Restoring Unitarity in BTZ Black Hole

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Abstract

Whether or not system is unitary can be seen from the way it, if perturbed, relaxes back to equilibrium. The relaxation of semiclassical black hole can be described in terms of correlation function which exponentially decays with time. In the momentum space it is represented by infinite set of complex poles to be identified with the quasi-normal modes. This behavior is in sharp contrast to the relaxation in unitary theory in finite volume: correlation function of the perturbation in this case is quasi-periodic function of time and, in general, is expected to show the Poincaré recurrences. In this paper we demonstrate how restore unitarity in the BTZ black hole, the simplest example of eternal black hole in finite volume. We start with reviewing the relaxation in the semiclassical BTZ black hole and how this relaxation is mirrored in the boundary conformal field theory as suggested by the AdS/CFT correspondence. We analyze the sum over $SL(2, \mathbf{Z})$ images of the BTZ space-time and suggest that it does not produce a quasi-periodic relaxation, as one might have hoped, but results in correlation function which decays by power law. We develop our earlier suggestion and consider a non-semiclassical deformation of the BTZ space-time that has structure of wormhole connecting two asymptotic regions semiclassically separated by horizon. The small deformation parameter λ is supposed to have non-perturbative origin to capture the finite N behavior of the boundary theory. The discrete spectrum of perturbation in the modified space-time is computed and is shown to determine the expected unitary behavior: the corresponding time evolution is quasi-periodic with hierarchy of large time scales $\ln 1/\lambda$ and $1/\lambda$ interpreted respectively as the Heisenberg and Poincaré time scales in the system.

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1 Introduction

That unitarity is violated in black hole was suggested by Hawking [1] following his remarkable discovery that semiclassical black holes thermally radiate. This conclusion has been debated (see [2], [3]) ever since. In spite of considerable effort made to resolve this issue the black hole unitarity remains one of the most intriguing problems in physics. String theory offers a unifying approach to the short distance phenomena including the gravitational interaction that is based on intrinsically unitary quantum mechanical picture. This explains the long-standing belief that the unitarity problem can be resolved in string theory. One of the most recent and most promising suggestions in this direction is the AdS/CFT correspondence (see [4] for review and [5] for more recent developments) which suggests that in certain limit string theory (or, semiclassically, supergravity theory) on asymptotically AdS space-time can be equivalently described in terms of conformal field theory (CFT) living on the boundary of the space-time. The boundary CFT is supposed to be unitary and thus the correspondence should provide us with complete unitary description of black holes semiclassically appearing as solutions to the supergravity theory.

Recently, Maldacena [6] made a proposal that unitarity restores if one takes into account the topological diversity of gravitational instantons approaching asymptotically same boundary manifold. This proposal was studied in [7] in the case of an analogous problem in de Sitter space. It was further discussed in [8], [9], [10], [11], [12], [13]. It was realized that the problem can be analyzed by studying the relaxation in black hole and boundary theory after small perturbation (in the context of the AdS/CFT correspondence the relaxation was first discussed in [14]). The appropriate quantity to look at is the correlation function of the perturbation taken at two different moments of time: when the perturbation was first applied and at a later time. Two types of relaxation were identified [8]: oscillatory (or, in general, quasi-periodic) and the exponential decay. In unitary theory in finite volume the correlation function is quasi-periodic and, in general, is expected to show the so-called Poincaré recurrences, i.e. the initial value is approached arbitrarily close in finite (although very large) period of time. In the case of black hole the correlation function, however, is exponentially decaying so that the initial configuration is never repeated. This temporal behavior is governed by infinite set of complex quasi-normal modes [14], [8]. This is a clear manifestation of non-unitary nature of semiclassical black hole. We stress that it is the presence of horizon that makes the spectrum of perturbation continuous [8], [10] and shifts the poles to the complex region. In the case of unitary theory in finite volume the spectrum is discrete and complex frequencies never appear. Note, however, that finding a way to assign a discrete spectrum to black hole is not the end of the story. The spectrum should be sufficiently non-trivial to generate the complex time evolution: black hole is supposed to have very large Poincaré recurrence time estimated as exponential in the entropy and also show exponential decay for intermediate time scales during which the usual thermodynamic description would be accurate. Thus, say, simple equidistant spectrum (like the one for a particle in a circle) would not work: the recurrence time would be too short and the time evolution be strictly periodic like that of a clock.

In this paper we study the eternal BTZ black hole [15] in three-dimensional AdS space. We start with reviewing the relaxation in the black hole and in the thermal AdS and how it is mirrored in the boundary CFT [8]. These two spaces are only two (dominating semiclassically) members of much larger family of gravitational instantons approaching asymptotically two-dimensional torus. The whole family includes the $SL(2, \mathbb{Z})$ images of

the BTZ black hole. These are that topologically distinct geometries which we are instructed to sum over in the AdS/CFT correspondence. We analyze whether the sum over these images may actually produce a quasi-periodic result. We conclude that even though one sums over exponentially decaying individual terms the sum decays much slower than exponent, namely it decays by power law. The resulting correlation function is however not quasi-periodic. It is in agreement with earlier analysis in [8], [10], [13]. This suggests that we should look for solution to the unitarity problem within a nonsemiclassical description of black hole.

We follow our earlier suggestion [9] and consider a nonsemiclassical deformation of the BTZ space-time. The deformation parameter λ is supposed to have non-perturbative origin, $\lambda \sim e^{1/4G}$, so that on the boundary side the deformation should account for finite N behavior of the boundary theory. In the deformed metric horizon is replaced by throat that connects two asymptotic regions separated by horizon in the semiclassical BTZ black hole. The complete geometry resembles that of wormhole. Once horizon has been removed the complex quasi-normal frequencies disappear and the spectrum becomes discrete and real. We compute the spectrum and find that it has the form of the massive spectrum with mass m proportional to λ . This spectrum appears to be universal since it relies only on modification of the geometry in the near-horizon region. The spectrum naturally incorporates hierarchy of two large time scales, namely $\ln 1/\lambda$ and $1/\lambda$. The second time scale is the largest in the system and describes the long time correlations. Moreover it has the right value and can be naturally identified with the black hole Poincaré recurrence time.

The paper is organized as follows. In section 2 we review the description of relaxation in the BTZ black hole and in boundary CFT and how this fits in the AdS/CFT correspondence. This part is based on the paper [8]. In subsection 2.3.2 we analyze the sum over $SL(2, \mathbb{Z})$ family. In section 3 we introduce the nonsemiclassical modification of the BTZ metric and explain how finite entropy originates in this metric. In section 4 we apply the AdS/CFT rules and calculate the conformal anomaly in the boundary theory. The anomaly occurs to be non-vanishing that we interpret as a manifestation of a (nonperturbative) mass gap in the boundary theory. In section 5 we compute the spectrum for scalar perturbation in the background of the modified BTZ metric and show that it determines the quasi-periodic time evolution with two large time scales interpreted as the Heisenberg and Poincaré time scales. We conclude in section 6.

2 Relaxation and the unitarity problem

The way how thermal system reacts on small pertubation and whether the subsequent time evolution drives the system back to thermal equilibrium gives us important information about the system and the nature of its thermal state. Two possible types of reaction are known: the perturbation may exponentially decay so that the system relaxes back to equilibrium shortly after the perturbation has been applied or the perturbation oscillates showing quasi-periodic or even chaotic behavior. In the second case the full equilibrium is never reached. The system always keeps information about the initial perturbation and releases it from time to time reproducing the shape of initial perturbation. This behavior is typical for unitary system in finite volume. General arguments show that time evolution of unitary system should show the so-called Poincaré recurrences although the recurrence time may be extremely large, this time is estimated as e^S where S is entropy of the system. The information is thus never lost in finite volume – the clear manifestation of the unitary evolution. The exponential decay may happen only in infinite volume or if the system is in finite volume but is non-unitary. The obvious example of the latter is black hole whose semiclassical behavior appears to violate unitarity.

2.1 Relaxation in black hole: quasi-normal modes

We consider (2+1)-dimensional BTZ black hole with metric given by

$$ds^{2} = -\sinh^{2} y \ dt^{2} + dy^{2} + \cosh^{2} y \ d\phi^{2} \quad , \tag{2.1}$$

where for simplicity we consider non-rotating black hole and set the size of the horizon $r_+ = L$ and AdS radius l = 1. The coordinate ϕ is periodic with period L so that the boundary has topology of cylinder and L sets the finite size for the boundary system. L is related to the mass of BTZ black hole as $L = \sqrt{MG}$. A bulk perturbation $\Phi_{(m,s)}$ of mass m and spin s should satisfy the quasi-normal boundary condition, i.e. it should be in-going at the horizon and have vanishing flux at the infinity. The latter condition comes from the fact that in the asymptotically AdS space-times the effective radial potential is growing at infinity so that there can be no propagating modes as well as no leakage of the energy through the boundary. The relevant radial equation takes the form of the hypergeometric equation which is exactly solvable. The quasi-normal modes in general fall into two sets [16, 17]

$$\omega = \frac{2\pi}{L} \mathbf{1} - 4\pi i T_L(n+\bar{h})$$

$$\omega = -\frac{2\pi}{L} \mathbf{1} - 4\pi i T_R(n+h) , \quad \mathbf{1} \in \mathbf{Z} , \quad n \in \mathbf{N}$$
(2.2)

where the left- and right-temperatures $T_L = T_R = 1/2\pi$ and (h, \bar{h}) have the meaning of the conformal weights of the dual operator $\mathcal{O}_{(h,\bar{h})}$ corresponding to the bulk perturbation $\Phi_{(m,s)}$, with $h + \bar{h} = \Delta(m)$, $h - \bar{h} = s$ and $\Delta(m)$ is determined in terms of the mass m.

The appearance of complex modes (2.2) does not come as a surprise – the quasinormal boundary conditions are dissipative in nature, they say that the perturbation once created should leave the region through all possible boundaries. Since no leakage of energy happens through the spatial infinity all the dissipation goes through horizon. For comparison, in the case of global anti-de Sitter space the horizon and respectively the quasi-normal modes are absent. But, instead, one can define the normalizable modes by imposing the Dirichlet condition at infinity as well as regularity in the origin. They form a discrete set of real frequencies [18]

$$\omega = 2\pi \mathbf{1}/L + 4\pi (n+h)/L , \quad \mathbf{1} \in \mathbf{Z} , \quad n \in \mathbf{N}$$
(2.3)

where the size of the boundary is also set to be L as in the black hole case.

A simple way to understand why black hole is not characterized by a set of real frequencies like (2.3) is to observe that the perturbation effectively propagates in the infinite volume in the case of black hole. Indeed, near horizon the wave propagates freely in the so-called optical metric defined as

$$ds^2 = \sinh^2 y \ ds^2_{\text{opt}} \quad . \tag{2.4}$$

The distance to horizon in this metric is measured in terms of coordinate $z = \int dy / \sinh(y)$. Obviously it diverges as y approaches horizon at y = 0. Thus, the perturbation sees effectively infinite volume and thus can not be characterized by a discrete set of real frequencies. We stress that it is the presence of horizon which creates this effective infinite volume and eventually leads to appearance of the complex frequencies (2.2).

2.2 Relaxation in two-dimensional Conformal Field Theory

The thermal state of the black hole in the bulk corresponds to the thermal state on the CFT side. In fact, the boundary CFT factorizes on left- and right-moving sectors with temperature T_L and T_R respectively. The bulk perturbation corresponds to perturbing the thermal field theory state with operator $\mathcal{O}_{(h,\bar{h})}$. The further evolution of the system is described by the so-called Linear Response Theory (see [19]). According to this theory one has to look at the time evolution of the perturbation itself. More precisely, the relevant information is contained in the retarded correlation function of the perturbation at the moments t and t = 0 (when the perturbation has been first applied). Since the perturbation is considered to be small the main evolution is still governed by the unperturbed Hamiltonian acting on the thermal state so that the correlation function is the thermal function at temperature T. Thus, the analysis boils down to the study of the thermal 2-point function of certain conformal operators. Such a function should be doubly periodic: with period 1/T in the direction of the Euclidean time and with period L in the direction of the compact coordinate ϕ . This can be first calculated as a 2-point function on the Euclidean torus and then analytically continued to the real time.

2.2.1 Large L/Small T universality

In general the correlation function on torus can be rather complicated since its form is not fixed by the conformal symmetry. The conformal symmetry however may help to deduce the universal form of the 2-point function in two special cases: when size L of the system is infinite (temperature T is kept finite) and when inverse temperature is infinite (the size L is finite). The universal form of the (real time) 2-point function in the first case is

$$\left\langle \mathcal{O}(t,\phi)\mathcal{O}(0,0)\right\rangle = \frac{(\pi T)^{2(h+\bar{h})}}{(\sinh \pi T(\phi-t))^{2h}(\sinh \pi T(\phi+t))^{2\bar{h}}}$$
(2.5)

which for large t decays exponentially as $e^{-2\pi T(h+\bar{h})t}$. The information about the perturbation is thus lost after characteristic time set by the inverse temperature. It is clear that this happens because in infinite volume the information may dissipate to infinity. In the second case correlator

$$\langle \mathcal{O}(t,\phi)\mathcal{O}(0,0)\rangle = \frac{(\pi/iL)^{2(h+h)}}{(\sin\frac{\pi}{L}(t+\phi))^{2h}(\sin\frac{\pi}{L}(t-\phi))^{2\bar{h}}}$$
(2.6)

has the oscillatory behavior. Notice that the oscillatory behavior in the second case should have been expected since the system lives on circle. Perturbation once created at the moment t = 0 at point $\phi = 0$ travels around the circle with the speed of light and comes back to the same point at t = L. Thus, the information about the perturbation is never lost. The correlation function (2.6) as function of time represents a series of singular peaks concentrated at $t = \pm \phi + nL$, $n \in \mathbb{N}$. In fact, this behavior should be typical for any system with unitary evolution in finite volume.

It is interesting to see what happens in the intermediate regime when both L and 1/T are kept finite. In this case the behavior of the correlation functions is not universal, may depend on the (self)interaction in the system and is known only in some cases.

2.2.2 Intermediate regime: (quasi-)periodicity and unitarity

As an example of a system in the intermediate regime when both L and T are kept fixed we consider the free fermions for which the correlation function on the torus is known explicitly (e.g. [20]). The real time correlation function is

$$\langle \psi(w)\psi(0)\rangle_{\nu} = \frac{\theta_{\nu}(wT|iLT)\partial_{z}\theta_{1}(0|LT)}{\theta_{\nu}(0|iLT)\theta_{1}(wT|iLT)} \quad , \tag{2.7}$$

were $w = i(t + \phi)$ and ν characterizes the boundary conditions for $\psi(w)$. For finite temperature boundary conditions we have $\nu = 3, 4$. Using the properties of θ -functions, it is then easy to see that (2.7) is invariant under shifts $w \to w + 1/T$ and $w \to w + iL$. It is then obvious that the resulting real time correlator (2.7) is a *periodic* function of t with period L. Zeros of the theta function $\theta_1(wT|iLT)$ are known [20] to lie at w = m/T + inL, where m, n are arbitrary relative integers. Therefore, for real time t, the correlation function (2.7) is a sequence of singular peaks located at $(t + \phi) = nL$. Using the standard representation [20] of the θ -functions, we also find that in the limit $LT \to \infty$ the correlation function (2.7) approaches the left-moving part of (2.3) with h = 1/2 that exponentially decays with time,

$$\langle \psi(w)\psi(0)\rangle_{3(4)} = \frac{\pi T}{4\sinh\pi T(t+\phi)} [1\pm 2e^{-\pi LT}\cosh 2\pi T(t+\phi) + ..]$$
 (2.8)

In the opposite limit, when $LT \rightarrow 0$, it approaches the oscillating function (2.6). A natural question is how the asymptotic behavior (2.8) when size of the system is taken to infinity can be consistent with the periodicity, $t \to t + L$, of the correlation function (2.7) at any finite L? In order to answer this question we have to observe that there are two different time scales in the game. The first time scale is set by the inverse temperature $\tau_1 = 1/T$ and while the second time scale is associated with the size of the system $\tau_2 = L$. When L is taken to infinity we have that $\tau_2 \gg \tau_1$. Now, when the time t is of the order of τ_1 but much less than τ_2 the asymptotic expansion (2.8) takes place. The corrections to the leading term are multiplied by the factor $e^{-\pi LT}$ and are small. The 2-point function thus is exponentially decaying in this regime. It seems that the system has almost lost information about the initial perturbation (at t = 0). But it is not the case: as time goes on and approaches the second time scale $t \sim \tau_2$ the corrections to the leading term in (2.8) become important and the system starts to collect its memory about the initial perturbation. The information is completely recovered as $t = \tau_2$ and the time-periodicity restores. This example is instructive. Provided the two scales τ_1 and τ_2 are widely separated, $\tau_1 \ll \tau_2$, the system would seem to relax exponentially fast back to thermal equilibrium for the moments of time t such that $t \gg \tau_1$ but $t \ll \tau_2$. Observing the system during these t we would have concluded that the information about the initial perturbation had been lost completely and that the system was showing a non-unitary behavior. The unitarity however completely restores if we wait long enough until t gets close to the second time scale τ_2 . In general we should expect that unitary system in finite volume is characterized by a set of periods so that its time evolution is quasi-periodic.

2.3 CFT₂ dual to AdS_3

2.3.1 Correlation functions

As an example of a strongly coupled theory we consider the supersymmetric conformal field theory dual to string theory on AdS_3 . This theory describes the low energy excitations of a large number of D1- and D5-branes [4]. It can be interpreted as a gas of strings that wind around a circle of length L with target space T^4 . The individual strings can be simply- or multiply wound such that the total winding number is $\mathbf{k} = \frac{c}{6}$, where $c \gg 1$ is the central charge. The parameter \mathbf{k} plays the role of N in the usual terminology of large N CFT.

According to the prescription (see [4]), each AdS space which asymptotically approaches the given two-dimensional manifold should contribute to the calculation, and one thus has to sum over all such spaces. In the case of interest, the two-manifold is a torus (τ_E, ϕ) , and $\beta = 1/T$ and L are the respective periods. There exist two obvious AdS spaces which approach the torus asymptotically. The first is the BTZ black hole and the second is the so-called thermal AdS space, corresponding to anti-de Sitter space filled with thermal radiation. Both spaces can be represented (see [21]) as a quotient of three dimensional hyperbolic space H^3 , with line element

$$ds^{2} = \frac{l^{2}}{y^{2}} (dz d\bar{z} + dy^{2}) \quad y > 0 \quad .$$
(2.9)

In both cases, the boundary of the three-dimensional space is a rectangular torus with periods L and 1/T. Two configurations (thermal AdS and the BTZ black hole) are T-dual to each other, and are obtained by the interchange of the coordinates $\tau_E \leftrightarrow \phi$ and $L \leftrightarrow 1/T$ on the torus. In fact there is a whole $SL(2, \mathbb{Z})$ family of spaces which are quotients of the hyperbolic space.

In order to find correlation function of the dual conformal operators, one has to solve the respective bulk field equations subject to Dirichlet boundary condition, substitute the solution into the action and differentiate the action twice with respect to the boundary value of the field. The boundary field thus plays the role of the source for the dual operator $\mathcal{O}_{(h,\bar{h})}$. This way one can obtain the boundary CFT correlation function for each member of the family of asymptotically AdS spaces. The total correlation function is then given by the sum over all $SL(2, \mathbb{Z})$ family with appropriate weight. Let us however first consider the contribution of only two terms [6]

$$\langle \mathcal{O}(t,\phi)\mathcal{O}(0,0)\rangle \simeq e^{-S_{\text{BTZ}}} \langle \mathcal{O} \ \mathcal{O}' \rangle_{\text{BTZ}} + e^{-S_{\text{AdS}}} \langle \mathcal{O} \ \mathcal{O}' \rangle_{\text{AdS}} \quad , \tag{2.10}$$

where $S_{\text{BTZ}} = -k\pi LT/2$ and $S_{\text{AdS}} = -k\pi/2LT$ are Euclidean action of the BTZ black hole and thermal AdS₃, respectively [22]. On the Euclidean torus $\langle \rangle_{\text{BTZ}}$ and $\langle \rangle_{\text{AdS}}$ are T-dual to each other. Their exact form can be computed explicitly [24]. After the analytical continuation $\tau_E = it$, the BTZ contribution

$$\langle \mathcal{O}(t,\phi)\mathcal{O}(0,0)\rangle_{\text{BTZ}} = \sum_{n} \frac{1}{(\sinh\frac{\pi}{\beta}(\phi-t+nL))^{\Delta}(\sinh\frac{\pi}{\beta}(t+\phi+nL))^{\Delta}} \quad (2.11)$$

exponentially decays with time. The result for the thermal AdS

$$\langle \mathcal{O}(t,\phi)\mathcal{O}(0,0)\rangle_{\text{AdS}} = \sum_{n} \frac{1}{(\sin\frac{\pi}{L}(t+\phi+i\beta n))^{\Delta}(\sin\frac{\pi}{L}(t-\phi+i\beta n))^{\Delta}}$$
(2.12)

is periodic in time with period L. It represents a periodic sequence of singular peaks at $t \pm \phi = nL$.

Thus, the total 2-point function (2.10) has two contributions: one is exponentially decaying and another is oscillating. So that (2.10) is *not* a quasi-periodic function of time t. This can be re-formulated in terms of the poles in the momentum representation of 2-point function (see [17] and [25]). The poles of $\langle \rangle_{BTZ}$ are exactly the complex quasi-normal modes (2.2) while that of $\langle \rangle_{AdS}$ are the real normalizable modes (2.3). Depending on the value of LT, one of the two terms in (2.10) dominates [22]. For high temperature (LT is large) the BTZ is dominating, while at low temperature (LT is small) the thermal AdS is dominant. The transition between the two regimes occurs at 1/T = L. In terms of the gravitational physics, this corresponds to the Hawking-Page phase transition [26]. This is a sharp transition for large k, which is the case when the supergravity description at low temperature and exponential decay at high temperature [8].

Whether including sum over $SL(2, \mathbb{Z})$ in (2.10) we can get a quasi-periodic result is discussed in the next subsection.

2.3.2 Can sum over $SL(2, \mathbb{Z})$ family produce quasi-periodic result?

In the AdS/CFT correspondence we are instructed to sum over all possible AdS metrics which approach same boundary manifold at infinity. In the case at hand the boundary manifold is two-dimensional torus (ϕ, τ_E) with periodicities $(\phi, \tau_E) \rightarrow (\phi + Ln, \tau_E + \beta m)$, $n, m \in \mathbb{Z}$. The complex holomorphic coordinate on the torus is $w = \phi + i\tau_E$. The torus is characterized by the modular parameter $\tau = i\beta/L$. The $SL(2, \mathbb{Z})$ modular transformations act as

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d}, \quad w \to w' = \frac{w}{c\tau + d}, \quad ad - bc = 1$$
, (2.13)

where group parameters a, b, c, d are integers. In fact we should be interested in the $SL(2, \mathbb{Z})/\mathbb{Z}$ transformations, i.e. modulo the parabolic group transformations $(a, b) \rightarrow (a + c, b + d)$. These transformations are completely determined by pairs of relatively prime (c, d). In the $SL(2, \mathbb{Z})$ family the choice (a = 1, b = c = 0, d = 1) corresponds to the thermal AdS_3 while the choice (a = 0, b = 1, c = -1, d = 0) describes BTZ black hole. The gravitational action for a AdS metric asymptotically approaching the torus characterized by modular parameter τ' takes the form [22]

$$S(\tau') = \frac{\pi \mathbf{k}i}{2} \left[\frac{a\tau + b}{c\tau + d} - \frac{a\bar{\tau} + b}{c\bar{\tau} + d} \right] = -\pi \mathbf{k} \frac{\beta L}{c^2 \beta^2 + d^2 L^2} \quad . \tag{2.14}$$

Applying now AdS/CFT rules for computing the 2-point function on the torus we have to sum over the $SL(2, \mathbf{Z})/\mathbf{Z}$ family with appropriate weight [13],

$$\langle \mathcal{O}(w,\bar{w})\mathcal{O}(0,0)\rangle_{\mathrm{SL}(2,\mathbf{Z})} = \sum_{(c,d)} \sum_{n\in\mathbb{Z}} \frac{e^{-S(\tau')}}{|c\tau+d|^{2\Delta}} \frac{(\pi/L)^{2\Delta}}{|\sin\pi(\frac{w'}{L}+n\tau')|^{2\Delta}} ,$$
 (2.15)

where w' and τ' are defined in (2.13) and for simplicity we take $h = \bar{h} = \Delta/2$. An important question is what to choose for the weights in (2.15). Explicit calculation [23] for the field theory elliptic genus shows that the weight of each geometry might be rather complicated and is not just the Euclidean action. In (2.15) however we made this simplest choice in order to simplify our analysis.

Since we are interested in the real time correlators, we should analytically continue $\tau_E = it$. This gives the substitution $w = \phi - t$, $\bar{w} = \phi + t$ in (2.15). The correlation function (2.15) then can be written in the form the

$$\langle \mathcal{O}(t,\phi)\mathcal{O}(0,0)\rangle_{\mathrm{SL}(2,\mathbf{Z})} = \sum_{(c,d)} \sum_{n\in\mathbb{Z}} e^{\frac{\pi k\beta L}{c^2\beta^2 + d^2L^2}} \left[\frac{L^2}{c^2\beta^2 + d^2L^2}\right]^{\Delta} \frac{(\pi/L)^{2\Delta}}{(\sinh\pi(x^+_{n(c,d)})\sinh\pi(x^-_{n(c,d)}))^{\Delta}} , \quad (2.16)$$

where

$$x_{n(c,d)}^{\pm} = \frac{\beta}{c^2 \beta^2 + d^2 L^2} (c(t \pm \phi) \pm nL) -i \frac{L}{c^2 \beta^2 + d^2 L^2} (d(\phi \pm t) + n \frac{(ac\beta^2 + bdL^2)}{L}) \quad .$$
(2.17)

For large c or d the quantities $x_{n(c,d)}^{\pm}$ accumulate near zero so that convergence of (2.16) is not obvious. (A particular divergent contribution is due to terms with c = d - 1, a = b = 1.) Some regularization of (2.16) may be needed. A possible one is to replace $x_{n(c,d)}^{\pm} \rightarrow x_{n(c,d)}^{\pm} \pm i\gamma$ where γ is real. Below some regularization of this sort is assumed. In the $SL(2, \mathbb{Z})$ family the configuration with c = 0 is thermal AdS for which correlation function (2.16) is oscillating (see (2.12)) while the ones with $c \neq 0$ are black holes for which the correlation function exponentially decays as in (2.11).

What can we say about the temporal behavior of sum (2.16)? Can it be quasi-periodic even though it is built out of the exponentially decaying pieces? Answering these questions it is instructive to look at the large t behavior of (2.16) at fixed n. We find (see also [13] for a related analysis)

$$\langle \mathcal{O}(t,\phi)\mathcal{O}(0,0)\rangle_{\mathsf{SL}(2,\mathbf{Z})} = \sum_{(c,d)} e^{\frac{\pi k\beta L}{c^2\beta^2 + d^2L^2}} \left[\frac{\pi^2}{c^2\beta^2 + d^2L^2}\right]^{\Delta} e^{-\frac{2\Delta\beta\pi |c|t}{c^2\beta^2 + d^2L^2}} .$$
 (2.18)

Each term in (2.18) is exponentially decaying. However, the characteristic decay time for term characterized by pair (c, d) becomes arbitrary large for large c and d. This means that such terms become relevant at later and later times. In fact it is an indication of that the sum (2.18) actually decays with time slower than an exponent. On the other hand, as was observed in [13], at certain critical time $t_c = \frac{kL}{2\Delta|c|}$ the decaying exponential factor compensates the exponent of action in (2.18) so that for time $t \geq \frac{kL}{2\Delta}$ there is no suppression in (2.18) and all terms in the sum are equally important. In [13] this was interpreted as a signal of breakdown of the semiclassical approximation.

It is instructive to analyze those issues in a simple example. Consider the sum

$$I(t) = \sum_{n=1}^{\infty} \frac{1}{n^2} e^{a^2/n^2} e^{-t/n}$$
(2.19)

which consists of exponentially decaying terms but the characteristic decay time grows with n. As a result, the sum decays slower than exponential function. In fact, as we show below, it falls off by power law. At $t = a^2/n$ the two exponential functions in (2.19) compensate each other so that for $t > a^2$ all terms in the sum are equally contributing. One might worry whether sum (2.19) is actually convergent at $t = a^2$. Obviously, this is a false alarm – sum (2.19) is convergent for any t due to factor $1/n^2$. In order to make all these points more transparent we approximate the sum (2.19) by integral

$$I(t) \simeq \int_{1}^{\infty} \frac{dn}{n^2} e^{a^2/n^2} e^{-t/n} = \frac{\sqrt{\pi}}{2} i \left[\Phi(-i\frac{t}{2a}) - \Phi(i(a - \frac{t}{2a})) \right] = \frac{a^2}{t} + O(\frac{a^2 e^{a^2}}{t} e^{-t}) , \quad (2.20)$$

where in the last passage we assumed that $t \gg 2a^2$ and approximation for the error function $\Phi(z)$ of large (complex) argument was used. Obviously, there is nothing special happening at $t = a^2$. For large $t \gg 2a^2$ the sum I(t) falls off by power law as anticipated.

Returning to our sum (2.18) we may expect that it actually falls off by a power law similarly to the sum I(t). In order to see this we can use similar trick and replace infinite sum $\sum_{(c,d)}$ by integral $\int dc \, dd$. Obviously, this will overestimate the actual sum and hence give an upper bound on it. Considering c and d as continuous variables it is useful to make a transformation to "polar" coordinate variables (ρ, φ)

$$c = \frac{\rho}{\beta}\cos\varphi , \quad d = \frac{\rho}{L}\sin\varphi$$
 (2.21)

Then we get that

$$\langle \mathcal{O}(t,\phi)\mathcal{O}(0,0)\rangle_{\mathrm{SL}(2,\mathbf{Z})} \simeq \frac{2\pi^{2\Delta}}{L\beta} \int_{R}^{\infty} \frac{d\rho}{\rho^{2\Delta-1}} e^{\frac{\pi k\beta L}{\rho^{2}}} \int_{0}^{\pi/2} d\varphi \ e^{-\frac{2\Delta\pi t}{\rho}\cos\varphi} \ ,$$

$$\int_{0}^{\pi/2} d\varphi \ e^{-\frac{2\Delta\pi t}{\rho}\cos\varphi} = \frac{\pi}{2} (I_{0}(\frac{2\Delta\pi t}{\rho}) - \mathbf{L}_{0}(\frac{2\Delta\pi t}{\rho})) \ ,$$

$$(2.22)$$

where $R \simeq \sqrt{\beta^2 + L^2}$. The ρ integral is peaked at lower limit and we can approximate $e^{\frac{\pi k \beta L}{\rho^2}} \sim e^{\frac{\pi k \beta L}{R^2}}$. The integration over ρ in (2.22) then can be performed explicitly for any integer Δ . The result is expressed in terms of Bessel $I_k(z)$ and Struve $\mathbf{L}_k(z)$ functions. For $\Delta = 2$ in particular we have

$$\langle \mathcal{O}(t,\phi)\mathcal{O}(0,0)\rangle_{\mathrm{SL}(2,\mathbf{Z})} \simeq \frac{1}{L\beta} \frac{e^{\frac{\pi k\beta L}{R^2}}}{4\pi tR} [I_1(\frac{4\pi t}{R}) - \mathbf{L}_1(\frac{4\pi t}{R})] \sim \frac{1}{t} \frac{1}{\beta LR} e^{\pi k\beta L/(\beta^2 + L^2)} , \quad (2.23)$$

where in the last passage we take the limit of large t. In fact, the analysis shows that the power law 1/t is universal large t behavior of (2.22) for all values $\Delta \geq 2$.

We see that sum over pairs (c, d) produces correlation function which decays by a power law. It is an improvement over the exponential decay of each individual contribution (due to black holes) in the sum. This, however, does not produce a quasi-periodic correlation function.

2.3.3 Unitarity: boundary theory and black hole

Thus, the AdS/CFT correspondence predicts that the relaxation in CFT dual to gravity on AdS_3 is combination of oscillating and decaying functions. Thus, the information about initial perturbation seems to be inevitably lost in the boundary system. Since the latter lives in finite volume (circle of size L) and is supposed to be unitary there must be a way to resolve this apparent contradiction. The resolution was suggested in [8] where it was noted that at finite k there should exist another scale in the system which is additional to and much large than L. This scale appears due to the fact that in the dual CFT at high temperature the typical configuration consists of multiply wound strings which effectively propagate in a much bigger volume, $L_{eff} \sim kL$. The gravity/CFT duality however is valid in the limit of infinite k in which this second scale becomes infinite. So that the exponential relaxation corresponds to infinite effective size L_{eff} that is in complete agreement with the general arguments. At finite k the scale L_{eff} would be finite and the correlation function is expected to be quasi-periodic with two periods 1/L and $1/L_{eff}$. The transition of this quasi-periodic function to combination of exponentially decaying and oscillating functions when L_{eff} is infinite then should be similar to what we have observed in the case of free fermions when L was taken to infinity.

On the gravity side the question of which type of relaxation occurs in the system is related to one of the most fundamental problems in physics – the problem of information loss and black hole unitarity. The unitarity problem was suggested to be resolved within the AdS/CFT correspondence [6]. Indeed, since the theory on the boundary is unitary it should be possible to reformulate all processes happening in the bulk of black hole space-time on the intrinsically unitary language of the boundary CFT. In particular it was suggested [6] that in order to restore unitarity of physics in the bulk and reproduce the expected unitary behavior of boundary theory we have to take into account the topological diversity of gravitational instantons that asymptotic to given boundary manifold. If worked this way the black hole unitarity would be resolved within semiclassical gravity appropriately redefined to account for all possible topologies. Alternatively and perhaps more traditionally the unitarity might be expected to restore as a result of fundamentally non-perturbative effects of Quantum Gravity. On the present stage of our understanding of Quantum Gravity this second way would inevitably involve making some guesses about the non-perturbative behavior of black hole.

The relaxation phenomenon gives us adequate language for analysis of the problem of black hole unitarity. That relaxation of black hole is characterized by a set of complex frequencies (quasi-normal modes) is mathematically precise formulation of the lack of unitarity in the semiclassical description of black holes. The loss of information in semiclassical black hole is indeed visible on the CFT side. It is encoded in the exponentially decaying contribution to the 2-point correlation function. For the CFT itself this however is not a problem. As we discussed above the finite size unitarity restores at finite value of k. This however goes beyond the limits where the gravity/CFT duality is formulated. Assuming that the duality can be extended to finite k an important question arises: What would be the gravity counter-part of the duality at finite k? Obviously, it can not be a semiclassical black hole or ensemble of topologically distinct semiclassical black holes. The black hole horizon should be somehow removed so that the complex quasi-normal modes (at infinite k) would be replaced with real (normal) modes when k is finite. This puts us on the second track of resolving the unitarity problem: within a non-perturbative treatment of Quantum Gravity. So that we should start with making our best guess about the nonsemiclassical description of black hole horizons.

3 Wormhole modification of near-horizon geometry

Before passing to our proposal for the nonsemiclassical black hole let us pause for a moment and discuss another proposal made almost twenty years ago by 't Hooft and called the "brick wall" [27] (see also [28]). It was an attempt to explain the entropy of black hole as entropy of thermal atmosphere of particles outside the black hole horizon. In D space-time dimensions the free energy of thermal gas at temperature T in finite volume V_{D-1} takes the form

$$F = -\pi^{-D/2} \Gamma(\frac{D}{2}) \zeta(D) T^D V_{D-1} \quad . \tag{3.1}$$

Near horizon the appropriate volume V_{D-1} is defined in the optical metric (2.4) and is infinite. This means that the spectrum of field excitations is continuous and the entropy is infinite. In order to regularize it 't Hooft suggested to cut the region just outside horizon by introducing boundary at small distance ϵ from horizon and imposing there Dirichlet boundary condition. This procedure has two important consequences: the spectrum now becomes discrete, $\omega_n \sim \pi n/L_{opt}$, $n \in \mathbb{Z}$ and $L_{opt} = \ln \frac{1}{\epsilon}$ is the "size" of the finite region; and the entropy which can be computed from (3.1) using standard formula $S = -\frac{\partial F}{\partial T}$ becomes finite. Moreover, due to remarkable property of the optical volume $V_{D-1} \sim \frac{A}{\epsilon^{D-2}}$. The entropy calculated this way is proportional to the horizon area $A, S \sim \frac{A}{\epsilon^{D-2}}$. The entropy diverges when ϵ is taken to zero and there was a lot of discussion in the literature in the 90's what this divergence should mean [29]. For our story it is important to note that once horizon has been removed the system now lives in finite optical volume and, most importantly, the complex quasi-normal modes disappear. This is exactly what we need in order to restore unitarity in black hole [10]. The brick wall model however is an artificial way to regularize the otherwise smooth black hole geometry. It can be considered as rather crude way of presenting the unknown non-perturbative Planckian physics.

By our earlier (unpublished) work [30] there however exists a smooth way of changing the near horizon geometry which would now look like a wormhole connecting two asymptotic regions semiclassically separated by horizon. This modification of black hole geometry does the same job as the brick wall, i.e. leads to discrete spectrum and finite entropy, but does it in a smooth way. In the context of the black hole relaxation which is subject of the present study the wormhole modification has some attractive features absent in the brick wall model and in fact describes the expected unitary relaxation quite naturally. We study these issues in the next sections. Here we first introduce the modified geometry for the BTZ black hole [9],

$$ds^{2} = -(\sinh^{2} y + \lambda^{2}(\mathbf{k})) dt^{2} + dy^{2} + \cosh^{2} y d\phi^{2} .$$
(3.2)

The deformation parameter $\lambda(\mathbf{k})$ is supposed to be some function of the large N parameter \mathbf{k} such that it vanishes when \mathbf{k} is infinite. Concrete form of this function is discussed later in this Section. The horizon located at y = 0 in classical BTZ black hole disappears in metric (3.2) if λ is non-vanishing. The whole geometry now is that of wormhole with the second asymptotic region at y < 0. The two asymptotic regions (y > 0 and y < 0) which were separated by horizon in classical BTZ metric (2.1) are now connected through narrow throat and thus can talk to each other exchanging information. The metric (3.2) is still asymptotically AdS although it is no more a constant curvature space-time.

Ricci scalar

$$R = -\frac{2}{(\sinh^2 y + \lambda^2)^2} [\lambda^2 + \lambda^4 + 3\sinh^4 y + 5\lambda^2 \sinh^2 y]$$
(3.3)

approaches value -6 at infinite y and $-2(1/\lambda^2 + 1)$ at y = 0 where the horizon used to stay. Notice that the parameter λ should account for the quantum Planckian effects. The curvature at the throat which replaced horizon is thus of the Planckian order. The metric (3.2) is an example that shows that horizon as causally special set in space-time can be seen as a place which accumulates the quantum effects so that under the small deformation there appears Planckian scale curvature concentrated in the Planck size region.

The metric (3.2) can be brought to the usual Schwarzschild like form introducing the radial coordinate $r = \cosh y$,

$$ds^{2} = -(r^{2} - 1 + \lambda^{2})dt^{2} + \frac{dr^{2}}{r^{2} - 1} + r^{2}d\phi^{2} \quad .$$
(3.4)

In the semiclassical case $(\lambda = 0)$ it was possible to extend the metric to include the region where $r^2 < 1$ that was joint to the region $r^2 > 1$ along the light-like horizon r = 1. In the non-semiclassical case (3.4) there appears an intermediate region $(1 - \lambda^2 < r^2 < 1)$ where the signature becomes (- - +) (i.e. it is spacetime with two time-like coordinates) and which can not be extended neither to region $r^2 \ge 1$ nor to region $r^2 \le 1 - \lambda^2$. The latter two regions are thus disconnected from each other and present two different space-times.

Let us now illustrate our point that the entropy of the thermal gas in the metric (3.2) has finite entropy. Indeed, the optical volume

$$V_{\text{opt}} = 2 \int_0^\infty \frac{dy \cosh y}{\sinh^2 y + \lambda^2(\mathbf{k})} \ L = \pi \lambda^{-1}(\mathbf{k}) A \quad , \tag{3.5}$$

where A = L is the "area" of horizon, is now finite. Applying now formula (3.1) for D = 3and taking into account that the Hawking temperature (we take the classical value for the temperature) is $T = 2\pi$ we find that the entropy

$$S = 6\pi^2 \zeta(3) \lambda^{-1}(\mathbf{k}) A \tag{3.6}$$

is finite. If there are N species of particles all of them should be taken into account so that the entropy (3.6) would be proportional to N. In principle, playing with two free parameters N and λ we can easily match the entropy (3.6) with the Bekenstein-Hawking entropy of BTZ black hole. This however is not our primary goal in this paper.

It is clear that the spectrum of field excitations in the metric (3.2) should be discrete. This is just because the size of the space-time in the optical metric measured from one boundary to another is finite

$$L_{\text{opt}} = 2 \int_0^\infty \frac{dy}{\sqrt{\sinh^2 y + \lambda^2}} = 2\mathbf{K}(\sqrt{1 - \lambda^2}) = 2\ln\frac{4}{\lambda} + O(\lambda^2\ln\lambda) \quad , \tag{3.7}$$

where $\mathbf{K}(z)$ is elliptic integral and we used one of its expansions. The expected spectrum in the limit of small λ then reads

$$\omega_n \simeq \frac{\pi \ n}{L_{\text{opt}}} \simeq \frac{\pi \ n}{2 \ln \frac{4}{\lambda}} , \quad n \in \mathbf{Z}$$
(3.8)

for large n, that agrees with the earlier mode calculation in [12]. These are the normal frequencies in the metric (3.2). Comparing the two approaches, the brick wall and the wormhole modification, we see that there exists a correspondence between them provided we make a substitution $\lambda \leftrightarrow \epsilon$.

We finish this section with discussion on the possible form of the deformation parameter λ as function of the large N parameter k. One obvious choice is $\lambda = a/k$ where a is some unknown factor. The advantage of this choice is that the entropy (3.6) takes the form

$$S = \# \mathbf{k}A \quad . \tag{3.9}$$

The numerical factor in front of (3.9) can be chosen (by changing parameter a) in a way that (3.9) exactly reproduces the classical Bekenstein-Hawking entropy. The normal frequencies (3.8) then would scale as $\sim 1/\ln k$.

Another choice is $\lambda(\mathbf{k}) \sim e^{-\mathbf{k}}$. Recalling relation between \mathbf{k} and the Newton constant, k = 1/4G, the wormhole modification appears as non-perturbative Quantum Gravity effect, $\lambda \sim e^{-1/4G}$. The normal modes then scale as $1/\mathbf{k}$ and, choosing $\lambda \sim e^{-\mathbf{k}L}$, we can identify L_{opt} and L_{eff} introduced in section 2.3.3. Notice that $S = \mathbf{k}L$ is the entropy of BTZ black hole. This choice seems to be preferable since in this case the geometry (3.2) originates in completely non-perturbative fashion.

4 Applying the AdS/CFT rules: conformal anomaly

The metric (3.2) is asymptotically AdS and thus we can use the AdS/CFT rules and extract information about boundary theory from the asymptotic behavior of the metric. In particular we can calculate the conformal anomaly in the boundary theory (see [31], [32] for more details). For that we first introduce a new radial coordinate $\rho = e^{-2y}$ and re-write the metric (3.2) in the form

$$ds^{2} = \frac{d\rho^{2}}{4\rho^{2}} + \frac{1}{\rho}g_{ij}(x,\rho)dx^{i}dx^{j} ,$$

$$g_{ij}(x,\rho) = \sum_{n}g_{ij}^{(n)}(x)\rho^{2n} , \qquad (4.10)$$

where $x^i = (t, \phi)$ are coordinates on the boundary. Clearly the metric (3.2) takes the form (4.10) with

$$g_{\phi\phi}^{(0)} = -g_{tt}^{(0)} = 1 ,$$

$$g_{\phi\phi}^{(2)} = \frac{1}{2} , \quad g_{tt}^{(2)} = \frac{1}{2} - \lambda^{2}(\mathbf{k}) ,$$

$$g_{\phi\phi}^{(4)} = -g_{tt}^{(4)} = \frac{1}{16} .$$
(4.11)

The vacuum expectation value of the boundary (quantum) stress tensor can be calculated using the formula [32]

$$\langle T_{ij} \rangle = \frac{2l}{16\pi G} (g_{ij}^{(2)} - g_{ij}^{(0)} \operatorname{Tr} g^{(2)}) \quad .$$
 (4.12)

In particular the trace of the stress tensor that represents the conformal anomaly in the boundary theory is given by the formula

$$\operatorname{Tr} \langle T \rangle = -\frac{2l}{16\pi G} \operatorname{Tr} g^{(2)} \quad . \tag{4.13}$$

Substituting here the asymptotic expansion (4.11) for the metric (3.2) we find

$$\operatorname{Tr}\left\langle T\right\rangle = \frac{c}{24\pi} [-2\lambda^2] \tag{4.14}$$

for the conformal anomaly, where c = 3l/2G is the central charge, c = 6k. The interesting fact about this conformal anomaly is that it is non-vanishing. Indeed, the conformal anomaly in two dimensions is given by the Ricci-scalar. The two-dimensional space-time lying in the boundary of the three-dimensional space-time (3.2) is flat. In the Euclidean case, when the Euclidean time is compactified, this space is two-dimensional torus. Since the Ricci-scalar is vanishing in this case we should not normally expect any conformal anomaly to appear. Surprisingly, the above calculation gives us non-trivial anomaly determined by parameter λ and this needs to be explained. The explanation we can offer is rather simple. We suggest that the metric (3.2) effectively describes boundary theory with mass gap determined by parameter λ^{\dagger} . Indeed, in this case the conformal anomaly can be built from a pair of dimension two quantities: Ricci-scalar and the mass squared. For instance, a free massive scalar field (c = 1) has conformal anomaly given by

$$\operatorname{Tr} \langle T \rangle = \frac{1}{24\pi} [R - 6m^2]$$
 (4.15)

Similar expression exists for free massive fermions. In flat space-time, when Ricci-scalar is vanishing, the conformal anomaly is given by the mass squared only. Comparison with (4.14) suggests that there is a mass gap $m \sim \lambda$ in the boundary theory. The exact proportionality coefficient can not be determined from these arguments since the boundary theory is strongly interacting while the expression (4.15) is given for non-interacting scalar field. Recalling that $\lambda \sim e^{-k}$ and $k = \frac{1}{4G}$ we see that the appearance of the mass gap is yet another non-perturbative effect encoded in the shape of the metric (3.2).

5 The quasi-periodicity: time scales and the spectrum

Demonstrating the quasi-periodicity in the time evolution of field perturbations in the metric (3.2) we first notice that there exist two time scales associated with the metric (3.2). The first one is given by the optical length $L_{opt} \sim \ln \frac{1}{\lambda}$ and another is determined by the size λ of the throat. When λ is small the two time scales are widely separated, $1/\lambda \gg \ln \frac{1}{\lambda}$, so that we can talk about hierarchy of time scales. The time scale $1/\lambda$ appears when we look at the metric (3.2) in the throat region (y is close to zero) and find that it is basically flat with the "throat time" being rescaled with respect to the time t at infinity as $t_{thr} \sim \lambda t$. Assuming that $\lambda \sim e^{-k}$ we find that time flows in the throat

[†]Alternative interpretation of the trace anomaly (4.14) is that it appears due to non-vanishing twodimensional cosmological constant proportional to λ .

extremely slow. So that processes which are rapidly changing with respect to the time in the throat are practically frozen as measured by clocks at infinity. It is exactly this property of the wormhole geometry (3.2) that makes the black hole unitarity restored in the long period of time set by $1/\lambda$ which is the maximal time scale in the system. This is the Poincaré recurrence time for the black hole that was missing in the semiclassical description.

Let us now turn on a field perturbation in the background of metric (3.2) and see what are the frequencies which characterize the time evolution of the perturbation. For simplicity we consider minimally coupled scalar field with vanishing mass. Making anzats $\Phi = e^{-i\omega t} e^{ik\phi} (\cosh y)^{-1/2} \psi(y)$, where $k = 2\pi 1/L$ and 1 is any integer, and switching to a new radial coordinate

$$z = \int \frac{dy}{\sqrt{\sinh^2 y + \lambda^2}} = F(\arcsin(\frac{\sinh y}{\sqrt{\sinh^2 y + \lambda^2}}), \sqrt{1 - \lambda^2})$$
(5.16)

we find that the radial function $\psi(z)$ should satisfy effective Schrödinger equation

$$\partial_z^2 \psi(z) + (\omega^2 - U(z))\psi(z) = 0 \quad , \tag{5.17}$$

where

$$U(y) = \frac{3}{4}\cosh^2 y + k^2 + \frac{1}{4}(\lambda^2 - 2) - (1 - \lambda^2)(\frac{1}{4} + k^2)\frac{1}{\cosh^2 y}$$
(5.18)

is the effective radial potential. Since there is no horizon in the metric (3.2) the quasinormal boundary conditions are no more in place. Instead we demand that solution to the radial equation (5.17) be normalizable which means it should fall off appropriately at infinity.

The integration in (5.16) results in some elliptic function. For our purposes it is however convenient to use an approximation valid in the case when $y \ll 1$. Notice that even though y is small it can either be as small as λ , $y \sim \lambda$, or be much larger than λ , $y \gg \lambda$; in the second case we have $\lambda \ll y \ll 1$. So that regime of small (compared to 1) y gives us possibility to probe both the throat region (set by λ) and the outside the throat region with the size set by 1 in terms of the coordinate y. For $y \ll 1$ we can replace $\sinh^2 y \simeq y^2$ in (5.16). Then the integration in (5.16) is easily performed

$$z = \int \frac{dy}{\sqrt{y^2 + \lambda^2}} = \operatorname{arcsinh} \frac{y}{\lambda} \longrightarrow y = \lambda \sinh z \quad . \tag{5.19}$$

Notice that z defined this way can be both small and large.

In this approximation the radial potential takes the form

$$U(z) \simeq \lambda^2 (k^2 + \frac{1}{2}) + \lambda^2 (k^2 - k^2 \lambda^2 + 1 - \frac{\lambda^2}{4}) \sinh^2 z \quad . \tag{5.20}$$

The Schrödinger equation (5.17) with this potential can be written in the form

$$\partial_z^2 \psi(z) + (\tilde{\omega}^2 - B^2 \sinh^2 z)\psi(z) = 0$$
(5.21)

where we defined

$$\tilde{\omega}^2 = \omega^2 - \lambda^2 (k^2 + \frac{1}{2}) , \quad B^2 = \lambda^2 (k^2 - k^2 \lambda^2 + 1 - \frac{\lambda^2}{4}) .$$

We remind that in the absence of the throat the near-horizon potential is given by exponentially decaying function, $U(z) \sim e^{2z}$. The Schrödinger equation with such potential does not have a discrete spectrum. One of the effects which the throat has produced is to replace this exponential function by the potential $U(z) \sim \lambda^2 \sinh^2 z$ which has the form of the potential well and thus admits the discrete spectrum. Equation (5.21) describes the quantum mechanical Toda Lattice. The discrete spectrum can be found explicitly using technique developed[‡] in [33]. Rather than deal with exact analysis we however prefer to apply the WKB prescription and get the spectrum approximately. The approximation is accurate in the limit of small λ . In what follows we assume that k = 0 (1 = 0).

5.1 The spectrum in the WKB approximation

For the equation (5.21) the WKB prescription gives us the quantization condition

$$\int dz \sqrt{\tilde{\omega}^2 - B^2 \sinh^2 z} = \pi (n + \frac{1}{2}) , \quad n \in \mathbf{Z}$$
(5.22)

where the integration is taken over z for which the expression staying under the square root is positive. The integration can be performed explicitly in terms of elliptic functions,

$$J(a) = \int_0^{\arcsin(a)} \sqrt{a^2 - \sinh^2 z} dz = \sqrt{a^2 + 1} \left(\mathbf{K} \left(\frac{a}{\sqrt{a^2 + 1}} \right) - \mathbf{E} \left(\frac{a}{\sqrt{a^2 + 1}} \right) \right) \quad , \quad (5.23)$$

where we introduced $a = \tilde{\omega}/B$. For higher energy levels $\omega \gg \lambda$ we have that $a \simeq \frac{\omega}{\lambda} \gg 1$ and can use the asymptotic formula for elliptic functions to get asymptotic expression

 $J(a) \simeq a \ln(4a)$

valid for large values of a. Assuming that $\omega \ll (1/\lambda)$ the WKB quantization condition (5.22) then produces the spectrum

$$\tilde{\omega} = \frac{\pi}{2\ln\frac{4}{\lambda}}(n+\frac{1}{2}) , \quad n \in \mathbf{Z}$$
(5.24)

in agreement with our qualitative analysis in section 3. Notice that we could have used the WKB prescription for the whole potential (5.18) and then assumed that the frequency $\omega \ll 1$. This condition than would bring the essential region in the WKB integral close to the bottom of the potential where the potential can be approximated by (5.20).

The frequency $\tilde{\omega}$ is not the same as the frequency ω which appears in the radial Schrödinger equation (5.17), both are related as $\omega^2 = \tilde{\omega}^2 + \lambda^2/2$. The spectrum for the frequency ω then can be represented in the following suggestive form

$$\omega_n^2 = m^2 + p_n^2$$

$$m^2 = \frac{\lambda^2}{2} , \quad p_n = \frac{\pi}{2\ln\frac{4}{\lambda}}(n + \frac{1}{2}) , \quad n \in \mathbf{Z} \quad .$$
(5.25)

The spectrum thus is that of massive particle. By our assumption, (5.25) should be identified with spectrum (understood as poles in 2-point function) of the boundary theory

[‡]I thank M. Olshanetsky for pointing this reference to me.

at finite k and is our prediction. Note that the spectrum (5.25) relies only on the shape of the modified metric in the near-horizon region and in this sense is universal. The parameter $\lambda(\mathbf{k})$ appears both in the quantization of the momentum p_n in the z-direction and in the mass. The appearance of the mass $m \sim \lambda$ is in agreement with our analysis in section 4. Notice that the inverse mass $1/m = t_P$ and the inverse momentum $1/p_0 = t_H$ determine two radically different time scales[§]: $t_P \sim 1/\lambda \sim e^{\mathbf{k}L}$ and $t_H \sim \ln \frac{1}{\lambda} \sim \mathbf{k}L$, $t_P \gg t_H$. Comparing both scales let's assume that $\lambda \sim 10^{-10}$ then the time scale related to the inverse momentum, $t_H \sim 10$, while the time scale determined by the mass, $t_H \sim 10^{10}$. Since the mass m in (5.25) is extremely small the spectrum determined by (5.25) describes (almost) periodic evolution with the period set by the time scale t_H . The mass m is however non-vanishing and hence this evolution is not exactly periodic, the ratio of any two frequencies ω_n/ω_k is not given by rational number in general, and hence the time evolution of the system is actually quasi-periodic. It is important that on the time scale much larger than t_H the evolution is dominated by the periodicity with much larger period t_P . The latter is the longest period in the system and is thus naturally associated with the Poincaré recurrence time.

5.2 The mass and large time scale periodicity

In this subsection we want to illustrate this last statement, namely that in the case when $1/m \gg 1/p_0$ the large time scale evolution of the system is periodic with the period set by inverse mass 1/m. We will do this on the boundary theory side. For simplicity we consider massive scalar field on circle of size L (note that in this sub-section L is similar to what we earlier denoted as L_{eff}) with periodic boundary conditions. In two dimensions the conformal dimension of scalar field is zero. Therefore, the correlation function of two such fields has logarithmic divergence. Correlation function of fields with higher conformal dimension can be obtained by differentiation of the correlation function of scalar field. For instance, the 2-point function of massive fermions (conformal dimension 1/2) is given by

$$S_F(x,x') = (i\gamma^a\partial_a + m)G_F(x,x') \quad . \tag{5.26}$$

The time periodicity of G_F is preserved in S_F . We choose the massive scalar field with periodic boundary condition because the spectrum in this case

$$\omega_n^2 = m^2 + (\frac{2\pi n}{L})^2 , \quad n \in \mathbf{Z}$$
(5.27)

is similar to the spectrum (5.25) $(L \sim \ln \frac{1}{\lambda} \text{ and } m \ll 1/L)$. The 2-point function (or, in the case at hand, the Feynman propagator) of scalar field satisfying the periodic boundary condition takes the form

$$G_F(x,x') = -\sum_{n=-\infty}^{+\infty} \frac{1}{4} H_0^{(2)}(m\sqrt{(t-t')^2 - (\phi - \phi' + Ln)^2 - i\epsilon})$$
(5.28)

of sum over images to maintain the periodicity in ϕ . In momentum space the sum over n appears as sum over all poles (5.27). For simplicity we ignore the temperature which

[§]There is, of course, one more time scale in the game: it is set by size L (~ \sqrt{MG}) of black hole. This time scale appears due to $k \neq 0$ (1 $\neq 0$) part in spectrum of equation (5.17) and is much smaller than t_H and t_P .

otherwise should be taken into account by imposing the condition of periodicity in Euclidean time with period 1/T. Let us set $\phi = \phi'$ and t' = 0 then the correlation function (5.28) is function of only time

$$G_F(t) = -\sum_{n=-\infty}^{+\infty} \frac{1}{4} H_0^{(2)}(m\sqrt{t^2 - L^2 n^2 - i\epsilon}) \quad .$$
 (5.29)

Manipulating further with this expression we first go to Euclidean time $t = i\tau$, the Hankel function $H_0^{(2)}(m\sqrt{t^2 - L^2n^2})$ becomes MacDonald function $K_0(m\sqrt{\tau^2 + L^2n^2})$ under this analytic continuation. Then we replace the infinite sum over n by integral, this procedure gives a good approximation if $\tau/L \gg 1$. Thus we have that

$$\sum_{n=-\infty}^{+\infty} K_0(m\sqrt{\tau^2 + L^2 n^2}) \simeq 2 \int_0^\infty dn K_0(mL\sqrt{\frac{\tau^2}{L} + n^2}) = \frac{\pi}{mL} e^{-m\tau}$$

where in the last passage the integral over n was performed explicitly using formula (6.596.3) from [34]. Analytically continuing back to the real time we find that

$$G_F(t) \simeq -\frac{\pi}{4mL} e^{-imt} \quad . \tag{5.30}$$

This is the desired formula which describes large t ($t \gg L$) behavior of the correlation function. Clearly this behavior is periodic with the period set by inverse mass 1/m. This periodic behavior is a result of superposition of contributions from large number of images in (5.29).

5.3 Black Hole Poincaré recurrences

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In general, system with discrete frequency spectrum shows rather complicated time evolution. Being quasi-periodic in nature it may look dissipative on certain time scales. Example of this we have seen in section 2.2.2 for free fermions on circle. For more complicated system the characteristic time is the so called Heisenberg time (discussed in detail in [10]) which can be defined as $t_H = 1/\langle \delta \omega \rangle$ where $\delta \omega_{kn} = \omega_n - \omega_k$ is the transition frequency and some sort of averaging over the spectrum is assumed. The Heisenberg time is the time scale which characterizes the discreteness of the spectrum. For time $t \ll t_H$ the spectrum can be approximated as continuous. In particular, this means that for time $t \ll t_H$ the system may show dissipative behavior similar to the one we have observed in section 2.2.2. For larger time $t \gg t_H$ the intrinsic quasi-periodicity in the system becomes more visible and the time evolution of correlation functions starts to show long-period oscillations. The longest one is given by the Poincaré recurrence time t_P .

Returning to the spectrum (5.25) we see that it gives a particularly simple example of time evolution we have just described. To the leading order the spacing between energy levels is given by $\pi/\ln(1/\lambda)^2$ so that the Heisenberg time t_H is related to the time scale $\ln(1/\lambda)$ we defined in section 5.1. The black hole then can be approximated by continuous spectrum on the time scale $t \ll t_H = \ln \frac{1}{\lambda}$ and thus shows the usual nonunitary (thermodynamic) behavior typical for space-times with semiclassical horizons. In particular, it may be characterized by complex quasi-normal modes if observed during time $t \ll t_H$. Time set by size L of black hole is the main characteristic time scale in this regime. The discreteness of the spectrum becomes manifest on the time scale close to t_H so that the correlation functions start to demonstrate certain periodicity. The spectrum looks as equidistant on this time scale. But even then unitarity is not yet completely restored: there is more information hidden in much longer oscillations. These oscillations are due to the fine structure of the spectrum (5.25) which deviates from that of equidistant: the non-vanishing mass in (5.25) drives the largest time scale correlations in the system. The time scale $t_P = 1/m \sim e^{\frac{1}{4G}}$ is thus the Poincaré recurrence time during which all information available in the system (black hole) is released. This is the time scale on which evolution of black hole is *ultimately* unitary. Note that the brick wall produces exactly equidistant spectrum and thus does not give rise naturally to the hierarchy of time scales. In our picture the latter comes out as a result of the non-semiclassical (smooth) modification (3.2) of the near-horizon geometry.

6 Conclusions

We conclude with several remarks. As is well known the black hole in AdS space can be in thermal equilibrium with the Hawking radiation and thus represents a well-defined example of what is called eternal black hole. It is a great simplification to us since in the non-semiclassical modification of black hole geometry we may restrict our consideration to static case and do not consider dynamical evolution of black hole due to quantum evaporation. This evolution can be rather complicated and it is not yet clear how the non-semiclassical modification should work in this case. It is however an interesting problem which we are planning to analyze in the future.

It is of course natural to ask how our consideration extends to other spacetimes with horizons, for instance whether de Sitter space time could be understood along same lines. Cautiously, we might expect that our picture may be useful in this case as well although details may be more subtle and yet have to be worked out.

Semiclassical singularity at r = 0 did not play any role in our consideration. It is because, as we discussed this before, region with $r^2 \leq 1 - \lambda^2$ is now disconnected from the region $r^2 \geq 1$ which is main focus in this paper. On the other hand, space time (3.2) does contain region of trans-Planckian curvature which should manifest itself somehow. For instance, it may be useful to consider two copies of CFT living on two asymptotic boundaries ($y = -\infty$ and $y = +\infty$). Semiclassically they are separated by horizon but are able to communicate through semiclassical singularity, the correlation function between two CFT thus contains information about the singularity (see [35] for more details). In non-semiclassical spacetime (3.2) the two theories can talk to each other directly. Geodesics connecting two boundaries pass through the highly curved region so that the correlation functions between two theories should carry information about the trans-Planckian curvature. It would be interesting to see how this information is encoded in the correlation functions.

An interesting question is what happens to the semiclassical $SL(2, \mathbb{Z})$ symmetry and does it make sense to consider the T-dual of the metric (3.2). This boils down to clarifying the non-perturbative status of the $SL(2, \mathbb{Z})$ symmetry. Semiclassically the T-dual to BTZ metric is thermal AdS. The formal T-dual of the metric (3.2) is some deformation of AdS space-time with main deformation concentrated near the origin. Since the origin in AdS is not in any sense a special point this modification might be difficult to justify. One possibility can be that the deformation (3.2) of BTZ metric is a manifestation of particular choice of the quantum state of black hole, this choice of state may not be natural in the case of AdS spacetime. The symmetry then should refer to spacetimes with same choice of the quantum state.

Important open question is whether the metric (3.2) can be consistently justified within string theory. Whatever the answer is we believe that the findings in this paper provide us with sort of existence theorem to the solution of the unitarity problem.

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