

The masses and decay widths of heavy hybrid mesons

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Abstract

We first derive the mass sum rules for the heavy hybrid mesons to obtain the binding energy and decay constants in the leading order of heavy quark effective theory. The pionic couplings between the lightest 1^{-+} hybrid ($Q\bar{q}g$) and the lowest three heavy meson doublets are calculated with the light cone QCD sum rules. With $SU_f(3)$ flavor symmetry we calculate the widths for all the possible two-body decay processes with a Goldstone boson in the final state. The total width of the 1^{-+} hybrid is estimated to be around 300 MeV. We find that the dominant decay mode of the 1^{-+} heavy hybrid is $1^{-+} \rightarrow \pi + 1^+$ where the 1^+ heavy meson belongs to the $(1^+, 2^+)$ heavy meson doublet. Its branching ratio is about 80% so this mode can be used for the experimental search of the lowest heavy hybrid meson.

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1. Introduction

From quark model we know that a $q\bar{q}$ meson with orbital angular momentum l and total spin s must have $P = (-1)^{l+1}$ and $C = (-1)^{l+s}$. Thus a resonance with $J^{PC} = 0^{-+}, 0^{+-}, 1^{-+}, 2^{+-}, \dots$ must be exotic. Such a state could be a gluonic excitation such as hybrids, glueballs or multiquark states. The hybrid and glueball has been a missing link in the hadron spectrum. Recently there appears experimental evidence for a $J^{PC} = 1^{-+}$ exotic [1, 2, 3]. The emergence of evidence for hybrids indicates the presence of dynamical glue in QCD and will be a direct test of low energy sector of QCD.

The light 1^{-+} hybrid meson mass has been found to lie between $1\text{GeV} \sim 1.7\text{GeV}$ [4]-[6] with QCD sum rules [7]. With the same method the hybrid meson containing one heavy quark and heavy hybrid quarkonium has been studied in [8]. Some decay modes of the light 1^{-+} hybrid meson are discussed in [9]. Recently the masses of the hybrid quarkonium ($Q\bar{Q}g$) states are calculated in the limit of $m_Q \rightarrow \infty$ [10].

The hybrid meson spectrum have been studied extensively with other theoretical approaches including the constituent gluon models [11], the flux tube models [12], the MIT bag model [13], and the lattice gauge theory [14]. Their decays have been studied in the

constituent gluon model via constituent gluon dissociation [15], in the different versions of the flux tube model with a 3P_0 pair creation mechanism [16]. The constituent gluon model predicts a significant width for the $\eta\pi$ channel for the light 1^{-+} hybrid meson [15] while the width of this channel is very small in other models [9, 16]. In contrast the flux tube model predicts very characteristic decay modes for the 1^{-+} decays. The dominant modes are $1^{-+} \rightarrow b_1\pi, f_1\pi$ or $a_1\pi$. In this model the width of the $\rho\pi$ channel is also significant.

The QCD sum rule was used to calculate the $\rho\pi, \eta\pi$ channels of the 1^{-+} hybrid mesons with the three point correlation functions at the symmetric point [9]. The final sum rule for the decay coupling constants suffers from large continuum and excited states contamination.

Since most of the decay calculation has focused on the light hybrid mesons, it will prove valuable to calculate the mass and decays of the hybrid meson with a heavy quark to see whether the same characteristic decay modes exist as in the flux tube model.

The combination of the heavy quark effective theory (HQET) [17] and the light cone QCD sum rules [18] provides a convenient framework to calculate the strong and electromagnetic decays of heavy hadrons containing one heavy quark [19, 20, 21].

In this work we first derive the mass sum rules for the heavy hybrid mesons and perform the numerical analysis. Then we calculate the pionic couplings between the lightest 1^{-+} hybrid ($Q\bar{q}g$) and the lowest three heavy meson doublets. Invoking $SU_f(3)$ flavor symmetry for the coupling constants we calculate the widths for all the possible two-body strong decay processes with a Goldstone boson in the final state.

2. The mass sum rules for the heavy hybrid mesons in HQET

2.1 Heavy quark effective theory

The effective Lagrangian of the HQET, up to order $1/m_Q$, is

$$\mathcal{L}_{\text{eff}} = \bar{h}_v iv \cdot D h_v + \frac{1}{2m_Q} \mathcal{K} + \frac{1}{2m_Q} \mathcal{S} + \mathcal{O}(1/m_Q^2), \quad (2.1)$$

where $h_v(x)$ is the velocity-dependent field related to the original heavy-quark field $Q(x)$ by

$$h_v(x) = e^{im_Q v \cdot x} \frac{1 + \not{v}}{2} Q(x), \quad (2.2)$$

v_μ is the heavy hadron velocity. \mathcal{K} is the kinetic operator defined as

$$\mathcal{K} = \bar{h}_v (iD_t)^2 h_v, \quad (2.3)$$

where $D_t^\mu = D^\mu - (v \cdot D) v^\mu$, with $D^\mu = \partial^\mu - ig A^\mu$ is the gauge-covariant derivative, and \mathcal{S} is the chromomagnetic operator

$$\mathcal{S} = \frac{g}{2} C_{\text{mag}}(m_Q/\mu) \bar{h}_v \sigma_{\mu\nu} G^{\mu\nu} h_v, \quad (2.4)$$

where $C_{\text{mag}} = \left(\frac{\alpha_s(m_Q)}{\alpha_s(\mu)} \right)^{3/\beta_0}$, $\beta_0 = 11 - 2n_f/3$.

2.2 The derivation of the sum rules

The interpolating current for the $J^{PC} = 1^{-+}, 0^{++}$ heavy hybrid mesons in HQET reads

$$J_\mu(x) = \bar{q}(x)g_s\gamma^\nu G_{\mu\nu}^a(x)\frac{\lambda^a}{2}h_v(x), \quad (2.5)$$

and for the $J^{PC} = 1^{+-}, 0^{--}$ ones

$$J_\mu^5(x) = \bar{q}(x)g_s\gamma^\nu\gamma_5 G_{\mu\nu}^a(x)\frac{\lambda^a}{2}h_v(x). \quad (2.6)$$

We consider the correlators

$$i \int d^4x e^{ikx} \langle 0 | T \{ J_\mu(x), J_\nu^\dagger(0) \} | 0 \rangle = -(g_{\mu\nu} - v_\mu v_\nu) \Pi_1(\omega) + v_\mu v_\nu \Pi_2(\omega), \quad (2.7)$$

$$i \int d^4x e^{ikx} \langle 0 | T \{ J_\mu^5(x), J_\nu^{5\dagger}(0) \} | 0 \rangle = -(g_{\mu\nu} - v_\mu v_\nu) \Pi_3(\omega) + v_\mu v_\nu \Pi_4(\omega) \quad (2.8)$$

with $\omega = k \cdot v$. The imaginary parts of $\Pi_1, \Pi_2, \Pi_3, \Pi_4$ receive contributions from the $J^{PC} = 1^{-+}, 0^{++}, 1^{+-}, 0^{--}$ hybrid intermediate states respectively.

To simplify the notations we denote the $J^{PC} = 1^{-+}, 0^{++}, 1^{+-}, 0^{--}$ hybrid mesons with up or down quark by H_1, H_2, H_3, H_4 and those with strange quark by $H_1^s, H_2^s, H_3^s, H_4^s$ respectively. We define the overlapping amplitude f_i as

$$\langle 0 | J_\mu(0) | H_1 \rangle = f_1 \epsilon_\mu^1, \quad (2.9)$$

$$\langle 0 | J_\mu(0) | H_2 \rangle = f_2 v_\mu, \quad (2.10)$$

$$\langle 0 | J_\mu^5(0) | H_3 \rangle = f_3 \epsilon_\mu^3, \quad (2.11)$$

$$\langle 0 | J_\mu^5(0) | H_4 \rangle = i f_4 v_\mu, \quad (2.12)$$

where $\epsilon_\mu^1, \epsilon_\mu^3$ is the H_1, H_3 polarization vectors.

The dispersion relation for $\Pi_i(\omega)$ reads

$$\Pi_i(\omega) = \int \frac{\rho_i(s)}{s - \omega - i\epsilon} ds, \quad (2.13)$$

where $\rho(s)$ is the spectral density in the limit $m_Q \rightarrow \infty$.

At the phenomenological side

$$\Pi_i(\omega) = \frac{1}{2} \frac{f_i^2}{\Lambda_i - \omega} + \text{excited states} + \text{continuum}. \quad (2.14)$$

In order to suppress the continuum and higher excited states contribution we make Borel transformation with the variable ω to (2.13). We have

$$\frac{1}{2} f_i^2 e^{-\frac{\Lambda_i}{T}} = \int_0^{s_0} \rho_i(s) e^{-\frac{s}{T}} ds, \quad (2.15)$$

where s_0 is the continuum threshold. Starting from s_0 we have modeled the phenomenological spectral density with the parton-like ones including both the perturbative term and various condensates.

We need the spectral density $\rho_i(s)$ at the quark level. The relevant feynman diagrams for the derivation of $\rho_i(s)$ are depicted in FIG. 1. The first line is the heavy quark propagator. The broken solid line, broken curly line and a broken solid line with a curly line attached in the middle stands for the quark condensate, gluon condensate and quark gluon mixed condensate respectively. We consider condensates with dimension less than seven. The last four diagrams in FIG. 1 involve with quark gluon mixed condensates. They appear as $\alpha_s \langle \bar{q} g_s \sigma \cdot G q \rangle$. Compared with the quark condensate, they are typically suppressed by a factor $\frac{m_0^2}{16T^2}$, where $m_0^2 = \frac{\langle \bar{q} g_s \sigma \cdot G q \rangle}{\langle \bar{q} q \rangle} = 0.8 \text{GeV}^2$ and $T \sim 1 \text{GeV}$. Its contribution is negligible. At dimension six there are two condensates, the four quark condensate and triple gluon condensate, corresponding to the sixth and fourth diagram in FIG. 1. But in the present case the four quark condensate appears as $\alpha_s^2 \langle \bar{q} q \rangle^2$. It is of high order in α_s so it can be omitted safely. Recall in the QCD sum rule analysis of the light hybrid meson masses, the four quark condensate plays a crucial role [5]. Up to now there exists no reliable way to estimate its value. Two approaches are commonly used to deal with this problem. One is to invoke the vacuum saturation hypothesis [7] and use the factorization approximation. The other is to scale the value derived from the vacuum saturation hypothesis by a number. In [5] a factor two is used to estimate the four quark condensate value, which introduces large uncertainty. In contrast, in the framework of HQET, the dominant nonperturbative corrections in the QSR analysis of heavy hybrid meson masses are due to the quark condensate and gluon condensate, which have been determined rather precisely.

In the calculation we need the following formulas for the gluon condensates.

$$\langle g_s^2 G_{\alpha\beta}^m G_{\mu\nu}^m \rangle = \frac{\delta^{mn}}{96} (g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu}) \langle g_s^2 G^2 \rangle, \quad (2.16)$$

$$\begin{aligned} \langle g_s^3 f^{abc} G_{\mu\nu}^a G_{\alpha\beta}^b G_{\rho\sigma}^c \rangle &= \frac{1}{24} \langle g_s^3 f^{abc} G_{\gamma\delta}^a G_{\delta\epsilon}^b G_{\epsilon\gamma}^c \rangle (g_{\mu\sigma} g_{\alpha\nu} g_{\beta\rho} + g_{\mu\beta} g_{\alpha\rho} g_{\sigma\nu} \\ &+ g_{\alpha\sigma} g_{\mu\rho} g_{\nu\beta} + g_{\rho\nu} g_{\mu\alpha} g_{\beta\sigma} - g_{\mu\beta} g_{\alpha\sigma} g_{\rho\nu} - g_{\mu\sigma} g_{\alpha\rho} g_{\nu\beta} - g_{\alpha\nu} g_{\mu\rho} g_{\beta\sigma} - g_{\beta\rho} g_{\mu\alpha} g_{\nu\sigma}) \quad . \end{aligned} \quad (2.17)$$

$$\text{Tr} \left(\frac{\lambda^a}{2} \frac{\lambda^b}{2} \frac{\lambda^c}{2} \right) = \frac{1}{4} (d_{abc} + i f_{abc}), \quad (2.18)$$

where d_{abc}, f_{abc} are the symmetric and anti-symmetric $SU(3)$ color group structure constants.

The heavy quark propagator has a simple form in coordinate space.

$$\langle 0 | T \{ h_v(x), \bar{h}_v(0) \} | 0 \rangle = \int_0^\infty dt \delta(x - vt). \quad (2.19)$$

It's convenient to calculate the Feynman diagrams directly in coordinate space. Then we perform Wick rotation and make Borel transformation with the variable ω using the Borel transformation formula

$$\hat{\mathcal{B}}_\omega^T e^{\alpha\omega} = \delta(\alpha - \frac{1}{T}). \quad (2.20)$$

As a last step we make a second Borel transformation to $\Pi_i(T)$ with respect to $\tau = \frac{1}{T}$ to get the spectral density,

$$\rho_i(s) = \hat{\mathcal{B}}_\tau^s \Pi_i(T = 1/\tau). \quad (2.21)$$

Finally we get

$$\begin{aligned} \rho(s) = & \frac{\alpha_s}{15\pi^3}s^6 + c_1\frac{\alpha_s}{15\pi^3}m_qs^5 + c_2\frac{\alpha_s}{9\pi^3}a_qs^3 + c_3\frac{1}{48\pi^2}\langle g_s^2G^2 \rangle s^2 \\ & + c_4\frac{1}{192\pi^2}\langle g_s^3G^3 \rangle + c_5\frac{1}{192\pi^2}\langle g_s^2G^2 \rangle a_q\delta(s), \end{aligned} \quad (2.22)$$

where $a_q = -4\pi^2\langle\bar{q}q\rangle$, $q = u, d, s$ and $\langle g_s^3G^3 \rangle = \langle g_s^3 f_{abc}G_{\mu\nu}^a G_{\nu\alpha}^b G_{\alpha\mu}^c \rangle$. The coefficients c_i are collected in TABLE I for H_i .

The contribution from the nonzero light quark mass is calculated up to the order $\mathcal{O}(m_q)$. This term is negligible for sum rules involved with up and down quark but it is important when the heavy hybrid meson contains a strange quark as we shall see later.

For the leading order binding energy of heavy hybrid meson we have

$$\Lambda_i = \frac{\int_0^{s_0} s\rho(s)e^{-\frac{s}{T}} ds}{\int_0^{s_0} \rho(s)e^{-\frac{s}{T}} ds}. \quad (2.23)$$

2.3 Numerical analysis of the mass sum rules

We use $\alpha_s = \frac{4\pi}{(11-\frac{2}{3}n_f)\ln(\frac{s_0/2}{\Lambda_{\text{QCD}}})^2}$ and kept four active flavors with $\Lambda_{\text{QCD}} = 220$ MeV. The tiny up and down quark mass is taken to be zero. The quark and gluon condensates adopts the standard values

$$\begin{aligned} \langle\bar{u}u\rangle &= -(0.225 \text{ GeV})^3, \\ \langle\alpha_s G^2\rangle &= 0.038 \text{ GeV}^4. \end{aligned} \quad (2.24)$$

The value of the triple gluon condensate is not well known. In fact several values exist in literature. We use [7]

$$\langle g^3G^3 \rangle = (1.2 \text{ GeV}^2)\langle\alpha_s G^2\rangle = 0.045\text{GeV}^6, \quad (2.25)$$

which is smaller than the value $0.06 - 0.1 \text{ GeV}^6$ in [22] and a even larger value 0.4 GeV^6 from the "instanton liquid" approach to the QCD vacuum [4]. Later we will enlarge (2.25) by a factor of ten to see the uncertainty from this source.

There are two commonly used methods to extract the masses, the derivative method and the fitting method. With the derivative method we arrive at (2.23). The fitting method involves with fitting the left hand side (L.H.S.) and right hand side (R.H.S.) of Eq. (2.15) with the most suitable parameters Λ_i, f_i, s_i^0 directly in the working region of the Borel parameter. In the numerical analysis, we invoke both methods to crosscheck our results. We find both methods yield nearly the same results.

We require that (1) the absolute value of each condensate contribution be less than 30% of the leading perturbative term with continuum subtracted and (2) the sum of the power corrections be less than one third of the whole sum rule. This requirement leads to the lower limit of the continuum threshold. Typically the triple gluon condensate is less than 0.5% of the leading term with the value in (2.25). The series of the operator expansion converge fast.

We present the numerical values of Λ, f, s_0 in TABLE II. It is understood that there is an error of 0.1 GeV for Λ, s_0 . With these values the left hand side and right hand side of

(2.15) agree within one percent in a large interval of T . Correspondingly, the ratio between the quark, gluon, triple gluon condensate and the perturbative term is collected in TABLE III.

The dependence of the binding energy on the Borel parameter T with different continuum threshold s_0 is shown in FIG. 2 and 3 for the two exotic hybrid mesons. We have also plotted the fitting lines and the curves of the right hand side of Eq. (2.15) for H_1, H_4 in FIG. 4 and 5.

Varying Λ_{QCD} from 220MeV to 300MeV the final result changes within 5%. in our numerical analysis.

If we use $\langle g^3 G^3 \rangle = 0.45 \text{GeV}^6$, the values of Λ, s_0 is shifted upwards by 2% for H_1 . For the other three channels the best fitting parameters changes little. In other words, the uncertainty due to the triple gluon condensate is small and included in the errors given already. This is in strong contrast with the case for the light hybrid mesons, where the hybrid masses depend crucially on the value of the triple gluon condensate. Varying its value from 0.045GeV^6 to 0.4GeV^6 , the 1^{-+} hybrid meson mass increases from 1.0 GeV to 1.5 GeV [4].

For the numerical analysis of the strange heavy hybrid meson, we use $\langle \bar{s}s \rangle = 0.8 \langle \bar{u}u \rangle$, $m_s = 150 \text{ MeV}$. The results are collected in TABLE IV and V. As can be seen from TABLE V the strange quark mass correction is very important. The binding energy of the light component of H_i^s is roughly 130 MeV larger than that for H_i . The continuum threshold increases by about 150 MeV. The decay constants increase typically by 25% due to the strange quark mass correction.

It is interesting to note that the central value of Λ_H is much greater than that for the $(0^-, 1^-)$ doublet of $Q\bar{q}$ meson, $\Lambda_{-, \frac{1}{2}} = (0.5 \pm 0.1) \text{GeV}$. And the continuum starts at rather large s_0 due to the presence of the dynamical gluon in H_Q .

The masses of heavy hybrid mesons shall be around $(m_b + \Lambda_H)$. If we can derive Λ_H reliably, we have a good estimate of H_Q mass. Especially for the bottom quark system, the $1/m_Q$ correction is not large. This point has been the motivation of our considering heavy hybrid mesons with one heavy quark in the framework of HQET.

The uncertainty due to the dimension six condensates renders the reliable extraction of the light hybrid meson masses rather difficult. In this section we have calculated the binding energy of the heavy hybrid meson masses in the leading order of HQET with QCD sum rules. Within the present approach (1) the heavy quark mass is disentangled; (2) the four quark condensate is of higher order in α_s , hence can be neglected safely; (3) the dominant power corrections are from the quark condensate and the gluon condensate, which is well known. The triple gluon condensate shall at most affect the binding energy by 3% varying its value from 0.045GeV^6 to 0.4GeV^6 .

3. Light cone QCD sum rules for the pionic couplings

Note H_1 is the lightest exotic heavy hybrid meson while H_4 lies about 1.4 GeV higher than H_1 . Experimental discovery of the 0^{++} and 1^{+-} heavy hybrid mesons will be difficult since they have the same non-exotic quantum numbers as the radial excitations of ordinary heavy mesons. In this section we discuss the decay modes and widths of the lowest heavy hybrid meson H_1 .

Denote the doublet $(1^+, 2^+)$ with $j_\ell = 3/2$ by (B_1, B_2^*) , the doublet $(0^+, 1^+)$ with

$j_\ell = 1/2$ by (B'_0, B'_1) and the doublet $(0^-, 1^-)$ by (B, B^*) .

We need the following interpolating currents [23]

$$J_{1,+,\frac{3}{2}}^{\dagger\alpha} = \sqrt{\frac{3}{2}} \bar{h}_v \gamma^5 (-i) \left(\mathcal{D}_t^\alpha - \frac{1}{3} \gamma_t^\alpha \mathcal{D}_t \right) q, \quad (3.1)$$

$$J_{2,+,\frac{3}{2}}^{\dagger\alpha_1, \alpha_2} = \bar{h}_v \frac{(-i)}{2} \left(\gamma_t^{\alpha_1} \mathcal{D}_t^{\alpha_2} + \gamma_t^{\alpha_2} \mathcal{D}_t^{\alpha_1} - \frac{2}{3} g_t^{\alpha_1 \alpha_2} \mathcal{D}_t \right) q, \quad (3.2)$$

$$J_{1,-,\frac{1}{2}}^{\dagger\alpha} = \bar{h}_v \gamma_t^\alpha q, \quad J_{0,-,\frac{1}{2}}^{\dagger\alpha} = \bar{h}_v \gamma_5 q, \quad (3.3)$$

$$J_{0,+,\frac{1}{2}}^{\dagger} = \bar{h}_v q, \quad J_{1,+,\frac{1}{2}}^{\dagger\alpha} = \bar{h}_v \gamma^5 \gamma_t^\alpha q, \quad (3.4)$$

where $\gamma_t^\mu = \gamma^\mu - \hat{v}v^\mu$, $g_{\mu\nu}^t = g_{\mu\nu} - v_\mu v_\nu$. We use "t" to denote the transverse index.

We take the p-wave decay process $H_1 \rightarrow B\pi$ as an example to illustrate the method.

- $H_1 \rightarrow B\pi$

The decay amplitude is

$$M(H_1 \rightarrow B\pi) = \epsilon_1^\mu q_\mu^t g_1, \quad (3.5)$$

with $q_\mu^t = q_\mu - (q \cdot v)v_\mu$.

For deriving the sum rules for the coupling constant g_1 we consider the correlator

$$i \int d^4x e^{ik \cdot x} \langle \pi(q) | T \left(J_{0,-,\frac{1}{2}}(x) J^{\dagger\mu}(0) \right) | 0 \rangle = G_1(\omega, \omega') q_\mu^t. \quad (3.6)$$

The function $G_1(\omega, \omega')$ in (3.6) has the following double dispersion relation

$$\frac{f_{-,\frac{1}{2}} f_H g_1}{4(\Lambda_{-,\frac{1}{2}} - \omega)(\Lambda_H - \omega')} + \frac{c}{\Lambda_{-,\frac{1}{2}} - \omega} + \frac{c'}{\Lambda_H - \omega'}, \quad (3.7)$$

where $\Lambda_{P,j_\ell} = m_{P,j_\ell} - m_Q$ and f_{P,j_ℓ} are constants defined as:

$$\langle 0 | J_{j,P,j_\ell}^{\alpha_1 \dots \alpha_j}(0) | j', P', j'_\ell \rangle = f_{P,j_\ell} \delta_{jj'} \delta_{PP'} \delta_{j_\ell j'_\ell} \eta^{\alpha_1 \dots \alpha_j}. \quad (3.8)$$

Keeping the three particle component of the pion wave function, the expression for $G_1(\omega, \omega')$ with tensor structure reads

$$G_1(\omega, \omega') = \int d^4x e^{ikx} \delta(x - vt) \mathbf{Tr} \left\{ \gamma_5 \frac{1 + \hat{v}}{2} \gamma^\nu \langle \pi(q) | q(0) g_s G_{\mu\nu}(0) \bar{q}(x) | 0 \rangle \right\}, \quad (3.9)$$

where we have not included the two particle component of the pion wave function since they are of higher order in α_s and suppressed by a large factor $\frac{g_s^2}{(4\pi)^2}$, which arises from the additional loop integration with the gluon attached to one of the quark line.

The light cone three particle pion wave functions are defined as [19]:

$$\begin{aligned} & \langle \pi(q) | \bar{d}(x) \sigma_{\alpha\beta} \gamma_5 g_s G_{\mu\nu}(ux) u(0) | 0 \rangle = \\ & i f_{3\pi} [(q_\mu q_\alpha g_{\nu\beta} - q_\nu q_\alpha g_{\mu\beta}) - (q_\mu q_\beta g_{\nu\alpha} - q_\nu q_\beta g_{\mu\alpha})] \int \mathcal{D}\alpha_i \varphi_{3\pi}(\alpha_i) e^{iqx(\alpha_1 + v\alpha_3)} \end{aligned} \quad (3.10)$$

$$\begin{aligned}
& \langle \pi(q) | \bar{d}(x) \gamma_\mu \gamma_5 g_s G_{\alpha\beta}(vx) u(0) | 0 \rangle = \\
& f_\pi \left[q_\beta \left(g_{\alpha\mu} - \frac{x_\alpha q_\mu}{q \cdot x} \right) - q_\alpha \left(g_{\beta\mu} - \frac{x_\beta q_\mu}{q \cdot x} \right) \right] \int \mathcal{D}\alpha_i \varphi_\perp(\alpha_i) e^{iqx(\alpha_1 + v\alpha_3)} \\
& + f_\pi \frac{q_\mu}{q \cdot x} (q_\alpha x_\beta - q_\beta x_\alpha) \int \mathcal{D}\alpha_i \varphi_\parallel(\alpha_i) e^{iqx(\alpha_1 + v\alpha_3)} \quad (3.11)
\end{aligned}$$

and

$$\begin{aligned}
& \langle \pi(q) | \bar{d}(x) \gamma_\mu g_s \tilde{G}_{\alpha\beta}(vx) u(0) | 0 \rangle = \\
& i f_\pi \left[q_\beta \left(g_{\alpha\mu} - \frac{x_\alpha q_\mu}{q \cdot x} \right) - q_\alpha \left(g_{\beta\mu} - \frac{x_\beta q_\mu}{q \cdot x} \right) \right] \int \mathcal{D}\alpha_i \tilde{\varphi}_\perp(\alpha_i) e^{iqx(\alpha_1 + v\alpha_3)} \\
& + i f_\pi \frac{q_\mu}{q \cdot x} (q_\alpha x_\beta - q_\beta x_\alpha) \int \mathcal{D}\alpha_i \tilde{\varphi}_\parallel(\alpha_i) e^{iqx(\alpha_1 + v\alpha_3)}. \quad (3.12)
\end{aligned}$$

The operator $\tilde{G}_{\alpha\beta}$ is the dual of $G_{\alpha\beta}$: $\tilde{G}_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\delta\rho} G^{\delta\rho}$; $\mathcal{D}\alpha_i$ is defined as $\mathcal{D}\alpha_i = d\alpha_1 d\alpha_2 d\alpha_3 \delta(1 - \alpha_1 - \alpha_2 - \alpha_3)$. Due to the choice of the gauge $x^\mu A_\mu(x) = 0$, the path-ordered gauge factor $P \exp(i g_s \int_0^1 du x^\mu A_\mu(ux))$ has been omitted.

The function $\varphi_{3\pi}$ is of twist three, while all the wave functions appearing in eqs.(3.11), (3.12) are of twist four. The wave functions $\varphi(x_i, \mu)$ (μ is the renormalization point) describe the distribution in longitudinal momenta inside the pion, the parameters x_i ($\sum_i x_i = 1$) representing the fractions of the longitudinal momentum carried by the quark, the antiquark and gluon.

The wave function normalizations immediately follow from the definitions $\int \mathcal{D}\alpha_i \varphi_{3\pi}(\alpha_i) = 1$, $\int \mathcal{D}\alpha_i \varphi_\perp(\alpha_i) = \int \mathcal{D}\alpha_i \varphi_\parallel(\alpha_i) = 0$, $\int \mathcal{D}\alpha_i \tilde{\varphi}_\perp(\alpha_i) = -\int \mathcal{D}\alpha_i \tilde{\varphi}_\parallel(\alpha_i) = \delta^2/3$, with the parameter δ defined by the matrix element: $\langle \pi(q) | \bar{d} g_s \tilde{G}_{\alpha\mu} \gamma^\alpha u | 0 \rangle = i \delta^2 f_\pi q_\mu$.

Expressing (3.9) with the pion wave functions we arrive at:

$$G_1(\omega, \omega') = - \int_0^\infty dt \int \mathcal{D}\alpha_i e^{i(kv + qv\alpha_1)t} \left\{ f_{3\pi} \varphi_{3\pi}(\alpha_i) (q \cdot v) + f_\pi \left[\varphi_\perp(\alpha_i) - \frac{\varphi_\parallel(\alpha_i)}{2} \right] \right\} + \dots \quad (3.13)$$

For large Euclidean values of ω and ω' this integral is dominated by the region of small t , therefore it can be approximated by the first a few terms.

After Wick rotations and making double Borel transformation with the variables ω and ω' the single-pole terms in (3.7) are eliminated. We arrive at:

$$\frac{1}{4} g_1 f_{-, \frac{1}{2}} f_H e^{-\left(\frac{\Lambda}{T_1} + \frac{1}{2} + \frac{\Lambda}{T_2}\right)t} = f_{3\pi} \Phi'_{3\pi}(u_0) T^2 - f_\pi \left[\Phi_\perp(u_0) - \frac{\Phi_\parallel(u_0)}{2} \right] T, \quad (3.14)$$

where $f_H = f_1$, $u_0 = \frac{T_1}{T_1 + T_2}$, $T \equiv \frac{T_1 T_2}{T_1 + T_2}$, T_1, T_2 are the Borel parameters. Note the sum rule is asymmetric with the Borel parameter T_1 and T_2 . The continuum subtraction is complicated in the present case. We shall discuss this point in section 5.. The right hand side of the sum rule is the result after integration of the double spectral density with respect to s_1, s_2 in the interval $(0, \infty)$ so it includes the continuum contribution.

The new wave functions introduced in (3.14) are defined as

$$\Phi_i(\alpha_1) = \int_0^{1-\alpha_1} \varphi_i(\alpha_1, \alpha_2, 1 - \alpha_1 - \alpha_2) d\alpha_2, \quad (3.15)$$

with $i = 3\pi, \perp, \parallel$ etc and

$$\Phi'_{3\pi}(u_0) = \frac{d\Phi_{3\pi}(\alpha_1)}{d\alpha_1} \Big|_{\alpha_1=u_0}. \quad (3.16)$$

We have used the Borel transformation formula: $\hat{\mathcal{B}}_\omega^T e^{\alpha\omega} = \delta(\alpha - \frac{1}{T})$. Integration by parts are employed to absorb the factor $(q \cdot v)$, which leads to the derivative in (3.14). In this way we arrive at the simple form after double Borel transformation.

- $H_1 \rightarrow B^* \pi$

$$M(H_1 \rightarrow B^* \pi) = i\epsilon_{\mu\alpha\sigma\beta}\epsilon_1^\mu \epsilon^\alpha q^\sigma v^\beta g_2, \quad (3.17)$$

where ϵ_μ is the polarization vector of B^* .

Similarly we consider the correlator

$$i \int d^4x e^{ik \cdot x} \langle \pi(q) | T \left(J_{1,-,\frac{1}{2}}^\alpha(x) J^{\dagger\mu}(0) \right) | 0 \rangle = i\epsilon_{\mu\alpha\sigma\beta} q^\sigma v^\beta G_2(\omega, \omega'). \quad (3.18)$$

$$G_2(\omega, \omega') = \frac{f_\pi}{2} \int_0^\infty dt \int \mathcal{D}\alpha_i e^{i(kv+qv\alpha_1)t} [\varphi_\perp(\alpha_i) - \tilde{\varphi}_\perp(\alpha_i) + \tilde{\varphi}_\parallel(\alpha_i)] + \dots. \quad (3.19)$$

$$\frac{1}{4} g_2 f_{-, \frac{1}{2}} f_H e^{-\left(\frac{\Lambda}{T_1} \frac{1}{2} + \frac{\Lambda_H}{T_2}\right)t} = -\frac{f_\pi}{2} [\Phi_\perp(u_0) - \tilde{\Phi}_\perp(u_0) + \tilde{\Phi}_\parallel(u_0)] T. \quad (3.20)$$

- $H_1 \rightarrow B'_0 \pi$

This process is forbidden due to the parity and angular momentum conservation.

- $H_1 \rightarrow B'_1 \pi$

There exist two independent coupling constants, corresponding to S-wave and D-wave decay. The decay amplitudes are:

$$M(H_1 \rightarrow B'_1 \pi) = \eta^\alpha \epsilon_1^\mu \{ g_{\mu\alpha}^t g_3 + (q_\alpha^t q_\mu^t - q_t^2 g_{\mu\alpha}^t) g_4 \}. \quad (3.21)$$

where η_μ is the polarization vector of B^* .

We consider the correlator

$$i \int d^4x e^{ik \cdot x} \langle \pi(q) | T \left(J_{1,+,\frac{1}{2}}^\alpha(x) J^{\dagger\mu}(0) \right) | 0 \rangle = g_{\mu\alpha}^t G_3(\omega, \omega') + (q_\alpha^t q_\mu^t - q_t^2 g_{\mu\alpha}^t) G_4(\omega, \omega'). \quad (3.22)$$

$$G_3(\omega, \omega') = \int_0^\infty dt \int \mathcal{D}\alpha_i e^{i(kv+qv\alpha_1)t} (q \cdot v) \{ -f_{3\pi} \varphi_{3\pi}(\alpha_i) (q \cdot v) + f_\pi [\tilde{\varphi}_\perp(\alpha_i) + \frac{\varphi_\parallel(\alpha_i)}{2}] \} + \dots. \quad (3.23)$$

$$G_4(\omega, \omega') = \int_0^\infty dt \int \mathcal{D}\alpha_i e^{i(kv+qv\alpha_1)t} \{ f_{3\pi} \varphi_{3\pi}(\alpha_i) (q \cdot v) + \frac{1}{2} \frac{f_\pi}{q \cdot v} [\varphi_\perp(\alpha_i) + \varphi_\parallel(\alpha_i) + \tilde{\varphi}_\perp(\alpha_i) + \tilde{\varphi}_\parallel(\alpha_i)] \} + \dots. \quad (3.24)$$

$$\frac{1}{4}g_3f_{+, \frac{1}{2}}f_H e^{-\left(\frac{\Lambda_{+, \frac{1}{2}}}{T_1} + \frac{\Lambda_H}{T_2}\right)} = -f_{3\pi} \frac{d^2\Phi_{3\pi}(\alpha_1)}{d\alpha_1^2} \Big|_{\alpha_1=u_0} T^3 - f_\pi \left[\Phi'_{\parallel}(u_0) + \frac{\tilde{\Phi}'_{\perp}(u_0)}{2} \right] T^2. \quad (3.25)$$

$$\frac{1}{4}g_4f_{+, \frac{1}{2}}f_H e^{-\left(\frac{\Lambda_{+, \frac{1}{2}}}{T_1} + \frac{\Lambda_H}{T_2}\right)} = f_{3\pi} \Phi_{3\pi}(u_0) T + \frac{f_\pi}{2} \Phi(u_0), \quad (3.26)$$

where

$$\Phi(u) = \int_0^u d\alpha [\varphi_{\perp}(\alpha) + \varphi_{\parallel}(\alpha) + \tilde{\varphi}_{\perp}(\alpha) + \tilde{\varphi}_{\parallel}(\alpha)]. \quad (3.27)$$

- $H_1 \rightarrow B_1\pi$

There also exist two independent coupling constants, corresponding to S-wave and D-wave decay. The decay amplitudes are:

$$M(H_1 \rightarrow B_1\pi) = \eta^\alpha \epsilon_1^\mu \{g_{\mu\alpha}^t g_5 + (q_\alpha^t q_\mu^t - q_t^2 g_{\mu\alpha}^t) g_6\}. \quad (3.28)$$

where e_μ is the polarization vector of B_1 .

We consider the correlator

$$i \int d^4x e^{ik \cdot x} \langle \pi(q) | T \left(J_{1,+, \frac{3}{2}}^\alpha(x) J^{\dagger\mu}(0) \right) | 0 \rangle = g_{\mu\alpha}^t G_5(\omega, \omega') + (q_\alpha^t q_\mu^t - q_t^2 g_{\mu\alpha}^t) G_6(\omega, \omega'). \quad (3.29)$$

$$G_5(\omega, \omega') = -\frac{\sqrt{6}}{3} \int_0^\infty dt \int \mathcal{D}\alpha_i e^{i(kv+qv\alpha_1)t} \alpha_1 (q \cdot v)^2 \{ f_{3\pi} \varphi_{3\pi}(\alpha_i) (q \cdot v) + f_\pi [\varphi_{\perp}(\alpha_i) - \frac{\varphi_{\parallel}(\alpha_i)}{2}] \} + \dots. \quad (3.30)$$

$$G_6(\omega, \omega') = \frac{\sqrt{6}}{12} \int_0^\infty dt \int \mathcal{D}\alpha_i e^{i(kv+qv\alpha_1)t} \alpha_1 \{ 4f_{3\pi} \varphi_{3\pi}(\alpha_i) (q \cdot v) + f_\pi [5\varphi_{\perp}(\alpha_i) - 2\varphi_{\parallel}(\alpha_i) - \tilde{\varphi}_{\perp}(\alpha_i) + \tilde{\varphi}_{\parallel}(\alpha_i)] \} + \dots. \quad (3.31)$$

$$\frac{1}{4}g_5f_{+, \frac{3}{2}}f_H e^{-\left(\frac{\Lambda_{+, \frac{3}{2}}}{T_1} + \frac{\Lambda_H}{T_2}\right)} = \frac{\sqrt{6}}{3} \left\{ f_{3\pi} \frac{d^3[\alpha_1 \Phi_{3\pi}(\alpha_1)]}{d\alpha_1^3} \Big|_{\alpha_1=u_0} T^4 - f_\pi \frac{d^2[\alpha_1 \Phi_{\perp}(\alpha_1) - \frac{\alpha_1 \Phi_{\parallel}(\alpha_1)}{2}]}{d\alpha_1^2} \Big|_{\alpha_1=u_0} T^3 \right\}. \quad (3.32)$$

$$\frac{1}{4}g_6f_{+, \frac{3}{2}}f_H e^{-\left(\frac{\Lambda_{+, \frac{3}{2}}}{T_1} + \frac{\Lambda_H}{T_2}\right)} = -\frac{\sqrt{6}}{12} \left\{ 4f_{3\pi} \frac{d[\alpha_1 \Phi_{3\pi}(\alpha_1)]}{d\alpha_1} \Big|_{\alpha_1=u_0} T^2 - f_\pi u_0 [5\Phi_{\perp}(u_0) - 2\Phi_{\parallel}(u_0) - \tilde{\Phi}_{\perp}(u_0) + \tilde{\Phi}_{\parallel}(u_0)] T \right\}. \quad (3.33)$$

- $H_1 \rightarrow B_2^* \pi$

There exists only one independent coupling constant, corresponding to D-wave decay. The decay amplitude is:

$$M(H_1 \rightarrow B_2^* \pi) = \eta^{\alpha_1 \alpha_2} \epsilon_\mu^1 [i \epsilon^{\alpha_1 \sigma \beta \mu} v_\sigma q_\beta q_{\alpha_2}^t + (\alpha_1 \leftrightarrow \alpha_2)] g_7 . \quad (3.34)$$

where $\eta_{\alpha_1 \alpha_2}$ is the polarization tensor of B_2^* .

We consider the correlator

$$i \int d^4 x e^{ik \cdot x} \langle \pi(q) | T \left(J_{2,+,\frac{3}{2}}^\alpha(x) J^{\dagger \mu}(0) \right) | 0 \rangle = [i \epsilon^{\alpha_1 \sigma \beta \mu} v_\sigma q_\beta q_{\alpha_2}^t + (\alpha_1 \leftrightarrow \alpha_2)] G_7(\omega, \omega') . \quad (3.35)$$

$$G_7(\omega, \omega') = \frac{f_\pi}{4} \int_0^\infty dt \int \mathcal{D}\alpha_i e^{i(kv + qv\alpha_1)t} \alpha_1 [\varphi_\perp(\alpha_i) - \tilde{\varphi}_\perp(\alpha_i) + \tilde{\varphi}_\parallel(\alpha_i)] + \dots . \quad (3.36)$$

$$\frac{1}{4} g_7 f_{+,\frac{3}{2}} f_H e^{-\left(\frac{\Lambda_{+,\frac{3}{2}}}{T_1} + \frac{\Lambda_H}{T_2}\right)t} = \frac{f_\pi}{4} u_0 [\Phi_\perp(u_0) - \tilde{\Phi}_\perp(u_0) + \tilde{\Phi}_\parallel(u_0)] T . \quad (3.37)$$

4. Determination of the parameters

4.1 The values of Λ , f

We need the mass parameters Λ 's and the coupling constants f 's of the corresponding interpolating currents as input. The results are [24, 25]

$$\Lambda_{-,\frac{1}{2}} = (0.5 \pm 0.10) \text{GeV} , \quad (4.1)$$

$$f_{-,\frac{1}{2}} = (0.35 \pm 0.04) \text{GeV}^{\frac{3}{2}} , \quad (4.2)$$

$$\Lambda_{+,\frac{1}{2}} = (0.85 \pm 0.10) \text{GeV} , \quad (4.3)$$

$$f_{+,\frac{1}{2}} = (0.36 \pm 0.04) \text{GeV}^{\frac{3}{2}} , \quad (4.4)$$

$$\bar{\Lambda}_{+,\frac{3}{2}} = (0.95 \pm 0.10) \text{GeV} , \quad (4.5)$$

$$f_{+,\frac{3}{2}} = (0.28 \pm 0.03) \text{GeV}^{\frac{5}{2}} . \quad (4.6)$$

The value of the continuum threshold for the $(0^-, 1^-)$, $(0^+, 1^+)$, $(1^+, 2^+)$ doublets is (1.1 ± 0.1) , (1.2 ± 0.1) , (1.3 ± 0.1) GeV respectively.

4.2 Expressions of PWFs

The detailed expressions of the pion wave functions relevant in our calculation are:

$$\varphi_{3\pi}(\alpha_i) = 360 \alpha_1 \alpha_2 \alpha_3^2 \left[1 + \frac{\omega_{1,0}}{2} (7\alpha_3 - 3) \right] , \quad (4.7)$$

$$\varphi_\perp(\alpha_i) = 30 \delta^2(\alpha_1 - \alpha_2) \alpha_3^2 \left[\frac{1}{3} + 2\epsilon(1 - 2\alpha_3) \right] , \quad (4.8)$$

$$\varphi_{\parallel}(\alpha_i) = 120\delta^2\epsilon(\alpha_1 - \alpha_2)\alpha_1\alpha_2\alpha_3, \quad (4.9)$$

$$\tilde{\varphi}_{\perp}(\alpha_i) = 30\delta^2(1 - \alpha_3)\alpha_3^2\left[\frac{1}{3} + 2\epsilon(1 - 2\alpha_3)\right], \quad (4.10)$$

$$\tilde{\varphi}_{\parallel}(\alpha_i) = -120\delta^2\alpha_1\alpha_2\alpha_3\left[\frac{1}{3} + \epsilon(1 - 2\alpha_3)\right], \quad (4.11)$$

where the coefficients $f_{3\pi}$, $\omega_{1,0}$ [26], δ^2 [27] and ϵ [28] have been determined from QCD sum rules. At the scale $\mu = 1.0$ GeV, $\omega_{1,0} = -2.88$, $f_{3\pi} = 0.0035\text{GeV}^2$, $\delta^2 = 0.2\text{GeV}^2$, $\epsilon = 0.5$.

With (4.7)-(4.11) we can calculate the explicit expressions of the other wave functions defined in this work. Note only the asymptotic form of these wave functions are exactly known. The pieces involved with ϵ , $\omega_{1,0}$ arise from nonperturbative corrections. They are estimated from the moments of the pion wave functions with QCD sum rules. For our purpose it's enough to keep the asymptotic form.

$$\Phi_{3\pi}(u) = 30u(1 - u)^4, \quad (4.12)$$

$$[\Phi_{3\pi}(u)]' = 30(1 - 5u)(1 - u)^3, \quad (4.13)$$

$$[u\Phi_{3\pi}(u)]' = 60u(1 - 3u)(1 - u)^3, \quad (4.14)$$

$$[\Phi_{3\pi}(u)]'' = -120(2 - 5u)(1 - u)^2, \quad (4.15)$$

$$[u\Phi_{3\pi}(u)]''' = 720(u - 1)(5u^2 - 5u + 1), \quad (4.16)$$

$$\Phi_{\perp}(u) = \frac{5}{6}\delta^2(5u - 1)(1 - u)^3, \quad (4.17)$$

$$[u\Phi_{\perp}(u)]'' = -\frac{10}{3}\delta^2(25u^3 - 48u^2 + 27u - 4), \quad (4.18)$$

$$\Phi_{\parallel}(u) = [\Phi_{\parallel}(u)]' = [u\Phi_{\parallel}(u)]'' = 0, \quad (4.19)$$

$$\tilde{\Phi}_{\perp}(u) = \frac{5}{6}\delta^2(3u + 1)(1 - u)^3, \quad (4.20)$$

$$[\tilde{\Phi}_{\perp}(u)]' = -10\delta^2u(1 - u)^2, \quad (4.21)$$

$$\tilde{\Phi}_{\parallel}(u) = -\frac{20}{3}\delta^2u(1 - u)^3, \quad (4.22)$$

$$[\tilde{\Phi}_{\parallel}(u)]' = -\frac{20}{3}\delta^2(1 - 4u)(1 - u)^2, \quad (4.23)$$

$$\Phi(u) = 0, \quad (4.24)$$

where $'$, $''$, $'''$ denotes the first, second and third derivative with respect to u .

5. Subtraction of the continuum

We have the double dispersion relation for $G_i(\omega, \omega')$,

$$\begin{aligned} \Pi(\omega, \omega') = & \int_0^\infty ds_1 \int_0^\infty ds_2 \frac{\rho(s_1, s_2)}{(s_1 - \omega - i\epsilon)(s_2 - \omega' - i\epsilon)} + \int_0^\infty ds_1 \frac{\rho_1(s_1)}{(s_1 - \omega - i\epsilon)} \\ & + \int_0^\infty ds_2 \frac{\rho_2(s_2)}{(s_2 - \omega - i\epsilon)} + \dots, \end{aligned} \quad (5.1)$$

where the ellipse denotes the subtraction terms.

Making double Borel transformation to the variables ω, ω' we get

$$\Pi(\tau_1, \tau_2) = \int_0^\infty ds_1 \int_0^\infty ds_2 e^{-s_1\tau_1} e^{-s_2\tau_2} \rho(s_1, s_2). \quad (5.2)$$

The single pole and subtraction terms have been eliminated in (5.2). Making a second double Borel transformation to τ_1, τ_2 , we get the double spectral density

$$\rho(s_1, s_2) = \mathcal{B}_{\tau_1}^{s_1} \mathcal{B}_{\tau_2}^{s_2} \Pi(\tau_1, \tau_2). \quad (5.3)$$

After expressing $G_i(\omega, \omega')$ with the PWFs and finishing Wick rotations and double Borel transformation, we can get the following general formula:

$$\begin{aligned} \Pi(\tau_1, \tau_2) = & \int_0^\infty dt \int_0^1 \delta[(1-u)t - \tau_1] \delta(ut - \tau_2) t^{-n} \psi(u) \\ = & \frac{1}{(\tau_1 + \tau_2)^{n+1}} \psi\left(\frac{\tau_2}{\tau_1 + \tau_2}\right) \\ = & \sum_{k=0}^\infty a_k \frac{\tau_1^k}{(\tau_1 + \tau_2)^{k+n+1}} \end{aligned} \quad (5.4)$$

if we assume

$$\psi(u) = \sum_{k=0}^\infty a_k (1-u)^k. \quad (5.5)$$

In order to facilitate the numerical analysis we collect the coefficients a_k in TABLE VI after expanding the PWFs into the polynomials of $(1-u)$.

$$\rho(s_1, s_2) = \sum_{k=0}^\infty a_k \frac{s_2^{k+n}}{\Gamma(k+n+1)} \left(-\frac{\partial}{\partial s_2}\right)^k \delta(s_2 - s_1). \quad (5.6)$$

Introducing new variables $s_+ = \frac{s_1+s_2}{2}$, $s_- = \frac{s_1-s_2}{2}$, $\frac{1}{T_-} = \frac{1}{T_1} - \frac{1}{T_2}$, we have

$$\Pi(T_1, T_2) = 2 \int_0^\infty ds_+ e^{-\frac{s_+}{T_-}} \int_{-s_+}^{s_+} ds_- e^{\frac{s_-}{T_-}} \rho(s_+, s_-), \quad (5.7)$$

where

$$\rho(s_+, s_-) = \sum_{k=0}^\infty \frac{a_k (s_+ + s_-)^{k+n}}{2^k \Gamma(k+n+1)} \left(-\frac{\partial}{\partial s_-}\right)^k \delta(2s_-). \quad (5.8)$$

The general quark-hadron duality holds only after the integration with respect to the variable s_- in (5.7). Finally we get

$$\Pi(T_1, T_2) = \sum_{k=0}^\infty \sum_{q=0}^k \frac{a_k}{2^k} \frac{k!}{(k-q)!q!(k+n-q)!} \left(\frac{1}{T_-}\right)^{k-q} \int_0^{E_c} s_+^{k+n-q} e^{-\frac{s_+}{T_-}} ds_+, \quad (5.9)$$

where E_c is the continuum threshold. From its definition we know that $E_c = \frac{s_0 + s_Q}{2}$ where s_0, s_Q is the continuum threshold for the mass sum rules of the hybrid and heavy meson respectively.

Letting $E_c \rightarrow \infty$ we recover (5.4). In the case of symmetric sum rules, only the term with $q = k$ survives since $T_1 = T_2 = 2T$ and $\frac{1}{T_-} = 0$. It is straightforward to get

$$\Pi(T) = \psi\left(\frac{1}{2}\right) \int_0^{E_c} s_+^n e^{-\frac{s_+}{T}} ds_+. \quad (5.10)$$

Now the subtraction of the continuum contribution does not depend on the wave functions.

In the present case the sum rules are asymmetric with T_1 and T_2 . It's reasonable to let $T_1 = 2\beta\Lambda_Q$, $T_2 = 2\beta\Lambda_H$, where Λ_Q is the binding energy of the heavy meson in the leading order of HQET and β is the dimensionless scale parameter. Then we have $u_0 = \frac{\Lambda_Q}{\Lambda_H + \Lambda_Q}$, $T = \frac{2\Lambda_Q\Lambda_H}{\Lambda_Q + \Lambda_H}\beta$.

We rewrite (5.9) as

$$\Pi(T) = T^{n+1} \sum_{k=0}^{\infty} \frac{a_k}{2^k} \left\{ \sum_{q=0}^k \frac{k!}{(k-q)!q!} \left(\frac{T}{T_-}\right)^q f_{n+q}\left(\frac{E_c}{T}\right) \right\}, \quad (5.11)$$

where $f_n(x) = 1 - e^{-x} \sum_{k=0}^n \frac{x^k}{k!}$ is the factor used to subtract the continuum. Note $\frac{T}{T_-} = \frac{\Lambda_H - \Lambda_Q}{\Lambda_H + \Lambda_Q} \approx \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$ for the $(1^+, 2^+)$, $(0^+, 1^+)$, $(0^-, 1^-)$ doublet respectively. So only the first few terms in the bracket in (5.11) is important. Replacing $f_{n+q}(\frac{E_c}{T})$ by $f_n(\frac{E_c}{T})$, we have

$$\begin{aligned} \Pi(T) &= T^{n+1} f_n\left(\frac{E_c}{T}\right) \sum_{k=0}^{\infty} \frac{a_k}{2^k} \left\{ \sum_{q=0}^k \frac{k!}{(k-q)!q!} \left(\frac{T}{T_-}\right)^q \right\} \\ &= T^{n+1} f_n\left(\frac{E_c}{T}\right) \psi(u_0). \end{aligned} \quad (5.12)$$

The above approximation is useful when $\frac{T}{T_-} \rightarrow 0$, i.e., the decay heavy meson mass is very close to the parent hybrid meson mass. We shall use the exact expression (5.11) in the numerical analysis below.

6. Numerical results and discussions

We now turn to the numerical evaluation of the sum rules for the coupling constants after the continuum is subtracted carefully.

The variation of the coupling constants g_i with the Borel parameter T and E_c is presented in FIG. 6-12. The curves correspond to $E_c = 1.4, 1.5, 1.6\text{GeV}$ respectively. Stability develops for these sum rules starting from 0.8 GeV. Numerically we have

$$g_1 f_{-, \frac{1}{2}} f_H = -(0.12 \pm 0.02)\text{GeV}^4, \quad (6.1)$$

$$g_2 f_{-, \frac{1}{2}} f_H = (0.078 \pm 0.016)\text{GeV}^4, \quad (6.2)$$

$$g_3 f_{+, \frac{1}{2}} f_H = (0.11 \pm 0.04)\text{GeV}^5, \quad (6.3)$$

$$g_4 f_{+, \frac{1}{2}} f_H = (0.046 \pm 0.01)\text{GeV}^3, \quad (6.4)$$

$$g_5 f_{+, \frac{3}{2}} f_H = (0.33 \pm 0.08) \text{GeV}^6, \quad (6.5)$$

$$g_6 f_{+, \frac{3}{2}} f_H = (0.026 \pm 0.006) \text{GeV}^4, \quad (6.6)$$

$$g_7 f_{+, \frac{3}{2}} f_H = -(0.018 \pm 0.004) \text{GeV}^4, \quad (6.7)$$

where the errors refers to the variations with T and E_c in this region. And the central value corresponds to $T = 1.1$ GeV, $\beta = 1$, and $E_c = 1.5$ GeV. Note the sum rules for g_1, g_4, g_6 changes less than 10% when either T or E_c varies in the working region. Variation with T is about 20% for those sum rules for g_2, g_3, g_7 . Only the sum rule for g_5 has a rather strong dependence on the continuum threshold E_c . It changes about 20% with $E_c = (1.5 \pm 0.1)$ GeV.

With the values of f_i given in the previous sections we get

$$g_1 = -(1.4 \pm 0.3) \text{GeV}^{-1}, \quad (6.8)$$

$$g_2 = (0.85 \pm 0.2) \text{GeV}^{-1}, \quad (6.9)$$

$$g_3 = (1.3 \pm 0.3), \quad (6.10)$$

$$g_4 = (0.5 \pm 0.2) \text{GeV}^{-2}, \quad (6.11)$$

$$g_5 = (5.0 \pm 1.0), \quad (6.12)$$

$$g_6 = (0.4 \pm 0.1) \text{GeV}^{-2}, \quad (6.13)$$

$$g_7 = -(0.3 \pm 0.05) \text{GeV}^{-2}. \quad (6.14)$$

With these coupling constants we can calculate the decay widths of heavy hybrid mesons. The leading order binding energy of H_1 meson is $\Lambda_H = 1.6$ GeV, which is not small compared with the bottom quark mass $m_b = 4.7$ GeV. We shall take into account of the corrections due to the finite m_b partly. In full QCD, we introduce the decay constant of the hybrid meson H_1 as:

$$\langle 0 | J_\mu(0) | H_1 \rangle = m_H F_H \epsilon_\mu^1, \quad (6.15)$$

where $F_H = \frac{f_H}{\sqrt{m_b}}$ as $m_b \rightarrow \infty$. The amplitude of the decay process $H_1 \rightarrow B\pi$ in full QCD is

$$M(H_1 \rightarrow B\pi) = \epsilon_1^\mu q_\mu \sqrt{m_H m_B} h_1. \quad (6.16)$$

In the limit $m_b \rightarrow \infty$ the decay constant h_1 in full QCD is equal to the decay constant g_1 defined in HQET. The decay width formulas in the leading order of HQET are not useful in the present case. We include the finite m_b correction and use the following decay width formulas.

$$\begin{aligned} \Gamma(H_1 \rightarrow B\pi) &= \frac{1}{16\pi} \left(\frac{m_B}{m_H}\right) \left(\frac{m_b^2}{m_B m_H}\right)^2 g_1^2 |\vec{q}|^3, \\ \Gamma(H_1 \rightarrow B^*\pi) &= \frac{1}{8\pi} \left(\frac{m_{B^*}}{m_H}\right) \left(\frac{m_b^2}{m_{B^*} m_H}\right)^2 g_2^2 |\vec{q}|^3, \\ \Gamma(H_1 \rightarrow B'_1\pi) &= \frac{1}{16\pi} \left(\frac{m_{B'_1}}{m_H}\right) \left(\frac{m_b^2}{m_{B'_1} m_H}\right)^2 (3g_3^2 + 4g_3 g_4 |\vec{q}|^2 + 2g_4 |\vec{q}|^4) |\vec{q}|, \\ \Gamma(H_1 \rightarrow B_1\pi) &= \frac{1}{16\pi} \left(\frac{m_{B_1}}{m_H}\right) \left(\frac{m_b^2}{m_{B_1} m_H}\right)^2 (3g_5^2 + 4g_5 g_6 |\vec{q}|^2 + 2g_6 |\vec{q}|^4) |\vec{q}|, \\ \Gamma(H_1 \rightarrow B_2^*\pi) &= \frac{1}{4\pi} \left(\frac{m_{B_2^*}}{m_H}\right) \left(\frac{m_b^2}{m_{B_2^*} m_H}\right)^2 g_7^2 |\vec{q}|^5, \end{aligned} \quad (6.17)$$

where $|\vec{q}| = \sqrt{(m_H^2 - (m_Q + m_\pi)^2)(m_H^2 - (m_Q - m_\pi)^2)}/2m_H$, m_H , m_Q is the hybrid and heavy meson mass. Note summation over charged and neutral pion final states has been performed. In the calculation we use $m_B = 5.28$ GeV, $m_{B^*} = 5.33$ GeV, $m_{B_1} = m_B + 0.37 = 5.7$ GeV, $m_{B_1} = m_{B_2^*} = 5.8$ GeV, $m_H = m_b + \Lambda_H = 6.3$ GeV with $m_b = 4.7$ GeV. For the heavy meson with strangeness, we use $m_{B_s} = m_B + 0.09 = 5.37$ GeV etc.

We employ the $SU_f(3)$ flavor symmetry to relate the coupling constants for the decay processes $H_1 \rightarrow B_s K$, $H_1 \rightarrow B \eta$ etc. For example, $g_{H_1 \rightarrow B_s K} = g_1$, $g_{H_1 \rightarrow B \eta} = \frac{g_1}{\sqrt{6}}$. We collect the numerical results of the decay widths for the different channels in TABLE VI.

Summing all the two-body strong decay channels with one Goldstone boson in the final state, we estimate the total decay width of the lowest lying hybrid meson to be around 300 MeV. The dominant decay mode is $H_1 \rightarrow B_1 \pi$. Its branching ratio is about 80%. Now we know B_1 is a narrow resonance with a width of ~ 20 MeV. So this mode can be used to detect the possible existence of H_1 experimentally. The same characteristic decay modes for the light hybrid mesons have been predicted in the flux tube model [16].

In summary we have calculated the binding energy of the heavy hybrid mesons in the leading order of HQET. With the help of the light cone QCD sum rules we have extracted the pionic couplings between the hybrid H_1 and lowest three heavy meson doublets. Invoking the flavor $SU_f(3)$ symmetry we have estimated the strong two-body decay widths of H_1 with one Goldstone boson in the final state. Our calculation yields very characteristic decay modes of H_1 and confirms the earlier predictions on the particular decay modes for the light hybrid mesons from the flux tube model. The mixing between η_8 and η_1 and the possible contribution due to the QCD anomaly will be topics of future work.

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TABLE I. The coefficients in the expressions for the spectral density $\rho(\epsilon)$.

	1^{-+}	0^{++}	1^{+-}	0^{--}
c_1	1	3	-1	-3
c_2	1	3	-1	-3
c_3	1	-3	1	-3
c_4	1	-3	1	-3
c_5	1	-1	-1	1

TABLE II. The values of Λ, f, s_0 for the nonstrange heavy hybrid meson ($Q\bar{q}g$). Λ, s_0 is unit of GeV, F is in unit of $\text{GeV}^{\frac{7}{2}}$.

	1^{-+}	0^{++}	1^{+-}	0^{--}
Λ	1.6	2.17	1.66	3.0
f	0.23	0.425	0.162	0.98
s_0	2.0	2.5	1.9	3.4

TABLE III. The ratio between various condensates and perturbative term after the continuum subtraction for the nonstrange heavy hybrid meson.

	1^{-+}	0^{++}	1^{+-}	0^{--}
$\langle \bar{q}q \rangle$	0.22	0.34	-0.26	-0.14
$\langle g_s^2 G^2 \rangle$	0.16	-0.23	0.19	-0.08
$\langle g_s^3 G^3 \rangle$	0.004	-0.004	0.005	-0.0006
$\langle \bar{q}q \rangle \langle g_s^2 G^2 \rangle$	0.02	-0.004	-0.026	0.0006

TABLE IV. The values of Λ , f , s_0 for the strange heavy hybrid meson ($Q\bar{q}g$).

	1^{-+}	0^{++}	1^{+-}	0^{--}
Λ	1.73	2.28	1.76	3.15
f	0.28	0.538	0.2	1.04
s_0	2.13	.63	2.03	3.53

TABLE V. The ratio between various condensates (including strange quark mass correction) and perturbative term for the strange heavy hybrid meson.

	1^{-+}	0^{++}	1^{+-}	0^{--}
$\langle \bar{s}s \rangle$	0.15	0.23	-0.17	-0.1
$\langle g_s^2 G^2 \rangle$	0.13	-0.19	0.15	-0.07
$\langle g_s^3 G^3 \rangle$	0.003	-0.003	0.004	-0.0005
$\langle \bar{s}s \rangle \langle g_s^2 G^2 \rangle$	0.01	-0.003	-0.014	0.0004
m_s	0.08	0.20	-0.09	-0.15

TABLE VI. The coefficients a_k when the PWFs are expanded into polynomials of $(1-u)$. They are in unit of δ^2 for the twist four PWFs.

	a_1	a_2	a_3	a_4	a_5
$\Phi_{3\pi}(u)$			-120	150	
$\Phi_{\perp}(u)$			$\frac{10}{3}$	$-\frac{25}{6}$	
$\Phi_{\perp}(u) - \tilde{\Phi}_{\perp}(u) + \tilde{\Phi}_{\parallel}(u)$			$-\frac{20}{3}$	5	
$\Phi_{3\pi}''(u)$		360	-600		
$\tilde{\Phi}'_{\perp}(u)$		-10	10		
$\Phi_{3\pi}(u)$				30	-30
$[u\Phi_{3\pi}(u)]'''$	-720	3600	-3600		
$\{u[\Phi_{\perp}(u) - \frac{\Phi_{\parallel}(u)}{2}]\}''$	20	-90	$\frac{250}{3}$		
$[u\Phi_{3\pi}(u)]'$			-120	300	-180
$u[5\Phi_{\perp}(u) - 2\Phi_{\parallel}(u) - \tilde{\Phi}_{\perp}(u) + \tilde{\Phi}_{\parallel}(u)]$			$\frac{20}{3}$	$-\frac{55}{3}$	$\frac{35}{3}$
$u[\Phi_{\perp}(u) - \tilde{\Phi}_{\perp}(u) + \tilde{\Phi}_{\parallel}(u)]$			$-\frac{20}{3}$	$\frac{35}{3}$	-5

TABLE VII. The decay widths of different two-body decay channels for the 1^{-+} hybrid meson H_1 , where combinations of Goldstone bosons and heavy mesons yield different final states. The unit is MeV. The minus sign means either such a decay mode is not allowed by the phase space or the decay width is negligible.

	B	B^*	B'_0	B'_1	B_1	B_2^*
π	11	9	0	25	230	0.05
η	1	1	0	1.6	-	-
K	5	5	0	10	-	-

Figure Captions

FIG. 1. The relevant feynman diagrams for the derivation of the QCD sum rule (2.15). The broken solid line, broken curly line and a broken solid line with a curly line attached in the middle stands for the quark condensate, gluon condensate and quark gluon mixed condensate respectively.

FIG. 2 The variations of Λ with T and s_0 for H_1 . From top to bottom the curves correspond to $s_0 = 2.1, 2.0, 1.9$ GeV. T is in unit of GeV.

FIG. 3 The variations of Λ with T and $s_0 = 3.5, 3.4, 3.3$ GeV for H_4 .

FIG. 4 The variation of the right and left hand side of Eq. (2.15) with T is plotted as solid and dotted curves respectively for H_1 with the values of Λ, f, s_0 in TABLE II.

FIG. 5 The variation of the right and left hand side of Eq. (2.15) with T for H_4 .

FIG. 6 The dependence of $g_1 f_{-, \frac{1}{2}} f_H$ on the Borel parameter T and the continuum threshold E_c after the factor $\frac{1}{4} e^{-\left(\frac{\Lambda}{T_1} \frac{1}{2} + \frac{\Lambda_H}{T_2}\right)}$ is moved to the right hand side of Eq. (3.14).

FIG. 7 The variation of $g_2 f_{-, \frac{1}{2}} f_H$ with T and E_c .

FIG. 8 The variation of $g_3 f_{+, \frac{1}{2}} f_H$ with T and E_c .

FIG. 9 The variation of $g_4 f_{+, \frac{1}{2}} f_H$ with T and E_c .

FIG. 10 The variation of $g_5 f_{+, \frac{3}{2}} f_H$ with T and E_c .

FIG. 11 The variation of $g_6 f_{+, \frac{3}{2}} f_H$ with T and E_c .

FIG. 12 The variation of $g_7 f_{+, \frac{3}{2}} f_H$ with T and E_c .

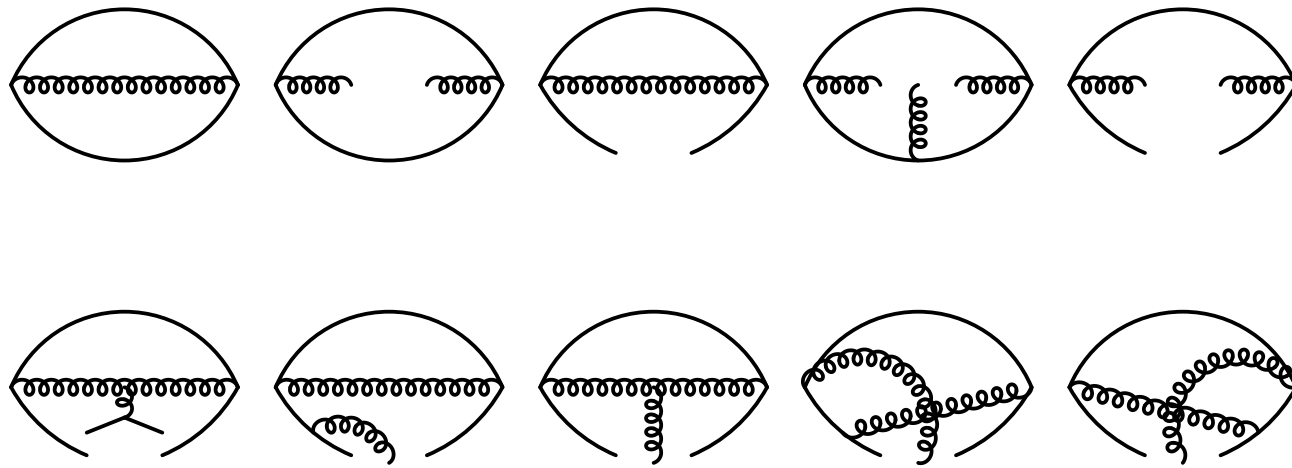


FIG. 1

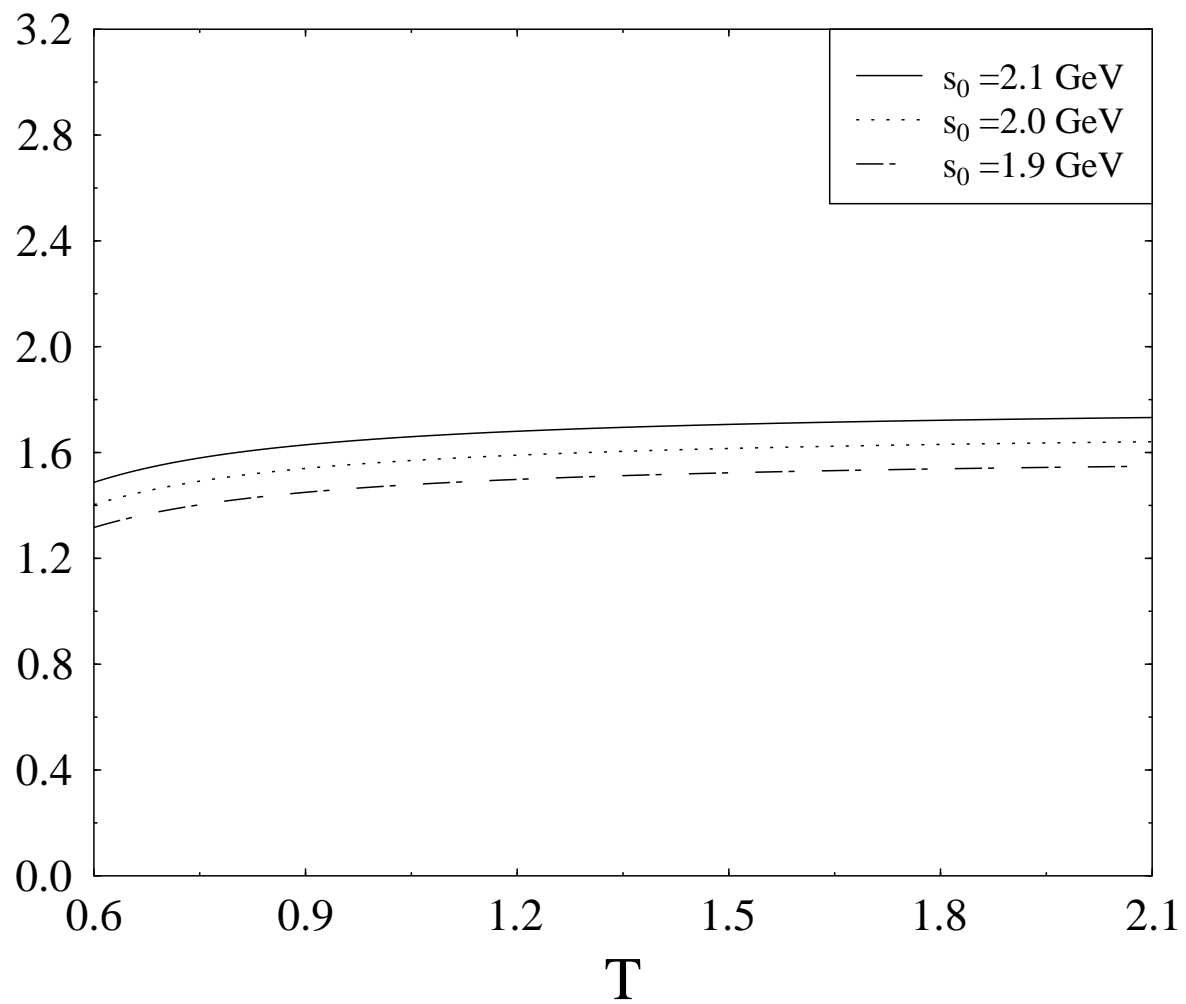


FIG. 2

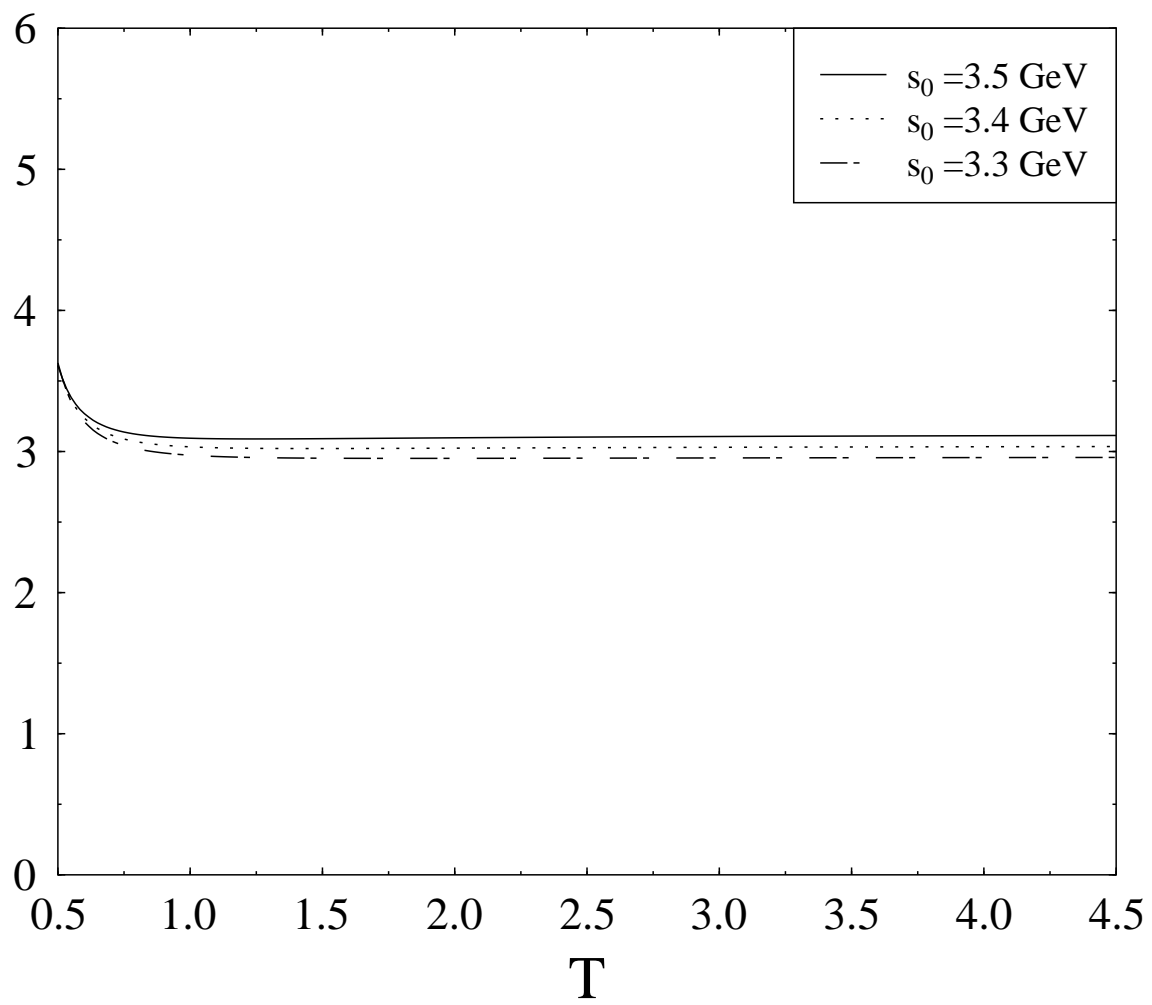


FIG. 3

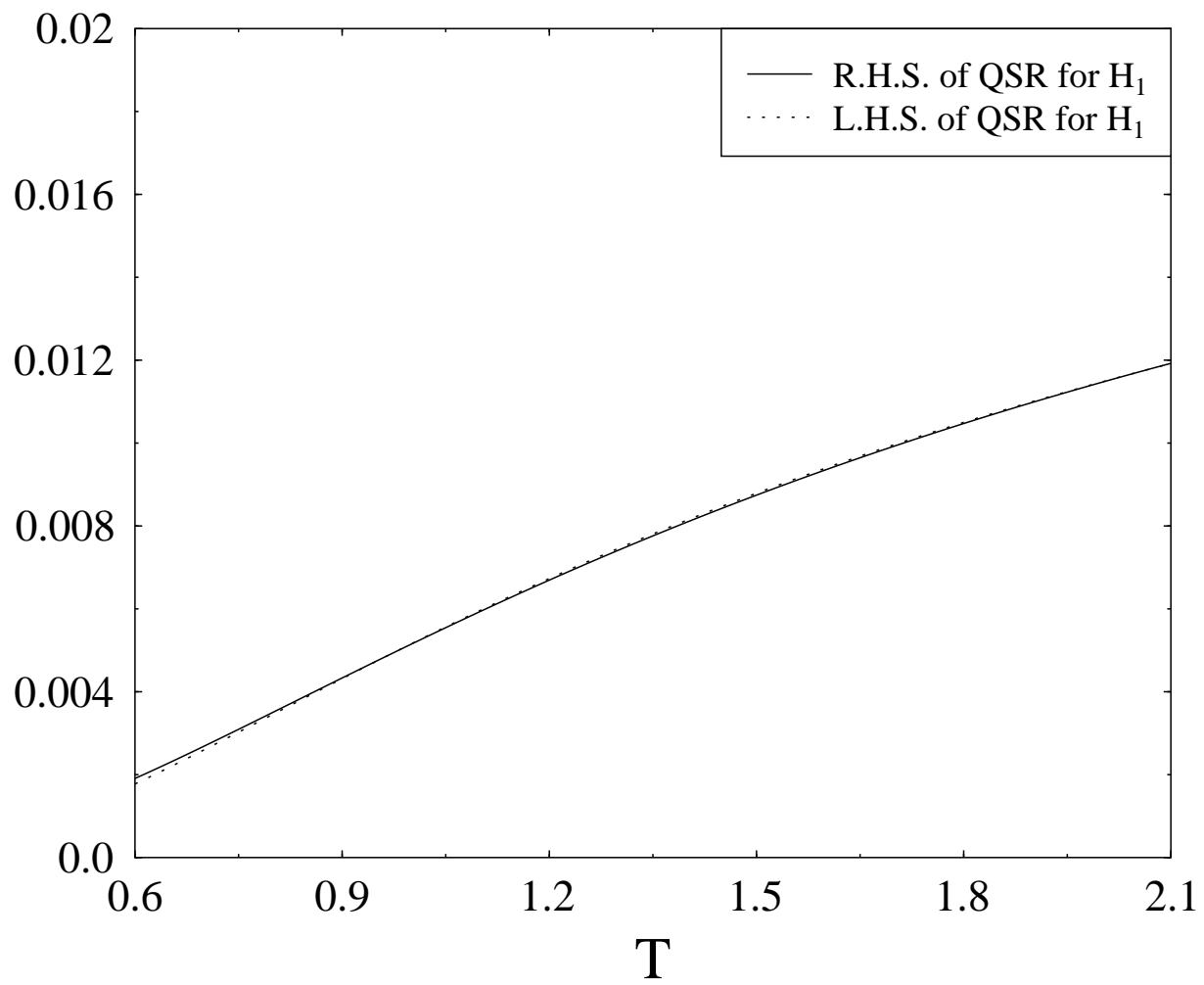


FIG. 4

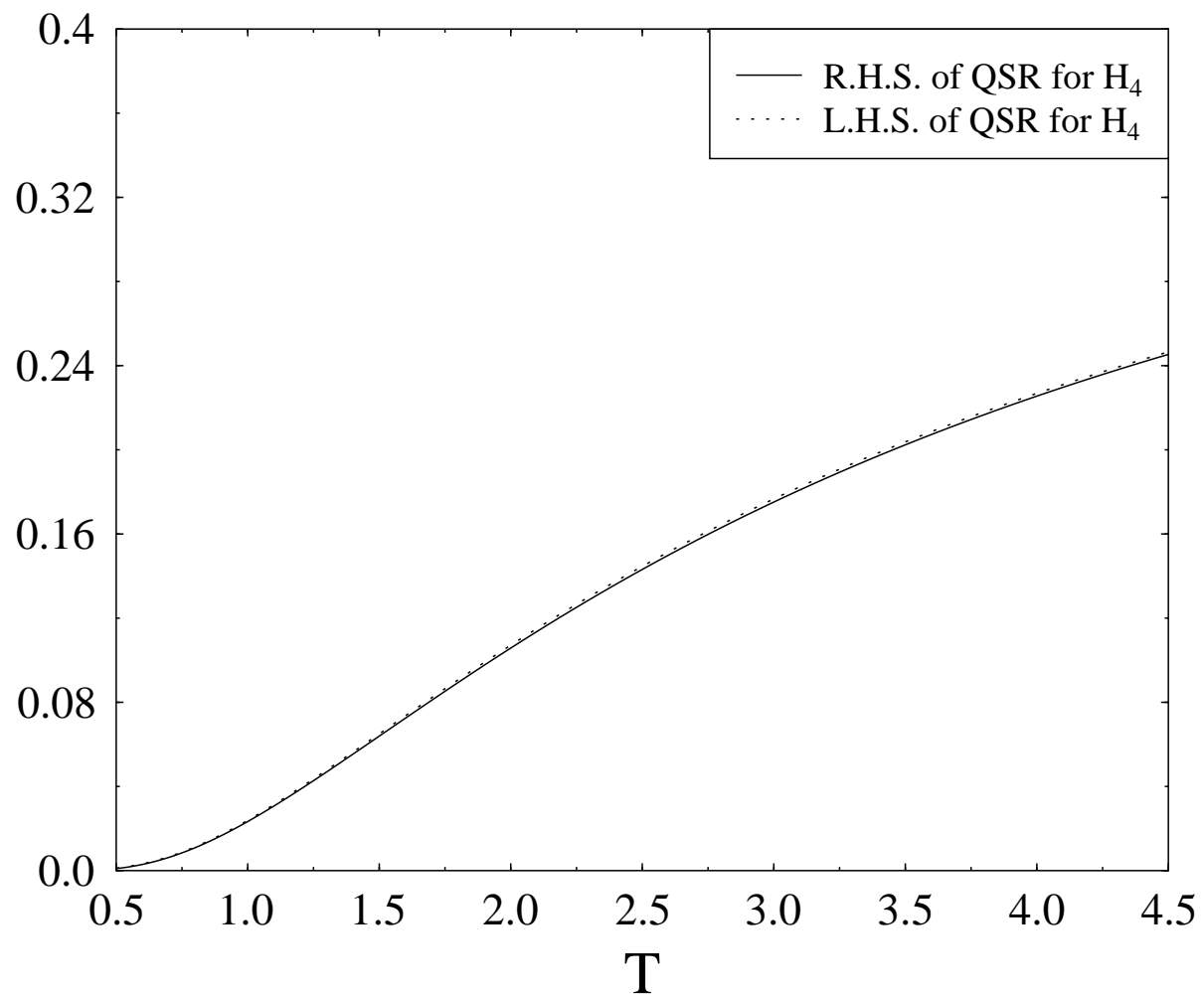


FIG. 5

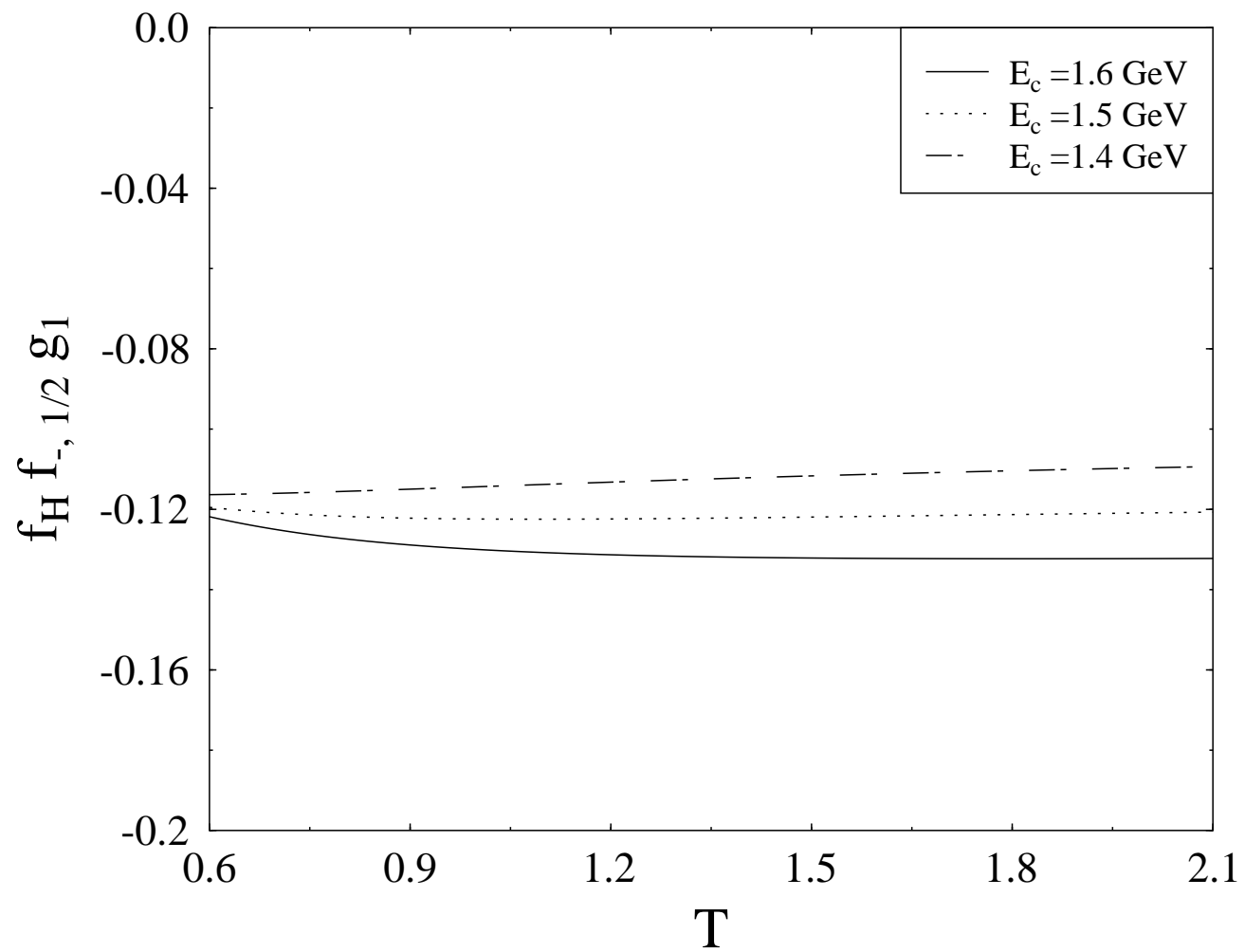


FIG. 6

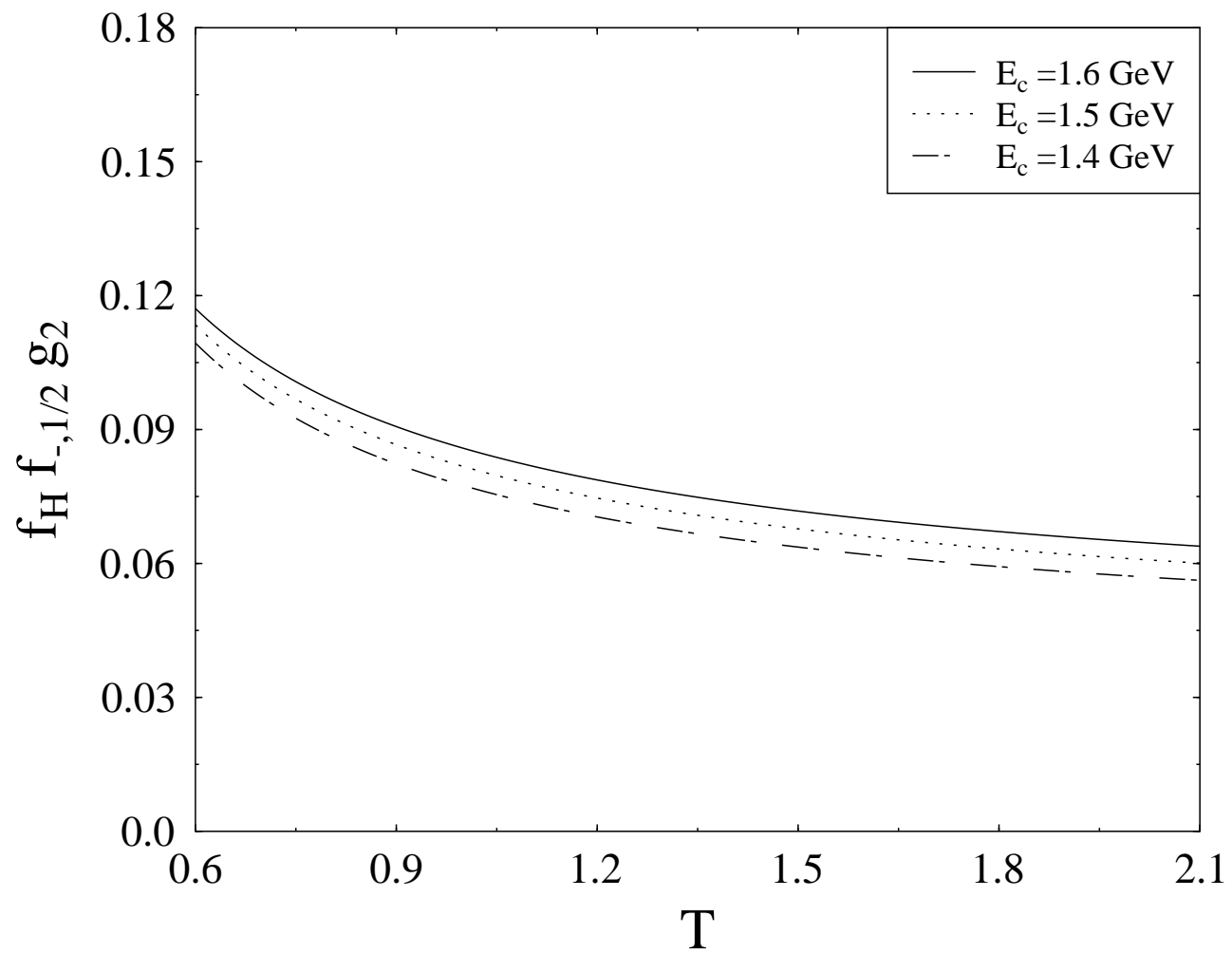


FIG. 7

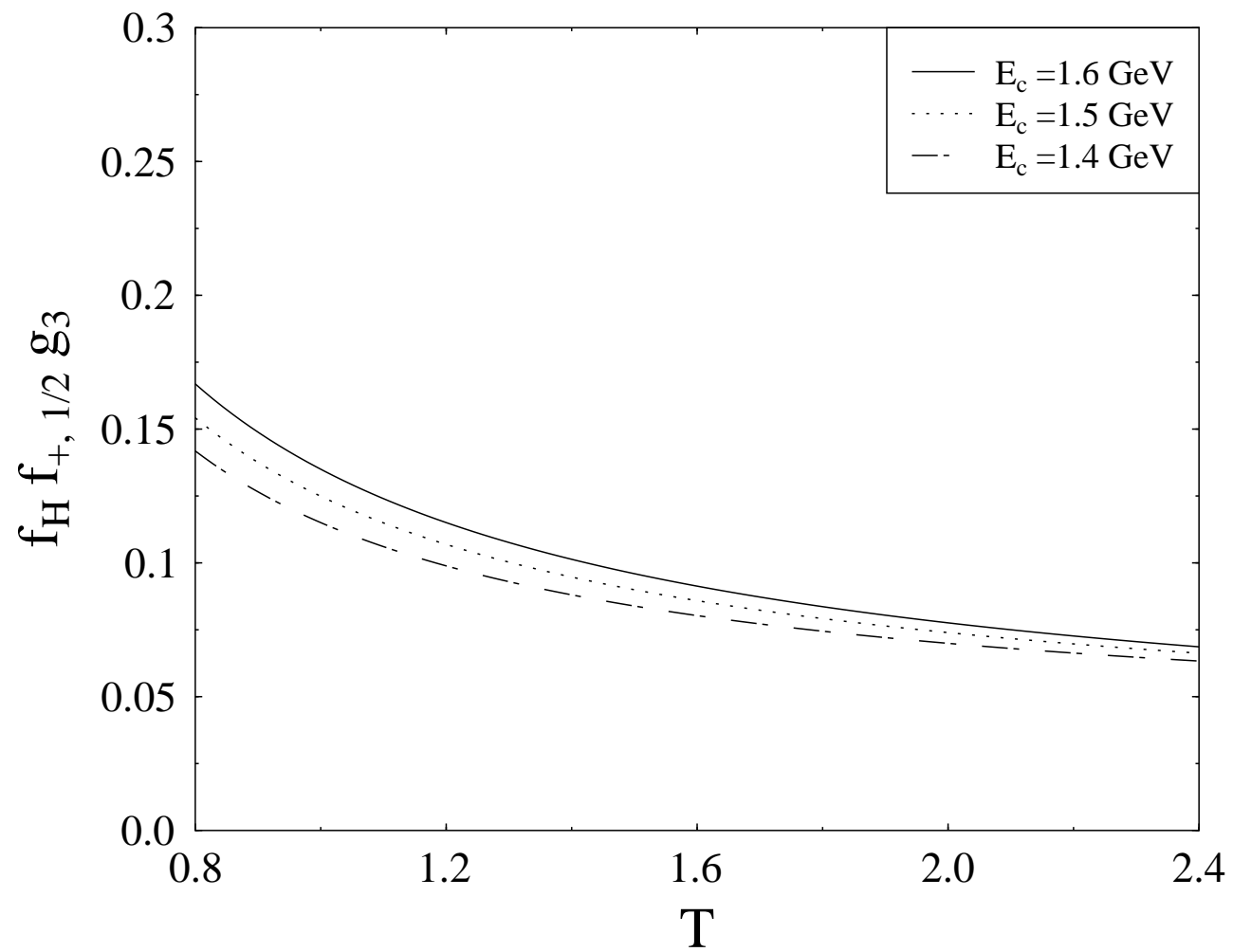


FIG. 8

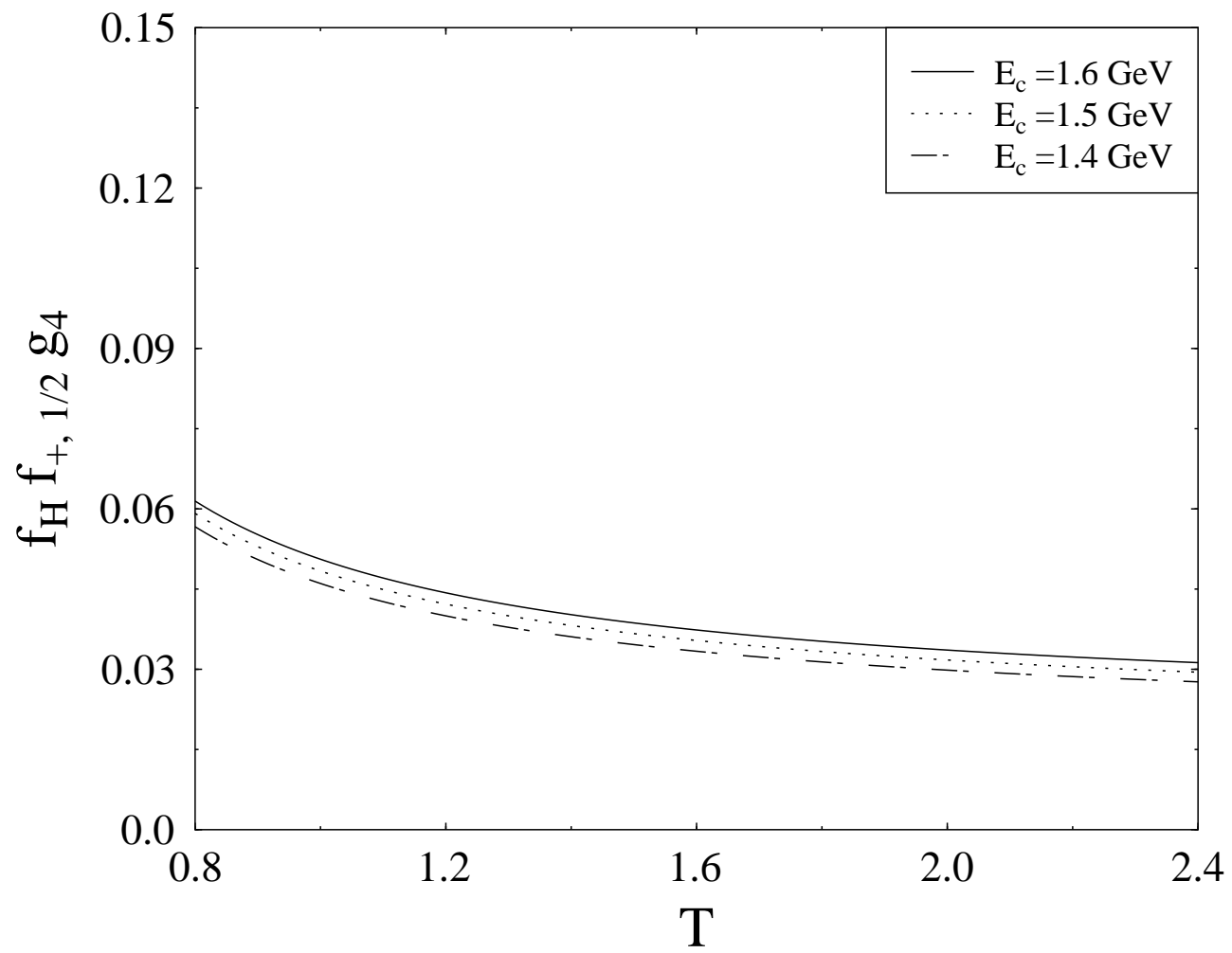


FIG. 9

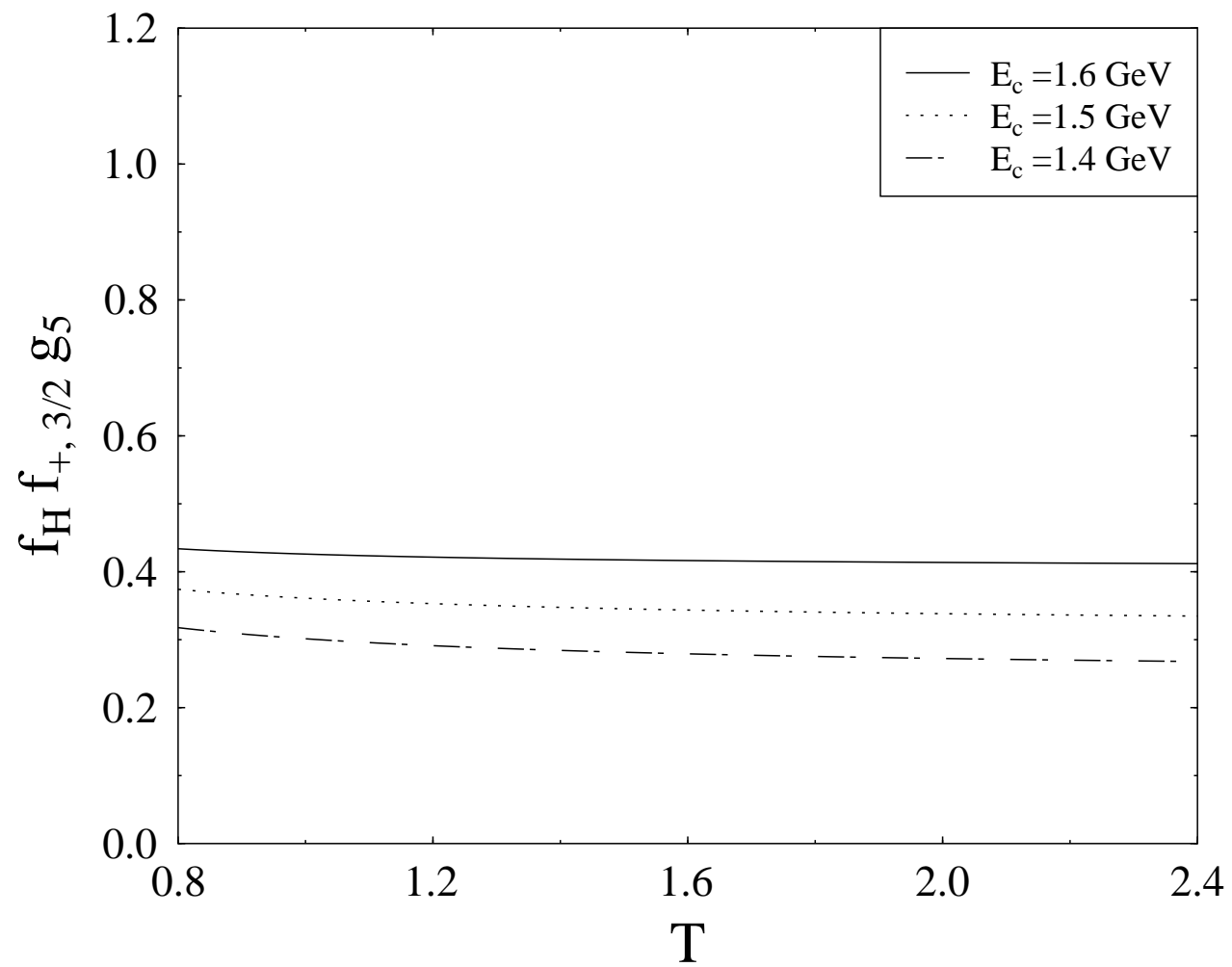


FIG. 10

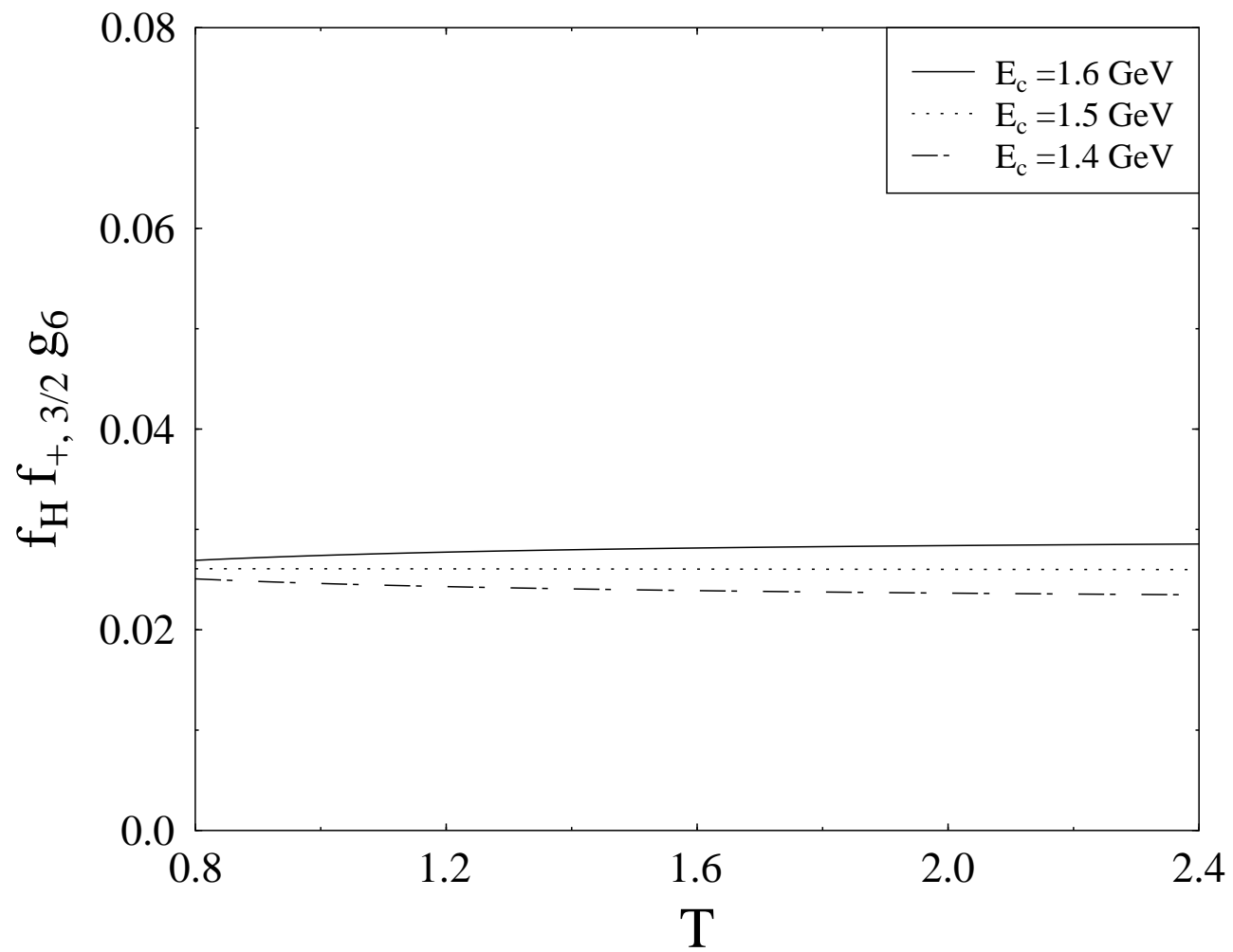


FIG. 11

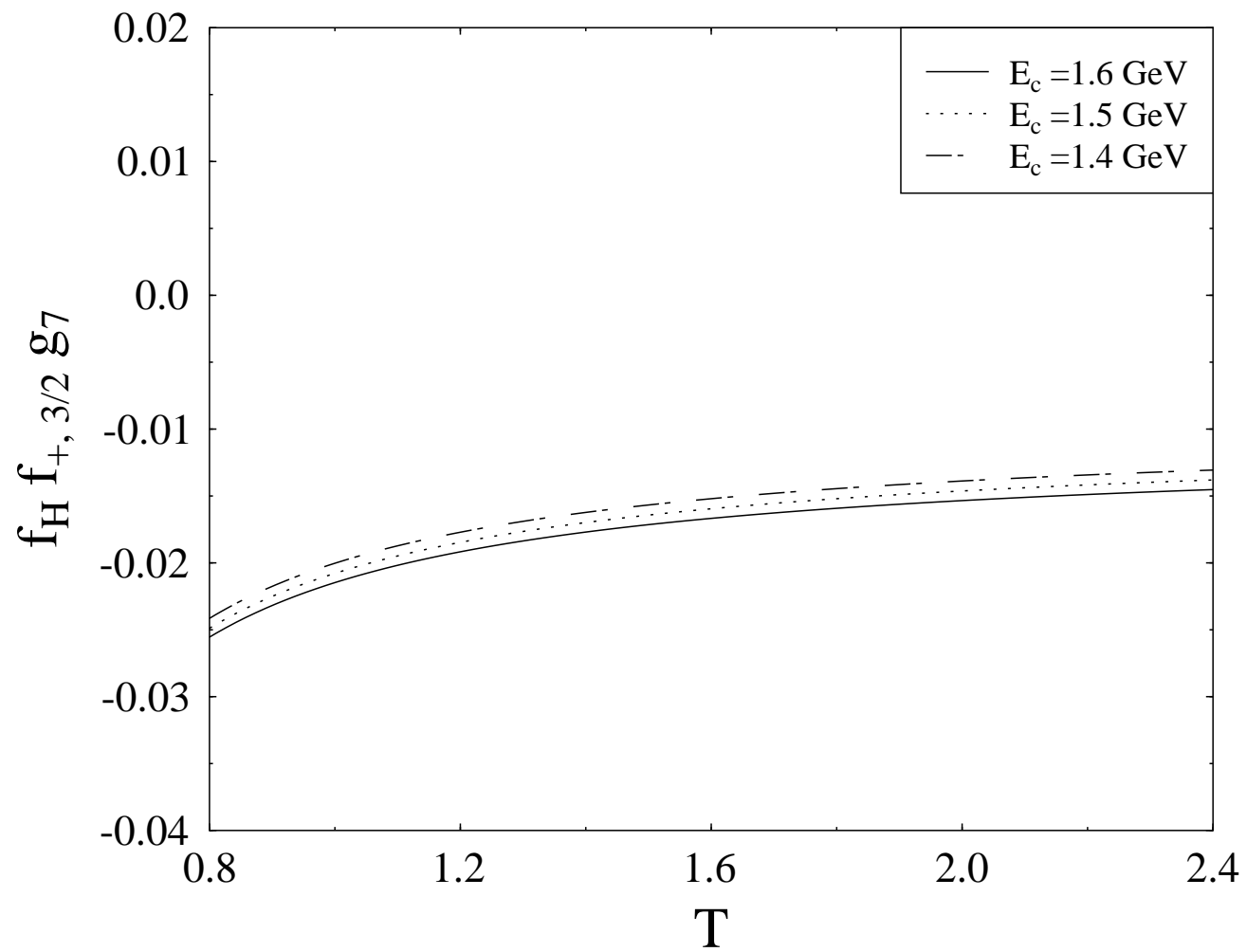


FIG. 12