

ORIGIN OF A CLASSICAL SPACE IN QUANTUM COSMOLOGIES¹

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The influence of vector fields on the origin of classical space in quantum cosmologies and on a possible compactification process in multidimensional gravity is investigated. It is shown that all general features of the transition between the classical and quantum evolution regimes can be obtained within the simplest Bianchi-I model for an arbitrary number of dimensions. It is shown that the classical space appears when the horizon size reaches the smallest of the characteristic scales (the characteristic scale of inhomogeneity or a scale associated with vector fields). In the multidimensional case the presence of vector fields completely removes the initial stage of the compactification process which takes place in the case of vacuum models [1].

One of the most important problems of quantum cosmology is an adequate description of the origin of the classical background space. Indeed, the transition from a pure quantum regime to a quasiclassical one forms the initial properties of the early Universe and therefore determines whether the subsequent stages contain an inflationary period and which kind of initial quantum state should be chosen for its realization [2]. In vacuum inhomogeneous models, the origin of classical background space was first investigated in Ref. [3]. It was found that the background metric belongs to the class of quasi-isotropic spaces, and the origin of the classical space corresponds to the instant when the horizon size matches the inhomogeneity scale of the space. However, a complete investigation of the problem requires studying the influence of different matter sources and also reserving the possibility that our Universe has extra dimensions, as predicted by a number of unified theories [4]. In the latter case, the transition between quantum and classical regimes should include the so-called compactification stage [5]. And indeed, as was shown in Refs. [1], quantum evolution of inhomogeneous models in dimensions smaller or equal than ten includes an initial stage of such a compactification. In this case the initial expansion of the Universe proceeds in an anisotropic way when, along the extra dimensions, scales decrease, and the expansion only occurs in three dimensions.

In this paper we study the influence of vector fields on the origin of classical space and on the possible compactification process in multidimensional gravity. It turns out that, in general, classical space appears when the horizon size reaches the minimal one among the scales: the characteristic inhomogeneity scale or the characteristic scale associated with the vector fields,

which is an analogue of Jeans' wavelength (in the present paper we do not consider the inhomogeneous case, and therefore we shall discuss the second possibility only). However, in multidimensional case the presence of vector fields completely removes the initial compactification stage, the process found in Refs. [1]. This may be a good reason why the vector fields should not be included as external fields, but should rather be composed from additional metric components in the compactification process.

First, we note that the origin of a background space is not a specific problem for quantum cosmology only. Such a problem does also exist in classical theory when the gravitational field and matter sources are described by a probabilistic measure distribution with unstable statistical properties. Just this case was shown to be realized in approaching the cosmological singularity [6]. It turns out that in both cases (classical or quantum cosmology) the background space formation mechanism is the same, while the nature of the statistical description is different. Moreover, estimates for the formation time of the background differ by a factor which collects all quantum corrections and depends on the choice of the initial conditions (and also on the specific scheme of quantization). This ensures the correctness of our consideration and provides a hope that whatever quantum gravity will be constructed in the future, our results will survive, (save, probably, minor corrections).

Let $A_\mu = (\varphi, A_\alpha)$ be a vector field ($\alpha = 1, 2, \dots, n$), and for the metric we shall use the standard decomposition

$$ds^2 = N^2 dt^2 - g_{\alpha\beta} (dx^\alpha + N^\alpha dt)(dx^\beta + N^\beta dt). \quad (1)$$

Then the action takes the form (in what follows we use the Planckian units)

$$I = \int d^n x dt \left\{ \pi^{\alpha\beta} \frac{\partial}{\partial t} g_{\alpha\beta} + \pi^\alpha \frac{\partial}{\partial t} A_\alpha \right\}$$

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$$+ \varphi \partial_\alpha \pi^\alpha - NH^0 - N^\alpha H_\alpha \}, \quad (2)$$

where

$$H^0 = \frac{1}{\sqrt{g}} \left\{ \pi_\beta^\alpha \pi_\alpha^\beta - \frac{1}{n-1} (\pi_\alpha^\alpha)^2 + \frac{1}{2} g_{\alpha\beta} \pi^\alpha \pi^\beta + V \right\}, \quad (3)$$

$$H_\alpha = -\nabla_\beta \pi_\alpha^\beta + \pi^\beta F_{\alpha\beta}, \quad (4)$$

here $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, $V = g(\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - R)$, and R is the scalar curvature with the metric $g_{\alpha\beta}$. Varying the action with respect to φ , we find the constraint $\partial_\alpha \pi^\alpha = 0$, so it suffices to consider only the transverse parts for A_α and π^α . Thus in what follows we set $\varphi = 0$ and $\partial_\alpha \pi^\alpha = 0$ to be satisfied.

It is convenient to use the so-called generalized Kasner-like parametrization of the dynamical variables [6, 1]. The metric components and their conjugate momenta are represented as follows:

$$g_{\alpha\beta} = \sum_a \exp\{q^a\} \ell_\alpha^a \ell_\beta^a, \quad \pi_\beta^\alpha = \sum_a p_a L_a^\alpha \ell_\beta^a, \quad (5)$$

where $L_a^\alpha \ell_\alpha^b = \delta_a^b$ ($a, b = 0, \dots, (n-1)$), and the vectors ℓ_α^a contain only $n(n-1)$ arbitrary functions of the spatial coordinates. A further parametrization may be taken in the form

$$\ell_\alpha^a = U_b^a S_\alpha^b, \quad U_b^a \in SO(n), \quad S_\alpha^a = \delta_\alpha^a + R_\alpha^a \quad (6)$$

where R_α^a denotes the triangle matrix ($R_\alpha^a = 0$ as $a < \alpha$). Substituting Eqs.(5), (6) into (2), we find the following expression for the action functional:

$$I = \int_S (p_a \frac{\partial q^a}{\partial t} + T_a^\alpha \frac{\partial R_\alpha^a}{\partial t} + \pi^\alpha \frac{\partial A_\alpha}{\partial t} - NH^0 - N_\alpha H^\alpha) d^n x dt, \quad (7)$$

where $T_a^\alpha = 2 \sum_b p_b L_b^\alpha U_a^b$, and the Hamiltonian constraint acquires the structure

$$H^0 = \frac{1}{\sqrt{g}} \left\{ \sum p_a^2 - \frac{1}{n-1} \left(\sum p_a \right)^2 + \frac{1}{2} \sum e^{q^a} (\pi^\alpha)^2 + V \right\}. \quad (8)$$

In the last equation the potential V collects all spatial derivatives, and we have used the notation $\pi^\alpha = \sum \pi^\alpha \ell_\alpha^a$. In case $n = 3$, the functions R_α^a are only connected by transformations of a coordinate system and may be removed by resolving the momentum constraints $H^\alpha = 0$ [6]. However, in the multidimensional case the functions R_α^a contain $\frac{1}{2}n(n-3)$ dynamical functions as well.

It can be shown that near the singularity, in the case $n > 3$, all spatial derivatives can be neglected in the leading order. Therefore, we neglect the potential V (the terms $F_{\alpha\beta}$ and R) in the action (8). In the case $n = 3$, the curvature term cannot be neglected. However, in this case the kinetic term of the vector field

($\frac{1}{2}g_{\alpha\beta}\pi^\alpha\pi^\beta$) induces precisely the same type of evolution (from qualitative and even quantitative viewpoints) as that of the curvature terms.

Thus, in this approximation, the Einstein equations formally coincide with the equations for homogeneous Bianchi-I model, and it is a remarkable fact that, near the singularity, a homogeneous Bianchi-I model with a vector field contains all qualitative features of the general inhomogeneous models. In what follows we shall use the gauge $N^\alpha = 0$ and, for the sake of simplicity, consider a homogeneous model, i.e., all functions depend on time only. We also use the normalization of the space volume $V^n = \int_S d^n x = 1$. Thus we find (within our approximation) the equations for the vector field

$$E_\alpha = \frac{\partial}{\partial t} A_\alpha = \frac{N}{\sqrt{g}} g_{\alpha\beta} \pi^\alpha, \quad (9)$$

$$\frac{\partial}{\partial t} \pi^\alpha = 0. \quad (10)$$

The last equation gives $\pi^\alpha = \text{const}$, and in what follows we shall treat the quantities π^α as external parameters (which are actually eigenstates of the respective operators). The rest of the present paper repeats mainly the method suggested in Ref. [3]. Near the singularity, it is convenient to make use of the following parametrization of the scale functions [6]:

$$q^a = \ln R^2 + Q_a \ln g; \quad \sum Q_a = 1, \quad (11)$$

where we distinguish a slow function of time R , which characterizes the absolute value of the metric functions [7, 8] and is specified by initial conditions (see below), and the anisotropy parameters Q_q and $\ln g = \sum q^a - 2n \ln R$ can be expressed in terms of the new set of variables τ , y^i ($i = 1, 2, \dots, (n-1)$), as follows:

$$Q_a(y) = \frac{1}{n} \left(1 + \frac{2y^i A_i^a}{1+y^2} \right), \quad \ln g = -n e^{-\tau} \frac{1+y^2}{1-y^2} \quad (12)$$

where A_i^a is a constant matrix, see, e.g., Refs. [6]. The parametrization (12) has the range $y^2 < 1$ and $-\infty < \tau < \infty$ ($0 \leq g \leq 1$), and an appropriate choice of the function R allows one to cover by this parametrization the whole classically allowed region of the configuration space.

The evolution (rotation) of the Kasner vectors results in a slow time dependence of the functions π^α , and it can be shown that these functions are completely determined by the momentum constraints, while the evolution of the scale functions is described by the action

$$I = \int \left\{ \left(\vec{P} \frac{\partial \vec{y}}{\partial t} + h \frac{\partial \tau}{\partial t} \right) - \frac{N}{n(n-1)R^n \sqrt{g}} e^{2\tau} [\varepsilon^2 - h^2 + U(\tau, \vec{y})] \right\} dt, \quad (13)$$

where $\varepsilon^2 = \frac{1}{4}(1-y^2)^2 \vec{P}^2$, and the potential term U (which comes from the kinetic term of the vector field)

has the following structure:

$$U = n(n-1)R^2 e^{-2\tau} \sum_{a=1}^n (\pi^a)^2 g^{Q_a}. \quad (14)$$

Here the coefficients π^a (projections of π^α on the Kasner vectors) are slow functions of $\ln g$ and characterize the initial intensity of the vector field. In the approximation of deep oscillations, when $g \ll 1$, this potential can be modelled by a set of potential walls:

$$g^{Q_a} \rightarrow \theta_\infty[Q_a] = \begin{cases} +\infty, & Q_a < 0, \\ 0, & Q_a > 0, \end{cases} \quad (15)$$

and is independent of the Kasner vectors $U_\infty = \sum \theta_\infty(Q_a)$.

By solving the Hamiltonian constraint $H = 0$ in (13) we determine the ADM action [9], reduced to the physical sector as follows:

$$I = \int (\vec{P}_{\vec{y}} \cdot \frac{d\vec{y}}{d\tau} - H_{ADM}) d\tau, \quad (16)$$

where $H_{ADM} \equiv -h = \sqrt{\varepsilon^2 + U}$ is the ADM Hamiltonian and τ plays the role of time ($\dot{\tau} = 1$), which corresponds to the gauge

$$N_{ADM} = n(n-1)R^n \sqrt{g} [2H_{ADM}]^{-1} e^{-2\tau}.$$

The applicability condition of the approximation (15) can be written as follows:

$$\varepsilon^2 \gg U \quad (17)$$

as $Q_a > \delta > 0$ ($\delta \ll 1$). Thus, from the condition that the approximation of deep oscillations (15) breaks at the instant $g \sim 1$, one finds that the function R should be chosen as follows: $R^2 = \varepsilon^2 [n(n-1)\pi^2]^{-1} e^{2\tau}$ (where $\pi^2 = \sum (\pi^a)^2$), and the inequality (17) reads $g \ll 1$.

The synchronous cosmological time is related to τ by means of the equation $dt = N_{ADM} d\tau$, from which we find the estimate $\sqrt{g} \sim t/t_0$, where $t_0 = cL^n \varepsilon^{n-1}$, (cf. Ref. [3]), $L \sim 1/\pi$ is a characteristic scale related to the vector field, ε is the ADM energy density ($\varepsilon = \text{const}$), and c is a slow (logarithmic) function of time ($c \sim 1$ as $g \rightarrow 1$). Thus, in the synchronous time, the upper limit of the approximation (15) is $t \sim t_0$. We note that from the physical viewpoint t_0 corresponds to the instant when the horizon size reaches the characteristic scale related to the energy of the vector field, and both terms in the Hamiltonian constraint (the kinetic energies of the anisotropy and of the vector field) acquire the same order.

The physical sector of the configuration space (the variables \vec{y}) is a realization of the Lobachevsky space, and the potential U_∞ bounds the part $K = \{Q_a \geq 0\}$. Quantization of this system can be carried out as follows. The ADM energy density represents a constant of motion, and therefore we can define stationary states

as solutions to the eigenvalue problem for the Laplace–Beltrami operator $-\varepsilon^2 = \Delta + \frac{(n-2)^2}{4}P$ (see Refs. [10, 1])

$$\left(\Delta + k_J^2 + \frac{(n-2)^2}{4}P \right) \varphi_J(y) = 0, \quad \varphi_J \Big|_{\partial K} = 0, \quad (18)$$

where the Laplace operator Δ is constructed via the metric $\delta l^2 = h_{ij} \delta y^i \delta y^j = 4(\delta y)^2 (1-y^2)^{-2}$. The eigenstates φ_J are classified by the integer number J and obey the orthogonality and normalization relations

$$(\varphi_J, \varphi_{J'}) = \int_K \varphi_J^*(y) \varphi_{J'}(y) D\mu(y) = \delta_{JJ'}, \quad (19)$$

where $D\mu(y) = \frac{1}{a_n} \sqrt{h} d^2 y(x)$ and a_n is the volume of K . Thus, an arbitrary solution Ψ to the Schrödinger equation $i\partial_\tau \Psi = H_{ADM} \Psi$ takes the form

$$\Psi = \sum_J \exp(-ik_J \tau) \varphi_J(y) C_J \quad (20)$$

where C_J are arbitrary constants which are to be specified by initial conditions. Within our approximation, these constants also represent arbitrary functions of the vector field

$$C_J(A) = \int d^{n-1} \pi C_{J,\pi} \exp(i\pi^\alpha A_\alpha), \quad (21)$$

and the normalization condition reads ($\int d^{n-1} A \sum_J |C_J(A)|^2 = 1$). The probabilistic distribution of the variables y has the standard form $P(y, \tau) = |\Psi(y, \tau)|^2$. The eigenstates φ_J determine stationary (in terms of the anisotropy parameters $Q(y)$) quantum states and describe an expanding universe with a fixed energy of the anisotropy.

For an arbitrary quantum state Ψ we can determine the background metric $\langle ds^2 \rangle$. However, such a background is stable and has sense only when quantum fluctuations around it are small. In case $g \ll 1$, fluctuations well exceed the average metric, and the background is hidden [10]. Indeed, consider the moments of scale functions $\langle a_i^M \rangle = \langle R^M g^{\frac{M}{2} Q_i} \rangle$. The leading contribution in $\langle a_i^M \rangle$ comes from those points of the billiard at which Q_i takes a minimal value. Such points are at the boundary of the billiard, and the minimal value of Q is $Q_i^{\min} = 0$. Since $\varphi_J(\partial K) = 0$, in the neighbourhood of ∂K we have $\varphi_J \approx \eta_J Q$, and the probability density is

$$P_\tau(Q) = \int_K P(y, \tau) \delta(Q - Q(y)) \sqrt{h} d^{n-1} y \approx B_J(\tau) Q^{f(n)} \quad (22)$$

as $Q \rightarrow 0$. Here $f(n) = 2$ for $n > 3$ and $f(3) = 3/2$ (since in case $n = 3$ we get $\sqrt{h} \sim 1/\sqrt{Q}$). Thus, in the limit $g \rightarrow 0$, the moments of the function $\langle a_i^M \rangle$ are given by ($M > 0$)

$$\langle a_i^M \rangle \simeq D_i(M, \tau) \frac{1}{(M \ln 1/g_*)^{f(n)+1}}, \quad (23)$$

where $g_* = g(\tau, y^*)$ and D_i is a function slowly varying in time, which collects the information of the initial quantum state.

Consider now an arbitrary stationary state φ_J which gives the stationary probabilistic distribution $P(y) = |\varphi_J|^2$. In this case $D \simeq bk_J^M (L^M)_J$ is a constant, where b comes from the uncertainty in the operator ordering. Thus, for the intensity of quantum fluctuations one finds the divergent, in the limit $g \rightarrow 0$ ($\tau \rightarrow -\infty$), expression $\langle \delta^2 \rangle = (\langle a^2 \rangle / \langle a \rangle^2 - 1) \sim (\ln 1/g_*)^{f(n)+1}$, which explicitly shows the instability of the average geometry as $g \ll 1$. The intensity of quantum fluctuations reaches the order $\delta \sim 1$ at the instant $t \sim (L^n)_J k_J^{n-1}$ ($g \sim 1$) when the anisotropy functions can be described by small perturbations $a_i^2 = R^2(1 + Q_i \ln g + \dots)$, and the Universe acquires a quasi-isotropic character. This instant can be considered as the origin of a stable classical background. It is important that strong anisotropy of a stable classical geometry turns out to be forbidden. We recall that the hypothesis that the very beginning of the Universe evolution should be described by quasi-isotropic models, while anisotropic models are forbidden, was first suggested in Ref. [11] from a different consideration (as a result of the impossibility of constructing a self-consistent theory which could account for back reaction of particle creation in such models).

To conclude, we make two important remarks. Firstly, in the presence of the vector field, the evolution of the metric undergoes spontaneous stochastization for an arbitrary number of dimensions [12]. This follows from the fact that potential walls always restrict a finite region of the configuration space ($\text{Vol.}(K) < \infty$ for arbitrary n), and the metric should be described by an invariant measure. In this case, estimates for the moments of scale functions can be obtained by setting $P(y, \tau) = 1$, which gives the replacement $f(n) \rightarrow f(n) - 2$ in (23). Thus the above picture of the formation of a stable background remains to be valid in classical theory as well (with minor corrections of the estimates). And secondly, the presence of a vector field drastically changes the structure of the configuration space in dimensions $n > 3$. The billiard turns out to be finite for an arbitrary number of dimensions [12], but the payment is the fact that the initial stage of the compactification process, which was found in vacuum models [1], is absent. Indeed, in the model considered above, the anisotropy parameters have the range $1 \geq Q \geq 0$ and remain always positive (we recall that in the vacuum case these functions could have negative values, e.g., they changed in the range $1 \geq Q \geq -(n-3)/(n+1)$). This means that the expansion always proceeds in such a way that the lengths monotonically increase in all spatial directions. This probably represents a rather serious reason why vector fields should not be included as external fields in multidimensional gravity.

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References

- [1] A.A. Kirillov, *Pis'ma Zh. Eksp. Teor. Fiz.* **62**, 81 (1995) [*JETP Lett.* **62**, 89 (1995)].
- [2] A.D. Linde, "Particle Physics and Inflationary Cosmology" (Harwood Academic, 1990).
- [3] A.A. Kirillov and G. Montani, *Pis'ma Zh. Eksp. Teor. Fiz.* **66**, 449 (1997).
- [4] M.B. Green, J.H. Schwarz, and E. Witten, "Superstring Theory", Cambridge Univ. Press, 1988.
- [5] A. Chodos and S. Detweiler, *Phys. Rev. D* **21**, 2176 (1980);
M. Demiański, Z. Golda, M. Heller and M. Szydlowski, *Clas. Qu. Grav.* **3**, 1196 (1986); *ibid.*, **5**, 733 (1988);
K. Maeda and P.Y. Pang, *Phys. Lett.* **180B**, 29 (1986).
- [6] A.A. Kirillov, *Zh. Eksp. Teor. Fiz.* **103**, 721 (1993) [*Sov. Phys. JETP* **76**, 355 (1993)];
A.A. Kirillov and V.N. Melnikov, *Phys. Rev. D* **51**, 723 (1995).
- [7] C.W. Misner, *Phys. Rev. Lett.* **22**, 1071 (1969).
- [8] C.W. Misner, K.S. Thorne and J.A. Wheeler, "Gravitation" (Freeman, San Francisco, 1973), Vol.2.
- [9] R. Arnowitt, S. Deser, and C.W. Misner, *in*: "Gravitation: An Introduction to Current Research", ed. L. Witten, Wiley, New York, 1962, p.227.
- [10] A.A. Kirillov, *Phys. Lett.* **399B**, 201 (1997).
- [11] V.N. Lukash and A.A. Starobinsky, *Zh. Eksp. Teor. Fiz.* **66**, 1515 (1974).
- [12] T. Damour and M. Henneaux, *Phys. Rev. Lett.* **85**, 920 (2000).