

A large N_c perspective on the QCD phase diagram

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The transition between the hadronic phase and the quark gluon plasma phase at nonzero temperature and quark chemical potentials is studied within the large- N_c expansion of QCD.

I. INTRODUCTION

The study of QCD at nonzero temperature and quark chemical potentials is of crucial importance to understand a wide range of different physical phenomena, from heavy-ion-collision experiments to neutron stars and cosmology. This has led to numerous theoretical investigations of the phase diagram of QCD at nonzero temperature and quark-chemical potentials.

Historically, numerical simulations using Monte Carlo techniques have been very fruitful for the study of QCD at nonzero temperature and zero quark chemical potentials. However, this approach faces a major problem at non-zero quark chemical potentials: In general, the fermion determinant is complex and Monte Carlo techniques do not work. This is the so-called 'sign problem'. There is one exceptional case where there is no sign problem: QCD at non-zero isospin chemical potential and zero baryon and strangeness chemical potentials. In this case standard Monte Carlo techniques can be used. Lattice simulations show that the QCD phase diagram in this particular case is very rich [1, 2]. The sign problem represents a severe challenge for our general understanding of the QCD phase diagram at nonzero temperature and quark chemical potentials. Eventhough there is no general method to solve the sign problem, recent advances have been made to circumvent it at small quark chemical potentials [3, 4, 5, 6]. These recent studies have concentrated on the transition between the hadronic phase and the quark-gluon-plasma phase at nonzero baryon chemical potential and zero isospin and strangeness chemical potentials, and in particular on the corresponding critical temperature as a function of quark chemical potentials, $T_c(\mu_u, \mu_d, \mu_s)$.

At small chemical potentials, two remarkable properties of the critical temperature seem to emerge from lattice studies. The critical temperature weakly depends on the chemical potentials, and $T_c(\mu, \mu, 0) \sim T_c(\mu, -\mu, 0)$ [1, 4]. These properties are rather puzzling, the second one in particular, since physics at nonzero baryon chemical potential and zero isospin chemical potential is rather different from physics at zero baryon chemical potential and nonzero isospin chemical potential. In this article we shall use the large- N_c expansion of QCD to shed light on these properties and show that they naturally emerge in this context. We shall also use the large- N_c expansion to show that if there is a first order phase transition between the hadronic phase and the quark-gluon-plasma phase at a critical temperature $T_c(\mu, -\mu, \mu_s) \lesssim 140$ MeV, then there is also a first order phase transition at a critical temperature $T_c(\mu, \mu, \mu_s)$, and vice versa. This is important since simulations at $\mu_u = -\mu_d$ can be readily performed using standard Monte Carlo techniques, whereas simulations at $\mu_u = \mu_d$ suffer from the sign problem.

II. LARGE- N_c EXPANSION

Since its inception, the large- N_c expansion in QCD has been a useful tool both conceptually and phenomenologically [7, 8]. It is insightful because it provides us with a small parameter expansion that distinguishes classes of Feynman diagrams. Although the theory has not yet been solved at leading order in $1/N_c$ in four dimension, properties of these classes of Feynman diagrams can be used to determine relations between some observables. In this article, we shall use the usual large- N_c expansion [8] augmented by recent developments on the large- N_c expansion at nonzero temperature and chemical potential [9]. In the large- N_c expansion, the leading-order contribution to the pressure is given by planar diagrams that contain only gluons, and the next-to-leading order contribution is given by planar diagrams with only one quark loop as a boundary. This expansion is still valid at nonzero temperature and quark chemical potentials, provided that the temperature and the quark chemical potentials are of order N_c^0 , and if the different superfluid and superconducting phases are avoided [9, 10]. Following the arguments developed for two flavors in [9], the large- N_c expansion leads to the following expression for the pressure in QCD with N_f quark flavors, f , in a finite volume V :

$$p(T, \{m_f, \mu_f\}) = \epsilon_0(\{m_f, \mu_f\}) + \frac{T}{V} \ln Z(T, \{m_f, \mu_f\}) \quad (1)$$

$$p(T, \{m_f, \mu_f\}) = N_c^2 \left(p_0(T) + \frac{1}{N_c} \sum_f p_1(T, m_f, \mu_f^2) + \mathcal{O}\left(\frac{1}{N_c^2}\right) \right), \quad (2)$$

where $Z(T, \{m_f, \mu_f\})$ is the QCD grand canonical partition function at nonzero temperature, T , for N_f quark flavors of mass m_f and chemical potential μ_f , and where $\epsilon_0(\{m_f, \mu_f\}) = -\lim_{T \rightarrow 0} \frac{T}{V} \ln Z(T, \{m_f, \mu_f\})$ is the vacuum energy density. In the large- N_c expansion in the equation above, $p_0(T)$ contains the leading-order diagrams, which are planar diagrams with gluons only, and thus depends on T only. At next-to-leading order there is no mixing between the different quark flavors. The diagrams that contribute to $p_1(T, m_f, \mu_f^2)$ are planar diagrams that contain gluons and only one quark loop as a boundary, and since the gluons do not mix the quark flavors, the quark chemical potentials do not mix at next-to-leading order. Furthermore, since $Z(T, \{m_f, \mu_f\}) = Z(T, \{m_f, -\mu_f\})$ because of CP, $p_1(T, m_f, \mu_f^2)$ is even in μ_f [9]. The mixing between the chemical potentials of the different quark flavors appears only at next-to-next-to-leading order.

The $U(1)$ -axial anomaly could spoil this property of the diagrammatic expansion of QCD in the large- N_c limit. However, as shown in [11], the triangle anomaly does not depend on the temperature. It is straightforward to generalize the argument developed in [11] to show that the anomaly is also independent of the quark chemical potentials. Another way to show that the anomaly does not mix the different quark chemical potentials is to use chiral perturbation theory in the hadronic phase at nonzero temperature, zero baryon chemical potential, nonzero isospin and strangeness chemical potentials [12, 13, 14, 15, 16]. Chiral perturbation theory is a low-energy effective theory of QCD solely based on the symmetry of QCD. It contains only the Goldstone modes due to the spontaneous breaking of chiral symmetry: π , η , and K . In the large- N_c limit, the chiral symmetry breaking is different, and the η' becomes a Goldstone boson as well. The large- N_c expansion and the inclusion of the η' have been implemented in chiral perturbation theory [12, 17]. The pressure of the hadronic phase can easily be calculated following [16]. In the hadronic phase, we find that the temperature and quark chemical potentials enter the pressure and mix at $\mathcal{O}(N_c^0)$ in chiral perturbation theory. Therefore, in chiral perturbation theory the quark chemical potentials mix only at order N_c^0 in the large- N_c expansion in the hadronic phase, in complete agreement with the diagrammatic argument sketched above and developed in detail in [9]. We thus conclude that the anomaly does not spoil the diagrammatic argument presented in [9].

III. CRITICAL TEMPERATURE

At finite volume V , the separation between the hadronic phase and the quark gluon plasma phase corresponds to a peak in the specific heat $C_V = \partial\epsilon/\partial T|_V$, where $\epsilon = -p + T\partial p/\partial T + \sum_f \mu_f \partial p/\partial \mu_f$ is the energy density. In the infinite volume limit the peak value of C_V stays finite if there is a crossover, and it diverges if the phase transition is first order or second order. The scaling of the peak value with the volume depends on the nature of the phase transition. Therefore, the critical temperature as a function of the quark chemical potential, $T_c(\mu_u, \mu_d, \mu_s)$, is implicitly given by

$$\left. \frac{\partial C_V}{\partial T} \right|_{T_c} = 0. \quad (3)$$

From (2) and (3) and at zero chemical potentials, we find that the difference between the critical temperature for very massive quarks, i.e. pure Yang-Mills theory, T_c^{YM} , and the critical temperature for QCD, T_c^{QCD} , should be of order $1/N_c$:

$$\frac{T_c^{\text{QCD}} - T_c^{\text{YM}}}{T_c^{\text{YM}}} = \mathcal{O}\left(\frac{1}{N_c}\right). \quad (4)$$

This is indeed what has been observed on the lattice for $N_c = 3$: $T_c \simeq 270$ MeV for pure Yang-Mills [18], and $T_c \simeq 175$ MeV for QCD with three flavors [19, 20, 21]. Similarly, the critical temperatures for pure Yang-Mills theories with different number of colors have been computed on the lattice, and they have been found to differ by $\mathcal{O}(1/N_c^2)$, in agreement with the large- N_c expansion,

$$\frac{T_c^{\text{YM}}(N_c)}{T_c^{\text{YM}}} = 1 + \frac{0.76(6)}{N_c^2} + \dots, \quad (5)$$

where T_c^{YM} is the critical temperature for pure Yang-Mills theory when $N_c \rightarrow \infty$ [18].

At nonzero quark chemical potentials the large- N_c expansion in (2) implies that for QCD with $m_u = m_d$, the critical temperature as a function of quark chemical potential, $T_c(\mu_u, \mu_d, \mu_s)$, must satisfy the following relation

$$\frac{T_c(\mu, \mu, \mu_s) - T_c(\mu, -\mu, \mu_s)}{T_c(\mu, \mu, \mu_s)} = \mathcal{O}\left(\frac{1}{N_c^2}\right). \quad (6)$$

Therefore, for $\mu_s = 0$, the critical temperature that separates the hadronic phase and the quark-gluon-plasma phase at nonzero baryon chemical potential, $\mu_b = (\mu_u + \mu_d)/2$, and zero isospin chemical potential, $\mu_i = (\mu_u - \mu_d)/2$, differs from the critical temperature at zero baryon chemical potential and nonzero isospin chemical potential by $1/N_c^2$. This is in complete agreement with recent results obtained in numerical lattice simulations [1, 4], and in various models [22, 23, 24]. This is a useful relation since lattice simulations at $\mu_b = 0$ and $\mu_i \neq 0$ do not suffer from the sign problem present at nonzero μ_b , and are therefore much easier to perform.

More explicitly and for QCD with three quark flavors with $m_u = m_d \neq m_s$ in a finite volume, we can perform a Taylor expansion of $\partial C_V/\partial T$ around the critical temperature that separates the hadronic phase from the quark-gluon-plasma phase at zero chemical potentials, T_0 . Using (2), we find that

$$\frac{\partial C_V}{\partial T} = a_1 \frac{T - T_0}{T_0} + \frac{1}{N_c} \left(\left[b_0 + b_1 \frac{T - T_0}{T_0} \right] \frac{\mu_u^2 + \mu_d^2}{T_0^2} + \left[b_{s0} + b_{s1} \frac{T - T_0}{T_0} \right] \frac{\mu_s^2}{T_0^2} \right) + \dots \quad (7)$$

Notice that the coefficients related to μ_s differ from those related to μ_u and μ_d since $m_u = m_d \neq m_s$. Using (7) to solve (3), we find that the critical temperature as a function of the quark chemical potentials is given by

$$\frac{T_c(\mu_u, \mu_d, \mu_s)}{T_0} = 1 - \frac{1}{N_c} \left(\frac{b_0}{a_1} \frac{\mu_u^2 + \mu_d^2}{T_0^2} + \frac{b_{s0}}{a_1} \frac{\mu_s^2}{T_0^2} \right) + \dots \quad (8)$$

Therefore, we find that the large- N_c expansion leads to interesting insight on the critical temperature as a function of quark chemical potentials. First the curvature of the critical temperature for small chemical potential is $1/N_c$ suppressed. Second, for a given number of colors N_c , the curvature of the critical temperature as a function of baryon chemical potential should increase with the number of flavors. For N_f degenerate quarks, this increase in the curvature should be linear in N_f up to $\mathcal{O}(1/N_c)$ corrections. This has been indeed observed in different lattice simulations for $N_f = 2, 3$, and 4 [5, 6]. Third, the leading-order dependence in the quark chemical potentials is even in μ_u , μ_d , and μ_s separately, and does not mix them. The mixing term appears at next-to-next-to-leading order only. Therefore effects due to the mixing of the quark chemical potentials should be of the order of $1/N_c^2$. This expression for the critical temperature in the large- N_c expansion agrees with and simply explains several results that have been found in numerical simulations and in various models [1, 2, 3, 4, 5, 6, 22, 23, 24].

IV. ORDER OF THE PHASE TRANSITION

The arguments developed above do not depend on the nature of the phase transition. This is, however, an important question to address. Indeed, for QCD with three colors and physical quark masses, the separation between the hadronic phase and the quark-gluon plasma phase at nonzero baryon chemical potential, zero isospin and strangeness chemical potentials is believed to be a crossover at low chemical potential and a first order phase transition at higher chemical potential [25, 26, 27]. Several recent lattice simulations have found the critical endpoint that corresponds to the end of this first order phase transition line [3, 4]. However, these different simulations yield results that significantly differ on the precise location of this critical endpoint.

Lattice simulations at nonzero baryon chemical potential suffer from the sign problem. No such problem is present at nonzero isospin chemical potential and zero baryon and strangeness chemical potentials. It is therefore easier to perform lattice simulations in the latter case. In the section above, we have shown that the large- N_c expansion leads to a relation between the critical temperature at $\mu_u = \mu_d$ with the critical temperature at $\mu_u = -\mu_d$ that is valid at next-to-leading order. It is therefore natural to investigate if such arguments can lead to a relation between the nature of the phase transition in these two cases.

In the infinite volume limit, if the phase transition is first order, then there is a latent heat, $L_h = T_c \text{disc } s$, where s is the entropy density. The latent heat is related to the discontinuity of the quark-antiquark condensate, $\langle \bar{q}q \rangle$, through the Clausius-Clapeyron relation derived in [28, 29]

$$L_h = \frac{T_c}{\partial T_c / \partial m_q|_{\{\mu_f\}}} \text{disc } \langle \bar{q}q \rangle, \quad (9)$$

where $Q|_x$ means that quantity Q is evaluated at constant x . Starting from (2), and using the same reasoning that leads to (6), we find that $\partial T_c / \partial m_q|_{\{\mu_f\}} \sim 1/N_c$. At zero temperature the large- N_c expansion leads to $\langle \bar{q}q \rangle_{T=0} \sim \mathcal{O}(N_c)$. At nonzero temperature, $\mu_u = -\mu_d$ and $\mu_s = 0$, based on lattice simulations [1, 2] and on chiral perturbation theory [16, 30], we know that $\langle \bar{q}q \rangle$ is almost independent of the chemical potential and of the temperature in the hadronic phase, provided $T \lesssim 140$ MeV. For instance in chiral perturbation theory at zero chemical potentials, it was found

that for $T \lesssim 140$ MeV, $\langle \bar{q}q \rangle / \langle \bar{q}q \rangle_{T=0} \gtrsim 80\%$ for QCD with two massive quarks, and with corrections due to massive non-Goldstone modes taken into account [30]. Thus, from both lattice simulations and chiral perturbation theory, we have that $\langle \bar{q}q \rangle \sim \langle \bar{q}q \rangle_{T=0} = \mathcal{O}(N_c)$ for $T \lesssim 140$ MeV in the hadronic phase. Therefore, we conclude that if a first order phase transition between the hadronic phase and the quark-gluon plasma phase takes place at $T \lesssim 140$ MeV, the corresponding latent heat is of order N_c^2 . This conclusion cannot be reached for higher temperatures since the value of the quark-antiquark condensate decreases and is no longer $\mathcal{O}(N_c)$.

In the large- N_c perspective described above and if there is a first order phase transition at $T \lesssim 140$ MeV, we can use the same reasoning that led to (4) to show that the latent heat as a function of quark chemical potentials, $L_h(\mu_u, \mu_d, \mu_s)$, has to satisfy the following relation

$$\frac{L_h(\mu, \mu, \mu_s) - L_h(\mu, -\mu, \mu_s)}{L_h(\mu, \mu, \mu_s)} = \mathcal{O}\left(\frac{1}{N_c^2}\right), \quad \text{for } T \lesssim 140\text{MeV}. \quad (10)$$

In other words, if there is a first order phase transition between the hadronic phase and the quark-gluon plasma phase at a temperature $T_c(\mu, -\mu, \mu_s) \lesssim 140$ MeV, with a latent heat $L_h = \mathcal{O}(N_c^2)$, or equivalently disc $\langle \bar{q}q \rangle = \mathcal{O}(N_c)$, then, according to the large- N_c expansion, equations (6) and (10), there should be a first order phase transition at a temperature $T_c(\mu, \mu, \mu_s) \sim T_c(\mu, -\mu, \mu_s) + \mathcal{O}(1/N_c^2)$. Thus if lattice simulations at $\mu_u = -\mu_d$ were to find a first order phase transition at temperatures below ~ 140 MeV, then the $1/N_c$ expansion predicts that there should also be a first order phase transition at the same $\mu_u = \mu_d$ and at the same temperature, up to $\mathcal{O}(1/N_c^2)$ corrections.

V. CONCLUSIONS

We have used the large- N_c expansion of QCD to study the phase transition between the hadronic phase and the quark-gluon plasma phase at nonzero temperature and quark chemical potentials. We have shown that the critical temperature depends on the chemical potentials at next-to-leading order in the large- N_c expansion. We have also shown that there are relations between the critical temperature at nonzero baryon chemical potential and zero isospin chemical potential, and the critical temperature at nonzero isospin chemical potential and zero baryon chemical potential. These relations are valid at next-to-leading order in the large- N_c expansion. Finally, based on large- N_c arguments, we have shown that it should be possible to determine relatively accurately the position of a first order phase transition between the hadronic phase and the quark-gluon plasma phase in the QCD phase diagram at nonzero baryon chemical potential and zero isospin and strangeness chemical potentials, by performing ordinary lattice simulations at nonzero isospin chemical potential and zero baryon and strangeness chemical potentials, as long as the latent heat is large enough. This should be a valid strategy up to temperatures of ~ 140 MeV where the quark-antiquark condensate is not significantly different from its value at zero temperature.

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- [1] J. B. Kogut and D. K. Sinclair, Phys. Rev. D **70**, 094501 (2004) [arXiv:hep-lat/0407027].
 - [2] J. B. Kogut and D. K. Sinclair, Phys. Rev. D **66**, 034505 (2002); Nucl. Phys. Proc. Suppl. **119**, 556 (2003); Nucl. Phys. Proc. Suppl. **129**, 542 (2004); D. K. Sinclair, J. B. Kogut and D. Toublan, Prog. Theor. Phys. Suppl. **153**, 40 (2004).
 - [3] Z. Fodor and S. D. Katz, Phys. Lett. B **534**, 87 (2002); JHEP **0203**, 014 (2002); JHEP **0404**, 050 (2004);
 - [4] C. R. Allton *et al.*, Phys. Rev. D **66**, 074507 (2002); Phys. Rev. D **68**, 014507 (2003); F. Karsch *et al.*, Nucl. Phys. Proc. Suppl. **129**, 614 (2004);
 - [5] P. de Forcrand and O. Philipsen, Nucl. Phys. B **642**, 290 (2002); Nucl. Phys. B **673** (2003) 170; Nucl. Phys. Proc. Suppl. **129**, 521 (2004).
 - [6] M. D'Elia and M. P. Lombardo, Phys. Rev. D **67**, 014505 (2003).
 - [7] G. 't Hooft, Nucl. Phys. B **72**, 461 (1974).
 - [8] For an introduction, see e.g., S. Coleman, *Aspects of Symmetry*, Cambridge University Press, Cambridge 1998.
 - [9] T. D. Cohen, arXiv:hep-ph/0410156.
 - [10] For a review see, K. Rajagopal and F. Wilczek, arXiv:hep-ph/0011333;
 - [11] H. Itoyama and A. H. Mueller, Nucl. Phys. B **218**, 349 (1983).
 - [12] J. Gasser and H. Leutwyler, Annals Phys. **158**, 142 (1984); Nucl. Phys. B **250**, 465 (1985).

- [13] J. B. Kogut, M. A. Stephanov and D. Toublan, Phys. Lett. B **464**, 183 (1999); J. B. Kogut *et al.*, Nucl. Phys. B **582**, 477 (2000);
- [14] D. T. Son and M. A. Stephanov, Phys. Rev. Lett. **86**, 592 (2001).
- [15] J. B. Kogut and D. Toublan, Phys. Rev. D **64**, 034007 (2001).
- [16] K. Splittorff, D. Toublan and J. J. M. Verbaarschot, Nucl. Phys. B **620**, 290 (2002).
- [17] A. V. Manohar, arXiv:hep-ph/9802419.
- [18] B. Lucini, M. Teper and U. Wenger, JHEP **0401**, 061 (2004).
- [19] F. Karsch, E. Laermann and A. Peikert, Nucl. Phys. B **605**, 579 (2001)
- [20] A. Ali Khan *et al.* [CP-PACS Collaboration], “Phase structure and critical temperature of two flavor QCD with Phys. Rev. D **63**, 034502 (2001).
- [21] C. Bernard *et al.* [MILC Collaboration], arXiv:hep-lat/0405029.
- [22] B. Klein, D. Toublan and J. J. M. Verbaarschot, Phys. Rev. D **68**, 014009 (2003);
- [23] D. Toublan and J. B. Kogut, Phys. Lett. B **564**, 212 (2003); M. Frank, M. Buballa and M. Oertel, Phys. Lett. B **562**, 221 (2003); A. Barducci, R. Casalbuoni, G. Pettini and L. Ravagli, Phys. Rev. D **69**, 096004 (2004).
- [24] A. Barducci, G. Pettini, L. Ravagli and R. Casalbuoni, Phys. Lett. B **564**, 217 (2003).
- [25] A. Barducci, R. Casalbuoni, S. De Curtis, R. Gatto and G. Pettini, Phys. Rev. D **41**, 1610 (1990); A. Barducci, R. Casalbuoni, G. Pettini and R. Gatto, Phys. Rev. D **49**, 426 (1994).
- [26] J. Berges and K. Rajagopal, Nucl. Phys. B **538**, 215 (1999).
- [27] M. A. Halasz, A. D. Jackson, R. E. Shrock, M. A. Stephanov and J. J. M. Verbaarschot, Phys. Rev. D **58**, 096007 (1998).
- [28] H. Leutwyler, Phys. Lett. B **284**, 106 (1992).
- [29] A. Barducci, R. Casalbuoni, G. Pettini and R. Gatto, Phys. Lett. B **301**, 95 (1993).
- [30] P. Gerber and H. Leutwyler, Nucl. Phys. B **321**, 387 (1989).