the constants  $\alpha$  and  $\beta$  in (28) must be pure imaginary). This is in sharp contrast with the  $q$ -MR (11), for which there exists an appealing possibility to fix the parameter q in a rigid way by taking zeros of relevant polynomial  $A<sub>q</sub>$ , see the discussion after (11) as well as [6,7].

**6.** It is clear that the r.h.s. of (29) is invariant under  $q \to q^{-1}$  only if  $q = q^{-1}$ , that is,  $q = \pm 1$ . Here we make an interesting observation.

Behind the "classical" mass formula of Gell-Mann and Okubo which obviously follows from (29) at  $q = 1$  and corresponds to the nondeformed unitary symmetries  $SU(4) \supset$  $SU(3) \supset SU(2)$ , there is also an unusual "hidden symmetry" reflecting the singular  $q = -1$  case of  $U_q(su_4) \supset U_q(su_3) \supset U_q(su_2)$  algebras (undefined in this case). However, the relevant objects exist as certain operator algebras. Let us describe them in the part corresponding to  $n = 2$  and  $n = 3$ .

At generic  $q, q \neq -1$ , the algebra  $U_q(su_2)$  is generated by the elements  $E_1^+, E_1^-$  and  $H_1$ , which satisfy the relations (here [A] denotes  $(q^A - q^{-A})/(q - q^{-1})$ :

$$
[H_1, E_1^{\pm}] = \pm 2E_1^{\pm}, \quad [E_1^+, E_1^-] = [H_1]. \tag{30}
$$

In the limit  $q \to 1$  it reduces to the classical algebra  $su_2$ . We take the representation spaces of the latter in order to construct operator algebras for the case  $q = -1$ . To each  $su_2$  representation space given by j (which takes integral or half-integral nonnegative values) with basis elements  $|jm\rangle$ ,  $m = -j, -j + 1, \ldots, j$ , there corresponds a concrete operator algebra generated by the operators defined according to the formulas

$$
H_1|jm\rangle = 2m|jm\rangle, \quad E_1^+|jm\rangle = \alpha_{j,m}|jm+1\rangle, \quad E_1^-|jm\rangle = \alpha_{j,m-1}|jm-1\rangle
$$

where

$$
\alpha_{j,m} = \begin{cases} \sqrt{-(j-m)(j+m+1)}, & j \text{ is an integer,} \\ \sqrt{(j-m)(j+m+1)}, & j \text{ is a half-integer.} \end{cases}
$$

The so defined operators  $E_1^+, E_1^-$  and  $H_1$  on the basis elements  $|jm\rangle$  satisfy the relations (compare with (30)):

$$
[H_1,E_1^\pm]=\pm 2E_1^\pm,\quad [E_1^+,E_1^-]=\left\{\begin{aligned}&-H_1,\quad \ j\quad\text{is an integer};\\&H_1,\quad \ j\quad\text{is a half-integer}. \end{aligned}\right.
$$

In order to consider the (singular) case  $q = -1$  of  $U_q(su_3)$ , it is more convenient to deal with  $U_q(u_3)$ . We take a representation space  $V_\chi$ , labelled by  $\{m_{13}, m_{23}, m_{33}\} \equiv \chi$ , of the nondeformed  $u_3$  and the Gel'fand-Tsetlin basis with elements  $|\chi; m_{12}, m_{22}; m_{11}\rangle$  in each  $V_{\chi}$ . Define the operators  $E_1^+, E_1^ L_1^-, H_1, E_2^+, E_2^ \mathcal{L}_2^-$ ,  $H_2$  that form the operator algebra of the  $\chi$ -type by their action according to the formulas

$$
H_2|\chi; m_{12}, m_{22}; m_{11}\rangle = (2m_{12} + 2m_{22} - m_{13} - m_{23} - m_{33} - m_{11})|\chi; m_{12}, m_{22}; m_{11}\rangle,
$$
  
\n
$$
E_2^+|\chi; m_{12}, m_{22}; m_{11}\rangle =
$$
  
\n
$$
a_{\chi,m_{11}}(m_{12}, m_{22})|\chi; m_{12} + 1, m_{22}; m_{11}\rangle + b_{\chi,m_{11}}(m_{12}, m_{22})|\chi; m_{12}, m_{22} + 1; m_{11}\rangle,
$$
  
\n
$$
E_2^-|\chi; m_{12}, m_{22}; m_{11}\rangle =
$$
  
\n
$$
a_{\chi,m_{11}}(m_{12} - 1, m_{22})|\chi; m_{12} - 1, m_{22}; m_{11}\rangle + b_{\chi,m_{11}}(m_{12}, m_{22} - 1)|\chi; m_{12}, m_{22} - 1; m_{11}\rangle,
$$

where

$$
a_{\chi,m_{11}}(m_{12}, m_{22}) =
$$
\n
$$
\left((-1)^{m_{11}+m_{13}+m_{23}+m_{33}}\frac{(m_{13}-m_{12})(m_{23}-m_{12}-1)(m_{33}-m_{12}-2)(m_{11}-m_{12}-1)}{(m_{22}-m_{12}-1)(m_{22}-m_{12}-2)}\right)^{1/2}
$$
\n
$$
b_{\chi,m_{11}}(m_{12}, m_{22}) =
$$
\n
$$
\left((-1)^{m_{11}+m_{13}+m_{23}+m_{33}}\frac{(m_{13}-m_{22}+1)(m_{23}-m_{22})(m_{33}-m_{22}-1)(m_{11}-m_{22})}{(m_{12}-m_{22}+1)(m_{12}-m_{22})}\right)^{1/2}.
$$

,

Action formulas for the operators  $E_1^{\pm}$  and  $H_1$  are completely analogous to the formulas given above for  $n = 2$  (with account of  $m_{11} - m_{22} = 2j$ ,  $2m_{11} - m_{12} - m_{22} = 2m$ ).

The presented action formulas for the operators that form the operator algebra of the  $\chi$ -type show that their matrix elements are, to some extent, similar to the "classical" matrix elements (i.e., to the matrix elements of the irrep  $\chi$  operators for  $su(n)$ ). However, there is an essential distinction: now we observe the important phase factors (namely,  $(-1)^{m_{11}+m_{13}+m_{23}+m_{33}}$  under the square root in  $a_{\chi,m_{11}}$  and  $b_{\chi,m_{11}}$ ) which depend on  $\chi$  and a concrete basis element. No such *basis-element dependent* factors exist in the  $su(n)$  case.

Let us illustrate the treatment with the example of the operator algebra which replaces the singular (undefined)  $q = -1$  case of  $U_q(su_3)$  and corresponds to octet representation of su<sub>3</sub>. We give explicitly those action formulas for  $E_1^{\pm}$  and  $E_2^{\pm}$  $x_2^{\pm}$ , in which matrix elements differ from their corresponding "classical" counterparts:

$$
E_1^- |\Sigma^+\rangle = \sqrt{-2} |\Sigma^0\rangle, \quad E_1^- |\Sigma^0\rangle = \sqrt{-2} |\Sigma^-\rangle,
$$
  
\n
$$
E_1^+ |\Sigma^-\rangle = \sqrt{-2} |\Sigma^0\rangle, \quad E_1^+ |\Sigma^0\rangle = \sqrt{-2} |\Sigma^+\rangle,
$$
  
\n
$$
E_2^- |n\rangle = \frac{1}{\sqrt{-2}} |\Sigma^0\rangle + \sqrt{-\frac{3}{2}} |\Lambda\rangle, \quad E_2^- |\Lambda\rangle = \sqrt{-\frac{3}{2}} |\Xi^0\rangle, \quad E_2^- |\Sigma^0\rangle = \frac{1}{\sqrt{-2}} |\Xi^0\rangle,
$$
  
\n
$$
E_2^+ |\Xi^0\rangle = \frac{1}{\sqrt{-2}} |\Sigma^0\rangle + \sqrt{-\frac{3}{2}} |\Lambda\rangle, \quad E_2^+ |\Lambda\rangle = \sqrt{-\frac{3}{2}} |n\rangle, \quad E_2^+ |\Sigma^0\rangle = \frac{1}{\sqrt{-2}} |n\rangle.
$$

To complete this operator algebra, we must add the rest of action formulas for  $E_1^{\pm}$  and  $E_2^{\pm}$  $2^{\pm}$  (i.e., action on those basis elements), which coincide with "classical" ones, as well as the action formulas for  $H_1$ ,  $H_2$  (these latter also coincide with "classical" formulas).

Analogously, proper operator algebras can be given which correspond to any other irrep of  $su_3$ .

7. We have demonstrated that applying, instead of customary hadronic flavor symmetries, the corresponding quantum algebras to derivation of baryon mass formulas takes effectively into account a high nonlinearity (even nonpolynomiality) of baryon masses in  $SU(3)$  breaking or, equivalently, in hypercharge Y and isospin I. This is clearly exhibited by formulas  $(7)-(10)$ ,  $(15)$  in the octet case and  $(19)$ ,  $(15)$  in the decuplet case. Using techniques of q-tensor operators, we obtained the new version (29) of  $q$ -MR. Unlike the previously obtained  $q$ -MRs  $(11)$ , this version does not respect the symmetry under  $q \to q^{-1}$ . We have found that, besides  $q = 1$ , the case  $q = -1$  also yields the standard GMO sum rule, requiring however a special treatment. The corresponding mathematical structure is supplied by operator algebras, as given in Sect.6.

Acknowledgements. We are thankful to Prof. A.Klimyk for valuable remarks. The research described in this publication was made possible in part by the Award No. UP1- 309 of the U.S. Civilian Research & Development Foundation for the Independent States of the Former Soviet Union (CRDF), and by the Grant INTAS-93-1038-Ext.

## References

- [1] V.G.Drinfeld, Sov. Math. Dokl. 32 (1985) 254. M.Jimbo, Lett. Math. Phys. 10 (1985) 63. L.D.Faddeev, N.Reshetikhin and L.Takhtajan, Leningrad Math. J. 1 (1990) 193.
- [2] C.Zachos, in: "Symmetries in Science V" (Eds. B.Gruber, L.C.Biedenharn and H.Doebner), Plenum, New York, 1991; M.Kibler, Lyon preprint LYCEN/9358, 1993.
- [3] M.Chaichian, J.M.Gomez, P.Kulish, Phys. Lett. B311 (1993) 93. D.B.Fairlie, J.Nuyts, Nucl. Phys. B 433 (1995) 26. L.Jenkovszki, M.Kibler, A.Mischenko, Mod. Phys. Lett. A 10 (1995), no.1, 51-60.
- [4] A.M.Gavrilik, J. Phys. A 27 (1994) L91; A.M.Gavrilik and A.V.Tertychnyj, Kiev preprint ITP-93-19E, 1993.
- [5] A.M.Gavrilik, I.I.Kachurik and A.V.Tertychnyj, Ukrainian J. Phys. 40 (1995) 645.
- [6] A.M.Gavrilik, in: "Symmetries in Science VIII" (B.Gruber, editor), Plenum, N.Y., 1995. A.M.Gavrilik, I.I.Kachurik, A.V.Tertychnyj, Kiev preprint ITP-94-34E, 1994; [hep](http://arxiv.org/abs/hep-ph/9504233)[ph/9504233](http://arxiv.org/abs/hep-ph/9504233).
- [7] A.M.Gavrilik, in: "Non-Euclidean Geometry in Modern Physics" (ed. by L.Jenkovszky), Kiev, 1997, p.183; [hep-ph/9712411.](http://arxiv.org/abs/hep-ph/9712411)
- [8] S.Biswas and H.S.Green, J. Phys. A 28 (1995) L339;
- [9] S.Okubo, Phys. Lett. 4 (1963) 14.
- [10] D.B.Lichtenberg, I.J.Tassie and P.J.Keleman, Phys. Rev. 167 (1968) 1535. I.Kokkedee, The Quark Model, Benjamin, New York, 1968. E.Jenkins, Nucl. Phys. B 368 (1992) 190. A.V.Manohar, 1997 Les Houches Lectures, [hep-ph/9802419](http://arxiv.org/abs/hep-ph/9802419).
- [11] M.Gell-Mann, Y.Ne'eman, The Eightfold Way, Benjamin, New York, 1964.
- [12] A.Klimyk, K.Schmüdgen, Quantum Groups and Their Representations, Springer, 1997.
- [13] A.M.Gavrilik and N.Z.Iorgov, in preparation.
- [14] Particle Data Group: Barnett R.M. et al. 1996, Phys. Rev. D 54 (1996) 1.
- [15] A.Klimyk, J. Phys. A 25 (1992) 2919-2927.
- [16] A.Klimyk, J. Math. Phys. 36 (1995) 508-523.