

# Bose-Einstein Condensation of Atoms and Thermal Field Theory<sup>1</sup>

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The Bose-Einstein condensation of atoms can be conveniently formulated as a problem in thermal quantum field theory. There are many properties of the equilibrium system and its collective excitations that can be studied experimentally. The remarkable experimental control over all aspects of the system make it an ideal testing ground for the methods of thermal field theory.

## I. INTRODUCTION

One of the most beautiful experimental developments in physics in this decade is the achievement of Bose-Einstein condensation in atomic gases. This involves trapping a large number of atoms and then cooling them to such extremely low temperatures that they undergo a phase transition to a Bose-Einstein condensate. This was first achieved in the summer of 1995 by a group at JILA in Colorado using <sup>87</sup>Rb atoms [1]. Bose-Einstein condensation was soon achieved also at Rice using <sup>7</sup>Li atoms [2] and at MIT using <sup>23</sup>Na atoms [3]. There was a gap of a year and a half before any other laboratories were able to achieve Bose-Einstein condensation, but by now the total number of groups that have succeeded is over a dozen.

To get a feeling for what is involved in the Bose-Einstein condensation of trapped atoms, consider the simple case of  $N$  identical noninteracting bosons trapped in an isotropic harmonic oscillator potential  $V(\mathbf{r}) = \frac{1}{2}m\omega^2 r^2$  at temperature  $T$ . These particles will form a cloud that is attracted to the deepest regions of the potential. If  $T$  is high enough that classical mechanics is applicable, the radius  $R$  of the cloud can be estimated using the equipartition theorem:

$$\frac{1}{2}m\omega^2 R^2 \sim \frac{p^2}{2m} \sim \frac{3}{2}kT. \quad (1)$$

We find  $R \sim (kT/m\omega^2)^{1/2}$ , so that the radius of the cloud decreases as it is cooled. At extremely low temperatures, it is necessary to take into account the quantum mechanical nature of the states in the potential. If  $kT$  is small

compared to the splitting  $\hbar\omega$  between the energy levels of the harmonic oscillator, all the particles will be in the ground state, which has a Gaussian wavefunction with width of order  $\sqrt{\hbar/m\omega}$ . Thus the radius of the cloud at zero temperature will be  $R \sim \sqrt{\hbar/m\omega}$ . Einstein realized in the 1920's that the cooling of noninteracting bosons involves a phase transition. The transition sets in when the wavefunctions of the particles begin to overlap, which is when their deBroglie wavelengths become comparable to their separations:

$$\hbar/p \sim (R^3/N)^{1/3}. \quad (2)$$

Using (1) and (2), we find that the radius of the cloud is  $R \sim N^{1/6} \sqrt{\hbar/m\omega}$  and that the transition temperature is roughly

$$kT \sim N^{1/3} \hbar\omega \quad (3)$$

At this temperature, atoms begin condensing into the ground state and forming a dense core with radius  $\sqrt{\hbar/m\omega}$  at the center of the cloud. As the temperature is lowered further, the core becomes more and more dense as more particles condense. Meanwhile the thermal cloud becomes less and less dense and then disappears altogether at  $T = 0$ .

We can use the example of the ideal gas of bosons to get some idea of the scales involved in experiments on the Bose-Einstein condensation of atoms. In a typical experiment,  $N \sim 10^6$  atoms are trapped in a potential that can be approximated by a harmonic oscillator with angular frequency  $\omega \sim 10^3$  Hz. From (3), the critical temperature for Bose-Einstein condensation is roughly  $kT_c \sim 10^{-10}$  eV, or  $T_c \sim 10^{-6}$  K. We might expect the size of the condensate at  $T = 0$  to be equal to that of the ground state wavefunction for a single atom, which is roughly 1 micron. However it is typically larger by almost an order of magnitude. This illustrates the important fact that many of the properties of a Bose-Einstein condensate of atoms are dramatically affected by the interactions between the atoms. This makes its behavior far more interesting than that of an ideal gas of noninteracting bosons.

## II. ATOMIC PHYSICS

There are a few aspects of atomic physics that one should be aware of in order to appreciate the experiments on Bose-Einstein condensation. I will describe these aspects using the <sup>87</sup>Rb atom as a specific example.

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## A. Atomic Structure

The  $^{87}\text{Rb}$  atom is an alkali atom with 36 electrons in closed shells with total angular momentum 0 and one valence electron in an S-wave outer shell with spin  $s = \frac{1}{2}$ . It has a nucleus containing 37 protons and 50 neutrons with total spin  $i = \frac{3}{2}$ . The total angular momentum vector of the atom is  $\mathbf{F} = \mathbf{I} + \mathbf{S}$ , where  $\mathbf{I}$  and  $\mathbf{S}$  are the angular momenta of the nucleus and the electrons. The spin-spin interaction between the magnetic moments of the nucleus and the valence electron splits the ground state of the atom into hyperfine states with total angular momentum  $f = 1$  and 2. The splitting between the two hyperfine energy levels is tiny (the  $f = 1$  state for  $^{87}\text{Rb}$  is lower by about  $3 \times 10^{-6}$  eV), but this is enormous compared to the kinetic energies of atoms undergoing Bose-Einstein condensation. Thus it is the hyperfine states of the atom that are relevant in Bose-Einstein condensation.

## B. Interactions between Atoms

The interaction between two atoms can be described by a two-body potential  $U(\mathbf{r}_1 - \mathbf{r}_2)$  with a repulsive core, a short-range attractive well, and a long-range van der Waals tail. The potential for  $^{87}\text{Rb}$  contains 123 molecular bound states, but the splittings between the bound state energy levels are enormous compared to the kinetic energies of atoms undergoing BEC. At these extremely low energies, the effects of the potential can be subsumed into a single number, the S-wave scattering length, which is  $a = 58\text{\AA}$  for  $^{87}\text{Rb}$ . It is at first surprising that  $a$  is an order of magnitude larger than the range of the attractive well, which is about  $5\text{\AA}$ . However, the scale of the scattering length is set not by the short-range potential, but instead by the van der Waals tail  $U(\mathbf{r}) \rightarrow \alpha/r^6$ . The van der Waals length  $(m\alpha/\hbar^2)^{1/4}$  is defined by the balance between the kinetic energy and the van der Waals energy, and has the value  $66\text{\AA}$  for  $^{87}\text{Rb}$ .

## C. Trapping Potentials

There are two kinds of traps that have been used to confine Bose-condensed atoms: magnetic traps and optical traps. A magnetic trap relies on the interaction between the magnetic moment of the valence electron and an applied magnetic field  $\mathbf{B}(\mathbf{r})$ . The relevant term in the hamiltonian is

$$H_{\text{magnetic}} = -\frac{e}{m_e} \mathbf{S} \cdot \mathbf{B}(\mathbf{r}_e), \quad (4)$$

where  $\mathbf{r}_e$  and  $\mathbf{S}$  are the position and spin of the valence electron. Using first order perturbation theory, the Wigner-Eckhart theorem, and the adiabatic approximation, the effects of the interaction can be represented by the external potential

$$V_{\text{magnetic}}(\mathbf{r}) = -m_f g_f \frac{e\hbar}{m_e} |\mathbf{B}(\mathbf{r})|, \quad (5)$$

where  $g_f = [f(f+1) + s(s+1) - i(i+1)]/2f(f+1)$  and  $m_f \hbar$  is the projection of the total angular momentum  $\mathbf{F}$  onto the local direction of the magnetic field  $\mathbf{B}(\mathbf{r})$ . Those hyperfine states  $|f, m_f\rangle$  for which  $m_f g_f$  is positive are attracted to the regions of strong field and therefore can be confined in such a region. In the case of  $^{87}\text{Rb}$ , these state are  $|2, +2\rangle$ ,  $|2, +1\rangle$ , and  $|1, -1\rangle$ .

Bose-condensed atoms can also be confined by an optical trap. Such a trap relies on the coupling of the electric dipole moment of the atom to an external electric field  $\mathbf{E}(\mathbf{r}, t)$  provided by laser beams. The relevant term in the hamiltonian is

$$H = -e(\mathbf{r}_e - \mathbf{r}_n) \cdot \mathbf{E}(\mathbf{r}_n, t), \quad (6)$$

where  $\mathbf{r}_e$  and  $\mathbf{r}_n$  are the positions of the valence electron and the nucleus respectively. At first-order in perturbation theory, the interaction averages to zero over time because the electric fields from the laser beams are rapidly oscillating:  $\langle \mathbf{E}(\mathbf{r}, t) \rangle_t = 0$ . However, at second order in perturbation theory, the time-average is nonzero and the effect of the interaction can be described by a potential

$$V_{\text{optical}}(\mathbf{r}) \propto \langle \mathbf{E}^2(\mathbf{r}, t) \rangle_t. \quad (7)$$

This optical potential can be used to trap any or all of the hyperfine states  $|f, m_f\rangle$ .

## III. THEORETICAL FORMULATION

A system consisting of a large number of low-energy atoms trapped in a potential can be conveniently described by a local quantum field theory. We sketch the derivation of this formulation starting from the conventional description using ordinary quantum mechanics.

### A. Many-body Quantum Mechanics

The problem of  $N$  identical bosonic atoms trapped in an external potential  $V(\mathbf{r})$  and interacting through an interatomic potential  $U(\mathbf{r}_1 - \mathbf{r}_2)$  can of course be formulated using ordinary quantum mechanics. The  $N$  atoms are described by a Schrödinger wavefunction  $\psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N; t)$ , which is a complex-valued function of  $N$  coordinates  $\mathbf{r}_i$  and the time  $t$ . The bosonic nature of the atoms requires the wavefunction to be totally symmetric under permutations of the  $N$  coordinates. The time evolution of the wavefunction is given by the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} \psi = \sum_{i=1}^N \left( -\frac{\hbar^2}{2m} \nabla_i^2 + V(\mathbf{r}_i) \right) \psi + \sum_{i < j} U(\mathbf{r}_i - \mathbf{r}_j) \psi. \quad (8)$$

This formulation of the problem is mathematically awkward if the number of atoms  $N$  is very large and if the effects of the interatomic interactions are important.

## B. Quantum Field Theory Formulation

There is an alternative formulation of this many-body quantum mechanics problem using the language of quantum field theory. This formulation is mathematically equivalent, but it is more convenient if  $N$  is very large and if interatomic interactions are important. In the quantum field theory formulation, the basic ingredient is a quantum field operator  $\Psi(\mathbf{r}, t)$ , which obeys the following commutation relations:

$$[\Psi(\mathbf{r}, t), \Psi(\mathbf{r}', t)] = 0, \quad (9)$$

$$[\Psi(\mathbf{r}, t), \Psi^\dagger(\mathbf{r}', t)] = \delta^3(\mathbf{r} - \mathbf{r}'). \quad (10)$$

The interpretation of these commutation relations is that the operator  $\Psi(\mathbf{r}, t)$  creates an atom at the point  $\mathbf{r}$  and time  $t$  while  $\Psi^\dagger(\mathbf{r}, t)$  annihilates an atom. They also implement the constraint that the atoms are identical bosons. The time evolution of the quantum field is described by the equation

$$i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) \right] \Psi(\mathbf{r}, t) + \left[ \int d^3\mathbf{r}' \Psi^\dagger(\mathbf{r}', t) U(\mathbf{r}' - \mathbf{r}) \right] \Psi(\mathbf{r}, t). \quad (11)$$

One can define a “number operator”,

$$\mathcal{N} = \int d^3r \Psi^\dagger \Psi(\mathbf{r}, t), \quad (12)$$

which is time independent as a consequence of (11). The eigenvalues of  $\mathcal{N}$  can be interpreted as the number of atoms. A state containing precisely  $N$  atoms is therefore represented in the quantum field theory by a state  $|X_N\rangle$  that is an eigenstate of  $\mathcal{N}$ :

$$\mathcal{N}|X_N\rangle = N|X_N\rangle. \quad (13)$$

It is also convenient to define a “vacuum state”  $|0\rangle$  containing  $N = 0$  atoms by the condition  $\Psi(\mathbf{r}, t = 0)|0\rangle = 0$  for all  $\mathbf{r}$ .

The quantum field theory formulation described above is completely equivalent to the quantum mechanics formulation. There is a one-to-one correspondence between Schrödinger wavefunctions for  $N$  particles and  $N$ -particle states in the quantum field theory. The Schrödinger wavefunction  $\psi(\mathbf{r}_1, \dots, \mathbf{r}_N, t)$  that corresponds to an  $N$ -particle state  $|X_N\rangle$  can be expressed as a particular vacuum-to- $|X_N\rangle$  matrix element:

$$\begin{aligned} & \psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t) \\ & \equiv \langle 0 | \Psi(\mathbf{r}_1, t) \Psi(\mathbf{r}_2, t) \cdots \Psi(\mathbf{r}_N, t) | X_N \rangle. \end{aligned} \quad (14)$$

One of the advantages of the quantum field theory formulation is that  $N$  enters into the problem as an eigenvalue rather than as the number of arguments of the wavefunction.

## C. Low-energy approximation

The evolution equation (11) for the quantum field operator  $\Psi(\mathbf{r}, t)$  is nonlocal in space. However, at the extremely low energies relevant to Bose-Einstein condensation, it can be replaced by a local equation. The reason for this is that at such low energies, the interactions due to the interatomic potential  $U(\mathbf{r}_1 - \mathbf{r}_2)$  can be described by a single number, the S-wave scattering length  $a$ . The low energy interactions will be unaffected if we replace  $U(\mathbf{r}_1 - \mathbf{r}_2)$  by any other potential that gives the same S-wave scattering length. In particular, we can replace  $U(\mathbf{r}_1 - \mathbf{r}_2)$  by the local potential  $(4\pi\hbar^2 a/m)\delta^3(\mathbf{r}_1 - \mathbf{r}_2)$ . The resulting quantum field equation for low energy atoms is

$$i\hbar \frac{\partial}{\partial t} \Psi = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) \right] \Psi + \frac{4\pi\hbar^2 a}{m} \Psi^\dagger \Psi \Psi. \quad (15)$$

## D. Thermodynamic Limit

Thus far, we have been considering quantum field equations that are appropriate for describing a fixed number  $N$  of atoms. If  $N$  is extremely large, it is mathematically more convenient to consider states with an indefinite number of particles but for which the average value is  $N$ . This can be accomplished by adding a chemical potential term to (15):

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \Psi &= \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) - \mu \right] \Psi \\ &+ \frac{4\pi\hbar^2 a}{m} \Psi^\dagger \Psi \Psi. \end{aligned} \quad (16)$$

The system described by this quantum field equation has a ground state  $|\Omega\rangle$ . The chemical potential  $\mu$  is to be adjusted so that the expectation value of the number operator  $\mathcal{N}$  in this state is equal to  $N$ :

$$\int d^3r \langle \Omega | \Psi^\dagger \Psi | \Omega \rangle = N. \quad (17)$$

We expect the RMS deviations of the operator  $\mathcal{N}$  to scale like  $\sqrt{N}$ , which is negligible compared to  $N$  if  $N$  is sufficiently large.

The quantum field equation (16) describes a system consisting of a large number of low energy atoms interacting through S-wave scattering. The lagrangian that summarizes this quantum field theory is

$$\mathcal{L} = \frac{i}{2}(\psi^*\dot{\psi} - \dot{\psi}^*\psi) - \frac{1}{2m}\nabla\psi^* \cdot \nabla\psi + [\mu - V(\mathbf{r})]\psi^*\psi - \frac{2\pi a}{m}(\psi^*\psi)^2, \quad (18)$$

where we have set  $\hbar = 1$ . This is a nonrelativistic field theory with a 4-point interaction. It has a  $U(1)$  internal symmetry:  $\psi \rightarrow e^{i\theta}\psi$ .

### E. Condensates with multiple spin states

The quantum field theory with lagrangian (18) describes a system consisting of identical atoms, all of which are in the same hyperfine spin state. However, it is also possible to have several spin states of an atom coexisting in a magnetic trap. The Colorado group has produced condensates containing a mixture of the  $|2, +2\rangle$  and  $|1, -1\rangle$  spin states of  $^{87}\text{Rb}$  [4]. Such a system would be described by a quantum field theory with two complex fields  $\psi_1$  and  $\psi_2$ . The lagrangian includes the interaction terms

$$\mathcal{L}_{\text{int}} = -g_1(\psi_1^*\psi_1)^2 - g_2(\psi_2^*\psi_2)^2 - g_{12}(\psi_1^*\psi_1)(\psi_2^*\psi_2). \quad (19)$$

The three interaction terms correspond to the S-wave scattering of  $|1, -1\rangle$  states, of  $|2, +2\rangle$  states, and of  $|1, -1\rangle$  states from  $|2, +2\rangle$  states.

In an optical trap, it is possible to confine all the spin states of a given hyperfine multiplet. The MIT group has produced condensates containing the  $|1, +1\rangle$ ,  $|1, 0\rangle$ , and  $|1, -1\rangle$  states of  $^{23}\text{Na}$  [5]. Such a system is described by a multiplet  $\Psi$  of complex fields, where  $\Psi^t = (\psi_{+1}, \psi_0, \psi_{-1})$ . The interaction lagrangian includes two terms:

$$\mathcal{L}_{\text{int}} = -g_0(\Psi^\dagger\Psi)^2 - g_1(\Psi^\dagger\mathbf{S}\Psi)^2, \quad (20)$$

where  $\mathbf{S}$  is the vector of  $3 \times 3$  spin matrices. This lagrangian has a nonabelian  $SO(3)$  internal symmetry. The interaction terms in (20) include all the S-wave interactions that respect this symmetry.

## IV. EQUILIBRIUM PROPERTIES

There are several equilibrium properties of a trapped gas of atoms that can be studied experimentally. We will contrast them with the equilibrium properties of a homogeneous Bose gas.

### A. Homogeneous Bose gas

The thermodynamic properties of a homogeneous Bose gas are determined by the free energy density. For a gas at zero temperature, the semiclassical expansion parameter is  $\sqrt{\rho a^3}$ . Thus if the gas is sufficiently dilute,

the system is accurately described by the classical or mean-field approximation in which the quantum field  $\psi(\mathbf{r}, t)$  is replaced by its ground-state expectation value:  $\phi = \langle \Omega | \psi(\mathbf{r}, t) | \Omega \rangle$ . The mean field  $\phi$  is also called the condensate. The number density and the free energy density are

$$\rho = |\phi|^2, \quad (21)$$

$$\mathcal{F} = -\mu|\phi|^2 + \frac{2\pi a}{m}|\phi|^4. \quad (22)$$

Since the chemical potential  $\mu$  is positive, the minimum of  $\mathcal{F}$  occurs at a nonzero value of  $|\phi|$ . There is a continuous set of degenerate minima  $\phi = \sqrt{\rho}e^{i\theta}$  parameterized by an angle  $\theta$ . The nonzero value of the condensate spontaneously breaks the  $U(1)$  symmetry. Solving for the minimum, we find that the chemical potential and the free energy density are

$$\mu = \frac{4\pi a}{m}\rho, \quad (23)$$

$$\mathcal{F} = -\frac{2\pi a}{m}\rho. \quad (24)$$

All the other thermodynamic variables at  $T = 0$  can be determined from  $\mu$  and  $\mathcal{F}$ . For example, the energy density is  $\mathcal{E} = \mathcal{F} + \mu\rho$  and the pressure is  $P = -\mathcal{F}$ .

As the temperature is increased from  $T = 0$  with  $\rho$  fixed, the chemical potential  $\mu$  decreases and the condensate  $\phi$  decreases. At some critical temperature  $T_c$ ,  $\mu$  vanishes. Above  $T_c$ ,  $\mu$  is positive and the condensate vanishes:  $\phi = 0$ . The  $U(1)$  symmetry is therefore manifest above  $T_c$ . The phase transition from the symmetric phase with manifest  $U(1)$  symmetry to the spontaneously broken phase as  $T$  decreases through  $T_c$  is Bose-Einstein condensation. It is a second-order phase transition. The mean-field approximation predicts a first order transition and therefore fails to provide an adequate description of the system in the critical region near  $T_c$ . Careful renormalization group studies of this transition have recently been carried out by Andersen and Strickland [6].

In summary, the equilibrium properties of a homogeneous Bose gas depend on the number density  $\rho$  and the temperature  $T$ . The properties of interest include the condensate  $\phi$ , the energy density  $\mathcal{E}$ , and the pressure  $P$ .

### B. Trapped Bose Gas

A Bose gas of  $N$  atoms trapped in a potential  $V(\mathbf{r})$  is a nonhomogeneous system. The ground state  $|\Omega\rangle$  of this system can still be described by the mean-field approximation, provided that the peak value of  $\sqrt{\rho(\mathbf{r})}a^3$  is sufficiently small. In this approximation, the quantum field  $\psi$  is replaced by its ground-state expectation value:

$$\phi(\mathbf{r}) = \langle \Omega | \psi(\mathbf{r}, t) | \Omega \rangle. \quad (25)$$

The condensate  $\phi(\mathbf{r})$  satisfies the Gross-Pitaevskii equation:

$$-\frac{1}{2m}\nabla^2\phi + [V(\mathbf{r}) - \mu]\phi + \frac{4\pi a}{m}|\phi|^2\phi = 0. \quad (26)$$

The number density is  $\rho(\mathbf{r}) = |\phi(\mathbf{r})|^2$ , and the chemical potential  $\mu$  in (26) must be adjusted so that  $\int d^3r \rho(\mathbf{r}) = N$ . In the ground state, almost all  $N$  atoms are in the same quantum state. The condensate  $\phi(\mathbf{r})$  can be interpreted as the Schrödinger wavefunction of that quantum state.

There are two limits in which the Gross-Pitaevskii equation (26) can be solved analytically. In the weak-interaction limit, the condensate is determined by a balance between kinetic and potential energy. If the interaction term in (26) is neglected, the Gross-Pitaevskii equation reduces to the Schrödinger equation with energy eigenvalue  $\mu$ . For an isotropic harmonic potential  $V(\mathbf{r}) = \frac{1}{2}m\omega^2r^2$ , the solution is the familiar Gaussian ground state wavefunction. The number density profile is

$$\rho(\mathbf{r}) = N \left( \frac{\pi}{m\omega} \right)^{1/2} e^{-m\omega r^2}. \quad (27)$$

The other limit in which the Gross-Pitaevskii equation can be solved analytically is the strong-interaction limit. The condensate in this limit is determined by a balance between the potential and interaction energies. Neglecting the kinetic term in (26), the Gross-Pitaevskii equation reduces to an algebraic equation. For an isotropic harmonic potential, the number density profile is

$$\rho(\mathbf{r}) = \frac{15N}{8\pi R^5} (R^2 - r^2), \quad r < R, \\ = 0, \quad r > R. \quad (28)$$

where  $R = (15Na/2m^2\omega^2)^{1/5}$  is the radius of the condensate. In contrast to the weak interaction limit, the condensate has a sharper edge and its size is larger by a factor of order  $(Na\sqrt{m\omega})^{1/5}$ .

The *density profile*  $\rho(\mathbf{r})$  of a condensate can be measured by illuminating it with a laser beam. The attenuation of the laser light increases with the column density, which is the integral of  $\rho(\mathbf{r})$  along the path of the laser beam. Measurements of the density profiles of the condensate produced in present experiments have shown that they are very close to the strong interaction limit. In typical experiments, the radius of the condensate is larger than that of ground-state wavefunction of a single atom by a factor of 5 to 10. One should not think of these systems as ideal gases of almost noninteracting atoms. The interactions between atoms have a dramatic effect on the condensate.

Given the density profile of a condensate at nonzero temperature, one can determine the *condensate fraction*  $N_0/N$ , where  $N_0$  is the number of atoms in the condensate. One simply fits the observed density profile  $\rho(\mathbf{r})$  to the sum of the density profiles of a condensate and a thermal cloud. The integral of the condensate density gives the number  $N_0$ . The condensate fraction  $N_0/N$

decreases from nearly 100% to 0 as the temperature is increased from zero to the critical temperature.

Another equilibrium property that can be measured is the *release energy*, which is the sum of the kinetic and interaction energies of the atoms:

$$E_R = \int d^3r \left( \frac{1}{2m} \nabla\phi^* \cdot \nabla\phi + \frac{2\pi a}{m} |\phi|^4 \right). \quad (29)$$

This can be measured by suddenly turning off the trapping potential  $V(\mathbf{r})$ . This sets the potential energy to 0, and the remaining energy is given by (29). Since the atoms are no longer confined by the potential, they will drift away. As the cloud of atoms expands, its density decreases and all its interaction energy is converted into kinetic energy. By measuring the density profile as a function of time, one can deduce the sum of the final kinetic energies, and this is equal to the release energy  $E_R$ . The evolution of the Bose-Einstein condensate after switching off the trapping potential has been recently studied using quantum field theory methods by Nakamura and Yamanaka [7].

In summary, the measurable equilibrium properties of a trapped Bose gas of atoms are rather different from those of a homogeneous Bose gas. They include the density profile  $\rho(\mathbf{r})$ , the condensate fraction  $N_0/N$ , and the release energy  $E_R$ .

## V. COLLECTIVE EXCITATIONS

Some of the most beautiful experiments on Bose-Einstein condensates involve studying their collective excitations. We will contrast the collective excitations of a trapped gas with those of a homogeneous Bose gas.

### A. Homogeneous Bose gas

The collective excitations of a homogeneous Bose gas at  $T = 0$  were first studied by Bogoliubov. He found that the quasiparticles in the gas have a dispersion relation of the form

$$\omega(k) = \frac{k\sqrt{k^2 + 1/\xi^2}}{2m}. \quad (30)$$

where  $\xi = (16\pi a\rho)^{-1}$  is the coherence length. For large  $k$ , this approaches the energy  $k^2/2m$  of a free atom. For small  $k$ , the dispersion relation is linear:  $\omega(k) \rightarrow k/(2m\xi)$ . This should not be a surprise, because our quantum field theory has a  $U(1)$  symmetry that is spontaneously broken. There must therefore be a Goldstone mode. The Bogoliubov dispersion relation interpolates between that of the Goldstone mode at small  $k$  and that of a free particle at large  $k$ , with the crossover occurring at  $k$  of order  $1/\xi$ .

At nonzero temperatures below  $T_c$ , the collective excitations are more complicated, because the condensate coexists with a thermal gas of atoms. One can think of this system as consisting of two interpenetrating fluids, with the thermal gas being a normal fluid and the condensate being a superfluid. The collective excitations involve coupled oscillations of the condensate and the thermal gas. A further complication is that the oscillations damp out with time, as energy in the collective modes is transferred to other modes of the system. Thus the basic properties of the collective excitations in the homogeneous gas are the dependence of the oscillation frequencies and their damping rates on the temperature and density.

## B. Trapped Bose Gas

For a Bose gas of  $N$  atoms trapped in a potential  $V(\mathbf{r})$ , the collective excitations at zero temperature can be described by the mean-field approximation, provided that the peak value of  $\sqrt{\rho(\mathbf{r})}a^3$  is sufficiently small. In the state  $|X\rangle$  in the quantum field theory that corresponds to a collective excitation, the expectation value of the quantum field  $\psi(\mathbf{r}, t)$  changes with time. The condensate is therefore time dependent:

$$\phi(\mathbf{r}, t) = \langle X | \psi(\mathbf{r}, t) | X \rangle \quad (31)$$

In the mean-field approximation, it satisfies the time-dependent Gross-Pitaevskii equation:

$$i \frac{\partial}{\partial t} \phi = -\frac{1}{2m} \nabla^2 \phi + [V(\mathbf{r}) - \mu] \phi + \frac{4\pi a}{m} |\phi|^2 \phi. \quad (32)$$

Almost all the atoms are in the same time-dependent quantum state, and the condensate  $\phi(\mathbf{r}, t)$  can be interpreted as the Schrödinger wavefunction for that quantum state.

For an isotropic harmonic potential  $V(\mathbf{r}) = \frac{1}{2}m\omega^2 r^2$ , the frequencies for small-amplitude oscillations can be calculated analytically in two limits. In the weak-interaction limit, (32) reduces to the Schrödinger equation for the harmonic oscillator. The normal modes are labelled by the principal quantum number  $n = 0, 1, 2, \dots$  and by the angular momentum quantum numbers  $\ell$  and  $m$ . The oscillation frequencies are

$$\omega_{n\ell} = (2n + \ell) \omega. \quad (33)$$

The other limit in which one can obtain analytic expression for the frequencies is the strong-interaction limit. The spectrum in this limit was first obtained by Stringari [8]:

$$\omega_{n\ell} = \sqrt{2n^2 + 3n + (2n + 1)\ell} \omega. \quad (34)$$

There are several normal modes of a Bose-Einstein condensate that can be excited by temporarily changing the trapping potential. Both the Colorado group [9]

and the MIT group [10] have measured the oscillation frequencies and damping rates of condensates excited in this way. For condensates containing large numbers of atoms, measurements of the oscillation frequencies have verified that these systems are close to the strong interaction limit. In some cases, the oscillation frequencies have been measured with precisions of better than a percent. This is approaching the accuracy necessary to see the corrections to the mean-field approximations that arise from quantum field fluctuations.

It is also possible to study the propagation of sound in a condensate. By using laser beams to modify the trapping potential, the MIT group [11] has created local density perturbations in a long cigar-shaped condensate. When the laser beams are turned off, the density perturbations propagate along the length of the condensate. The speed of propagation provides a measure of the speed of sound in the condensate.

## VI. CONCLUSIONS

In summary, the Bose-Einstein condensation of atoms is described by a thermal quantum field theory. The interactions between the atoms have dramatic effects on the condensate, and this makes the problem interesting and nontrivial. The interactions are local and have a simple structure, so the lagrangian that describes the system is known. The most remarkable feature is that there is experimental control over almost every aspect of the system. Bose-Einstein condensation has been achieved with several different atoms, and for each atom there are several choices of spin states. The trapping potential  $V$  can be controlled experimentally, not only as a function of  $\mathbf{r}$  but also as a function of  $t$ . The number of atoms in the trap and the temperature of the atoms can also be controlled. Even the interaction strength of the atoms can be controlled. The Colorado group [12] has demonstrated that by using laser pulses, all the atoms in a condensate consisting of the  $|1, -1\rangle$  state of  $^{87}\text{Rb}$  can be changed instantaneously into the  $|2, +2\rangle$  state. Since the  $|2, +2\rangle$  state has a different scattering length, this amounts to an instantaneous change in the scattering length. An even more dramatic development has come from the MIT group [13], which has demonstrated that, by using magnetic fields to adjust the atoms in an optical trap to a Feshbach resonance, the scattering amplitude can be adjusted to any desired value, even  $+\infty$ . In addition to marvelous experimental control, there are also beautiful experimental probes that can be used to study the equilibrium properties and the collective excitations of Bose-Einstein condensates. Thus this system provides an ideal testing ground for the methods of thermal field theory.

I hope this paper has whetted the appetites of thermal field theorists to learn more about the Bose-Einstein condensation of atoms. Fortunately, there are several

recent review articles that provide a good introduction to this subject. The Nordita lecture notes by Pethick and Smith [14] provide a good overview of the physics of Bose-Einstein condensation. A review article by Shi and Griffin [15] summarizes the current state of understanding of the homogeneous Bose gas. The theory of Bose-Einstein condensation in a trapped Bose gas has been reviewed by Dalfovo et al. [16].

Most of the important theoretical developments in this field thus far have come from physicists in the condensed matter physics community. The three review articles mentioned above include rather exhaustive references to this work. For a sampling of interesting work from a condensed matter perspective, I refer the reader to some papers by my colleague T.-L. Ho [17]. Researchers coming from a background in quantum field theory might find some of the following papers more accessible. There is a beautiful paper by Liu [18] that explains how effective-field-theory methods can be used to construct model-independent lagrangians that describe the low-energy collective excitations of a Bose-condensed gas at nonzero temperature. Haugset, Haugerud, and Ravndal [19] have calculated the one-loop effective potential for a Bose gas at nonzero temperature using functional methods. Nieto and I [20] have calculated the free energy of a Bose gas at zero temperature to two-loop accuracy. These papers all involve applications of quantum field theory to the homogenous Bose gas. However quantum field theory methods can also be useful for the experimentally achievable case of a trapped gas. For example, Nieto, Andersen, and I [21] have shown how semiclassical corrections to the properties of a Bose-Einstein condensate can be taken into account by adding local correction terms to the classical field equations of the mean-field approximation. These methods greatly simplify the study of semiclassical corrections.

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