

# Abelian Higgs Model Effective Potential in the Presence of Vortices

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We determine the contribution of nontrivial vacuum (topological) excitations, more specifically vortex–strings of the Abelian Higgs model in  $3 + 1$  dimensions, to the functional partition function. By expressing the original action in terms of dual transformed fields we make explicit in the equivalent action the contribution of the vortex–strings excitations of the model. The effective potential of an appropriately defined local vacuum expectation value of the vortex–string field in the dual transformed action is then evaluated both at zero and finite temperatures and its properties discussed in the context of the finite temperature phase transition.

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## I. INTRODUCTION

The study of phase transitions in quantum field theory has a long history, since the first works on the subject [1, 2, 3] (see also the Refs. [4, 5]) and it is still a highly active area of research motivated by several open problems in QCD phase transitions, grand-unified theory phase transitions and many other subject areas including also condensed matter physics problems [6]. One basic mechanism we are usually interested in these studies is how the variation of an external quantity like temperature, density or external fields may act and change different physical quantities in a given system or the study, for instance, of how symmetries may change under the variation of temperature, like in symmetry breaking phase transitions. One very common and extremely useful tool in the latter problem is the use of effective potentials for appropriate order parameters characterizing the possible phases of the system (at equilibrium), like the vacuum expectation value of a Higgs field in gauge field theories, determined as some constant (in space and time) solution of the effective field equations.

Around the same time of these studies on symmetry breaking/restoring phase transitions on gauge field theories, it was also realized that symmetry breaking in gauge field theories could give rise to nontrivial and nonperturbative stable solutions of the field equations of motion. This is the case, for example, of the magnetic vortex solutions in a  $U(1)$  symmetry broken Abelian gauge field theory [7] or magnetic monopoles in  $O(3)$  or  $SU(2)$  symmetry broken non-Abelian gauge field theories [8, 9], which are only a few examples among several other topological-like nontrivial vacuum field solutions that have been exhaustively studied to date (for reviews, see for instance Refs. [10, 11]). Extra interest on these field solutions is also due to the fact that, since these nonlocal vacuum structures are expected to emerge in most of the grand unified phase transitions in the early universe, they may have important cosmological consequences (for a detailed account see e.g. Ref. [12]).

In the present paper we consider the case of phase transition in the Abelian Higgs model from the viewpoint in which the phase transition at finite temperatures is driven by a condensation of magnetic vortices. This is not an entirely novelty in the sense that there are a lot of examples in which phase transitions are driven by topological defects in quantum field theory as well as in condensed-matter physics [13]. In fact, it has long been believed that, close to the critical point, the condensation of inhomogeneous configurations, solutions of the field equations, is able to provide a much better description of the phase transition as compared to mean field methods, e.g., using the sole contribution of constant, homogeneous field configurations in the partition function, as it is the case of the standard derivations of the finite-temperature effective potential in field theories. For instance, topological configurations, like strings in the Abelian Higgs model, have previously been studied in this context of phase transitions by computing the free energy associated to these configurations, e.g., by semiclassically expanding the quantum fields around the vortex-string *classical* solution [14, 15]. The problem with this approach of considering the contribution of topological configurations to the effective action in a semiclassical way is the intrinsic difficulty of computing the effective action,

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which becomes highly nonlocal, so only the first order loop terms can be computed analytically and to go beyond numerical methods have to be employed. An alternative approach to the semiclassical one that also has been used is directly quantizing the topological excitations and representing them as (nonlocal) quantum fields (see for instance the approach of Refs. [16] and references therein). But this is also problematic since we are only able to compute lowest-order correlation functions of the quantal topological field and even so, the still nonlocal character of these functions besets a simple derivation. To circumvent these problems, in this paper we adopt an alternative intermediate derivation between the latter two, which make use of the concept of duality [17]. Using this technique it is possible to conveniently rewrite the original action for the Goldstone modes of the broken symmetry, in terms of a dual action describing the topological defect currents and its interactions mediated by a dual *antisymmetric* tensor field.

We here consider the finite-temperature version of the Abelian Higgs model, which is then treated along a formalism developed long ago by the authors of Refs. [18, 19, 20]. In this formalism, a dual transformation is applied to the Higgs model partition function in order to show the contributions from topological excitations in a more explicit manner. An antisymmetric tensor auxiliary field is introduced and, after functional integration of the original electromagnetic vector field, the action of this dual model assumes the form of a relativistic hydrodynamics in the sense of Kalb–Ramond [21] and Nambu [22, 23]. The formalism may be generalized to non-Abelian gauge fields [24]. In more recent years this formalism has been generalized to extended objects in higher dimensions (D-branes) in string theory [25]. Also, in another kind of application, this duality approach has been used in the study of vortices in superfluidity models [26].

The next step in this mechanism, is to rewrite the sum over all possible distributions of the topological number density which appears in the partition function as a functional integration over some functional fields. This procedure was introduced previously in  $U(1)$  lattice gauge theory [27] and later used in the Abelian Higgs model by several authors [20, 28]. In this paper we use these techniques to calculate the contribution of the topological defects in the Abelian Higgs model to the one-loop effective potential, which can now be expressed directly in terms of the expectation value of a quantum vortex field. From this effective potential we have calculated the vortex condensation temperature obtaining a result compatible with previous estimations based on the statistical distribution of classical strings [15]. We have also checked that this temperature is different than the usual mean field critical temperature of the model when the inhomogeneous topological field configurations are neglected. This then makes possible to access in an analytical way the importance of these topological configurations during a phase transition.

The remaining of this paper is organized as follows. In Sec. II we introduce the model. In Sec. III we calculate the dual action showing how the topological defects explicitly show up in this formalism. We discuss the issue of gauge invariance and the equivalence between the original and the dual model at the effective potential level. In Sec. IV we calculate the contribution of the topological defects to the effective potential and evaluate the condensation temperature. Our final considerations and conclusions are given in Sec. V. An appendix is included to review and show some of the technical details of the formalism we have used here.

## II. THE MODEL

The model we consider is the Abelian Higgs model with Lagrangian density for a complex scalar field  $\phi$  and gauge field  $A_\mu$ ,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu\phi|^2 - V(\phi), \quad (2.1)$$

where, in the usual notation,  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ ,  $D_\mu = \partial_\mu - ieA_\mu$  and  $V(\phi)$  is a symmetry breaking potential given by

$$V(\phi) = -m_\phi^2 |\phi|^2 + \frac{\lambda}{3!} (|\phi|^2)^2. \quad (2.2)$$

The symmetry breaking  $U(1) \rightarrow 1$  with homotopy group  $\pi_1 \neq 1$  indicates the existence of string-like topological excitations in the system (for an extended introduction and review see e.g. Ref. [12]). For example, for a unit winding string solution along the  $z$  axis, the classical field equations of motion obtained from the Lagrangian density (2.1) admit a stable finite energy configuration describing the string and given by (using the cylindrical coordinates  $r, \theta, z$ )

$$\phi_{\text{string}} = \frac{\rho(r)}{\sqrt{2}} e^{i\theta}, \quad (2.3)$$

$$A_{\mu,\text{string}} = \frac{1}{e} A(r) \partial_\mu \theta, \quad (2.4)$$

where the functions  $\rho(r)$  and  $A(r)$  vanish at the origin and have the asymptotic behavior  $\phi(r \rightarrow \infty) \rightarrow \rho_v \equiv \sqrt{6m_\phi^2/\lambda}$  and  $A(r \rightarrow \infty) \rightarrow 1$ . The functions  $\rho(r)$  and  $A(r)$  are obtained (numerically) by solving the classical field equations. If we write the field  $\phi$  as  $\phi = \rho \exp(i\chi)/\sqrt{2}$ , then from (2.3) and (2.4) for the string, at spatial infinity  $\rho$  goes to the vacuum  $\rho_v$  and  $A_\mu$  becomes a pure gauge. This also gives, in order to get a finite energy for the string configuration, that  $\partial_\mu \chi = eA_\mu$  at  $r \rightarrow \infty$ , so  $D_\mu \phi = 0$ . This leads then that, by taking some contour  $C$  surrounding the symmetry axis, and using Stokes' theorem, to the nonvanishing magnetic flux

$$\Phi = \oint A_\mu dx^\mu = \oint \partial_\mu \chi dx^\mu = 2\pi/e. \quad (2.5)$$

Since  $\phi$  must be single-valued, the Eq. (2.5) implies that on the string  $\chi$  must be singular. Therefore, the phase  $\chi$  can be separated into two parts: in a regular part and in a singular one, due to the string configuration. We will use this latter fact in the next section when describing the topological vortex string contributions to the partition function, which are then characterized by multivalued (or singular) phases of the scalar field.

### III. THE DUAL-TRANSFORMED ACTION

Let us start by writing the partition function for the Abelian Higgs model (2.1), which, in Euclidean space-time is given by

$$Z[\beta] = \int \mathcal{D}A \mathcal{D}\phi \mathcal{D}\phi^* \exp \{ -S[A_\mu, \phi, \phi^*] - S_{GF} \}, \quad (3.1)$$

where in the above expression  $S$  denotes the Euclidean action,

$$S[A_\mu, \phi, \phi^*] = \int_0^\beta d\tau \int d^3x \left[ \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + |D_\mu \phi|^2 + V(\phi) \right], \quad (3.2)$$

where  $\beta = 1/T$  is the inverse of the temperature and  $S_{GF}$  in (3.1) is some appropriate gauge-fixing and ghost term that must be added to the action to perform the functional integral over the relevant degrees of freedom. We will come back later to this term and explicitly fix it within the formalism described below. Note also that the functional integral in Eq. (3.1) is to be performed over the bosonic scalar and gauge fields satisfying the usual periodic boundary conditions in imaginary time with period  $\beta = 1/T$  [4, 5].

By writing the complex Higgs field  $\phi$  in the polar parameterization form  $\phi = \rho e^{i\chi}/\sqrt{2}$ , the functional integration measure in Eq. (3.1) is changed to

$$\mathcal{D}\phi \mathcal{D}\phi^* \rightarrow \mathcal{D}\rho \mathcal{D}\chi \left( \prod_x \rho \right), \quad (3.3)$$

and the quantum partition function becomes

$$Z = \int \mathcal{D}A_\mu \mathcal{D}\rho \mathcal{D}\chi \left( \prod_x \rho \right) \exp \{ -S[A_\mu, \rho, \chi] - S_{GF} \}, \quad (3.4)$$

with

$$S[A_\mu, \rho, \chi] = \int d\tau d^3x \left[ \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{2} (\partial_\mu \rho)^2 + \frac{1}{2} \rho^2 (\partial_\mu \chi + eA_\mu)^2 - \frac{m_\phi^2}{2} \rho^2 + \frac{\lambda}{4!} \rho^4 \right]. \quad (3.5)$$

In order to make explicit the contribution of the nontrivial topological field configuration in the partition function (3.4), it is more convenient to work with the dual version of Eq. (3.5). To achieve this equivalent dual action we start by splitting the scalar phase field  $\chi$  in its regular and singular terms,  $\chi = \chi_{\text{reg}} + \chi_{\text{sing}}$ . Lets for now, for convenience, omit the gauge fixing term  $S_{GF}$  in Eq. (3.4) and re-introduce it again in the final transformed action. Following e.g. the procedure of Refs. [29, 30, 31, 32, 33], the functional integral over  $\chi$  in Eq. (3.4) can then be rewritten as

$$\begin{aligned} & \int \mathcal{D}\chi \exp \left[ - \int d^4x \frac{1}{2} \rho^2 (\partial_\mu \chi + e A_\mu)^2 \right] \\ &= \int \mathcal{D}\chi_{\text{sing}} \mathcal{D}\chi_{\text{reg}} \mathcal{D}C_\mu \left( \prod_x \rho^{-4} \right) \exp \left\{ - \int d^4x \left[ \frac{1}{2\rho^2} C_\mu^2 - i C_\mu (\partial_\mu \chi_{\text{reg}}) - i C_\mu (\partial_\mu \chi_{\text{sing}} + e A_\mu) \right] \right\} \\ &= \int \mathcal{D}\chi_{\text{sing}} \left( \prod_x \rho^{-4} \right) \mathcal{D}W_{\mu\nu} \exp \left\{ - \int d^4x \left[ \frac{\kappa^2}{2\rho^2} V_\mu^2 + e \kappa A_\mu V_\mu + i \pi \kappa W_{\mu\nu} \omega_{\mu\nu} \right] \right\}, \end{aligned} \quad (3.6)$$

where we have performed the functional integral over  $\chi_{\text{reg}}$  in the second line of Eq. (3.6). This gives a constraint on the functional integral measure,  $\delta(\partial_\mu C_\mu)$ , which can be represented in a unique way by expressing the  $C_\mu$  in terms of an antisymmetric field,  $C_\mu = -i \frac{\kappa}{2} \epsilon_{\mu\nu\lambda\rho} \partial_\nu W_{\lambda\rho} \equiv \kappa V_\mu$ , which then leads to the last expression in Eq. (3.6).  $\kappa$  is some arbitrary parameter with mass dimension and  $\omega_{\mu\nu}$  is the vorticity given only in terms of the singular phase part of  $\chi$ ,

$$\omega_{\mu\nu} \equiv \frac{1}{4\pi} \epsilon_{\mu\nu\lambda\rho} (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \chi(x). \quad (3.7)$$

Next, in order to linearize the dependence on the gauge field in the action we introduce a new antisymmetric tensor field  $G_{\mu\nu}$  through the identity

$$\exp \left( - \frac{1}{4} \int d^4x F_{\mu\nu}^2 \right) = \int \mathcal{D}G_{\mu\nu} \exp \left[ \int d^4x \left( - \frac{\mu_W^2}{4} G_{\mu\nu}^2 - \frac{\mu_W}{2} \tilde{G}_{\mu\nu} F_{\mu\nu} \right) \right], \quad (3.8)$$

with

$$\tilde{G}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\lambda\rho} G_{\lambda\rho}. \quad (3.9)$$

Substituting Eqs. (3.6) and (3.8) back into Eq. (3.4), we can immediately perform the functional integral over the  $A_\mu$  field. Taking also for convenience  $e\kappa = \mu_W$ , we then obtain for Eq. (3.4) the result

$$\begin{aligned} Z &= \int \mathcal{D}W_{\mu\nu} \mathcal{D}\chi_{\text{sing}} \mathcal{D}G_{\mu\nu} \delta \left[ \epsilon_{\mu\nu\alpha\beta} \partial_\mu \left( G_{\alpha\beta} - \frac{1}{2} W_{\alpha\beta} \right) \right] \mathcal{D}\rho \left( \prod_x \rho^{-3} \right) \\ &\times \exp \left\{ - \int d^4x \left[ \frac{\mu_W^2}{4} G_{\mu\nu}^2 + \frac{\mu_W^2}{2e^2 \rho^2} V_\mu^2 + \frac{1}{2} (\partial_\mu \rho)^2 - \frac{m_\phi^2}{2} \rho^2 + \frac{\lambda}{4!} \rho^4 + i \pi \frac{\mu_W}{e} W_{\mu\nu} \omega_{\mu\nu} \right] \right\}. \end{aligned} \quad (3.10)$$

The constraint  $\epsilon_{\mu\nu\alpha\beta} \partial_\mu (G_{\alpha\beta} - W_{\alpha\beta}) = 0$  can be solved by setting

$$G_{\mu\nu} = W_{\mu\nu} - \frac{1}{\mu_W} (\partial_\mu B_\nu - \partial_\nu B_\mu), \quad (3.11)$$

where  $B_\mu$  is an arbitrary gauge field, thus obtaining for the partition function the expression (and re-introducing the gauge fixing term)

$$Z = \int \mathcal{D}W_{\mu\nu} \mathcal{D}\chi_{\text{sing}} \mathcal{D}B_\mu \mathcal{D}\rho \left( \prod_x \rho^{-3} \right) \exp \{ - S_{\text{dual}} [W_{\mu\nu}, B_\mu, \rho, \chi_{\text{sing}}] - S_{GF} \}, \quad (3.12)$$

with

$$S_{\text{dual}} = \int d^4x \left[ \frac{\mu_W^2}{2e^2\rho^2} V_\mu^2 + \frac{1}{4} (\mu_W W_{\mu\nu} - \partial_\mu B_\nu + \partial_\nu B_\mu)^2 + \frac{1}{2} (\partial_\mu \rho)^2 - \frac{m_\phi^2}{2} \rho^2 + \frac{\lambda}{4!} \rho^4 + i\pi \frac{\mu_W}{e} W_{\mu\nu} \omega_{\mu\nu} \right]. \quad (3.13)$$

This dual model is completely equivalent to the original Abelian Higgs model in the polar representation given by Eqs. (3.4) and (3.5) and so, any calculations done using (3.12) must lead to the same results as those done with the original action. For example, if we compute the effective potential for a constant scalar field configuration  $\rho_c$  from the latter should be the same as the one obtained by the former. This we will check explicitly shortly. The advantage of the dual version is that it explicitly exhibits the dependence on the singular configuration of the Higgs field, making it appropriate to study phase transitions driven by topological defects. However, we need to be careful with gauge invariance, in special in the dual model (3.13), since it has more gauge freedom than the original model. Now we come to the part concerning the gauge fixing term  $S_{GF}$  in (3.12). From Eq. (3.13) we see that the dual action exhibits invariance under the double gauge transformation: the hypergauge transformation

$$\begin{aligned} \delta W_{\mu\nu}(x) &= \partial_\mu \xi_\nu(x) - \partial_\nu \xi_\mu(x), \\ \delta B_\mu &= \mu_W \xi_\mu(x), \end{aligned} \quad (3.14)$$

and the usual gauge transformation

$$\delta B_\mu = \partial_\mu \theta(x), \quad (3.15)$$

where  $\xi_\mu(x)$  and  $\theta(x)$  are arbitrary vector and scalar functions, respectively. Choosing  $\xi_\mu = B_\mu$  in the first transformation is equivalent to fix the gauge through the condition  $B_\mu = 0$  [30] and this is equivalent to choose the unitary gauge in Eq. (3.12).

At this point, it would be interesting to analyze the gauge fixing procedures for this model and to show that the resulting effective potential does not depend on the gauge fixing parameters within our parametrization choice for the complex scalar field. For simplicity, we neglect at this time the last term in the exponential in Eq. (3.13) due to the vorticity. In order to evaluate the effective potential we need to specify the gauge fixing term  $S_{GF}$ . To fix the gauge for the antisymmetric tensor field, associated to the first gauge transformation in Eq. (3.14), we need to introduce a vector ghost field. We here do this in the same way the gauge is fixed and corresponding ghost terms appear in the analogous case of choosing gauge terms for two-form gauge field models [34]. As we see below, this vector ghost also exhibits a gauge invariance which, therefore, need to be fixed. This leads to one more ghost field associated to this subsidiary gauge invariance. Next, we also need to fix the second gauge invariance associated to the transformation (3.15) and to add its corresponding ghost field. Therefore, three constants are needed to completely fix the gauge freedom [34]. This process leads to the following relevant additional terms that define the gauge-fixing term in the partition function,

$$\begin{aligned} S_{GF} &= \int d^4x \left\{ -\frac{1}{2\theta} (\partial^\mu W_{\mu\nu} + \partial_\nu \psi + u \mu_W B_\nu)^2 + i\bar{\zeta}^\nu [(\partial^2 + u \mu_W^2) \zeta_\nu - \partial_\nu \partial^\mu \zeta_\mu + \partial_\nu \vartheta + u \mu_W \partial_\nu c] \right. \\ &\quad \left. + i\zeta^\nu (\partial_\nu \bar{\vartheta} - \mu_W \partial_\nu \bar{c}) + \bar{\sigma} \partial^2 \sigma - i\bar{c} \partial^2 c + \frac{1}{2\xi} (\partial_\mu B^\mu)^2 \right\}, \end{aligned} \quad (3.16)$$

where  $\psi, \bar{c}, c, \bar{\sigma}, \sigma, \bar{\vartheta}, \vartheta$  are the ghost fields and  $\theta, u$  and  $\xi$  are the gauge parameters.

We can easily perform the functional integrals over the ghost fields appearing in Eq. (3.16). Besides an overall normalization factor independent of the action fields (and the background Higgs field) we get for the quantum partition function

$$\begin{aligned} Z &= N \int \mathcal{D}W_{\mu\nu} \mathcal{D}\rho \mathcal{D}B_\mu \mathcal{D}\bar{\eta} \mathcal{D}\eta \exp \left\{ - \int d^4x \left[ \frac{\mu_W^2}{2e^2\rho^2} V_\mu^2 + \frac{1}{4} (\mu_W W_{\mu\nu} - \partial_\mu B_\nu + \partial_\nu B_\mu)^2 \right. \right. \\ &\quad \left. \left. + \frac{1}{2} (\partial_\mu \rho)^2 - \frac{m_\phi^2}{2} \rho^2 + \frac{\lambda}{4!} \rho^4 - \bar{\eta} \rho^{-3} \eta - \frac{1}{2\theta} (\partial^\mu W_{\mu\nu})^2 + \frac{u}{2\theta} \mu_W W_{\mu\nu} (\partial^\mu B^\nu - \partial^\nu B^\mu) + \frac{1}{2\xi} (\partial_\mu B^\mu)^2 \right] \right\}. \end{aligned} \quad (3.17)$$

where  $\bar{\eta}, \eta$  are the ghost fields used to exponentiate the Jacobian  $\rho^{-3}$  in the functional integration measure in Eq. (3.12).

Let us now compute, for instance, the effective potential for a constant background field  $\rho_c$  from (3.17). The effective potential for  $\rho_c$  is defined as usual, by writing  $\rho$  in terms of the constant background field plus the quantum fluctuations around this constant field configuration,  $\rho = \rho_c + \rho'$ , and performing the functional integration over  $\rho'$  and remaining fields. In the usual derivation [2], the effective potential for interacting field theories is evaluated perturbatively as an expansion in loops, which is equivalent to an expansion in powers of  $\hbar$  [35]. The one-loop approximation for  $V_{\text{eff}}(\rho_c)$  is then equivalent to incorporating the first quantum corrections to the classical potential  $V(\rho_c)$ . For a general case of  $N$ -particle species interacting with the Higgs field, its one-loop effective potential can be written in the generic form (in Minkowski spacetime)

$$V_{\text{eff}}^{1\text{-loop}}(\rho_c) = V(\rho_c) \mp \frac{1}{2}i \sum_{j=1}^N g_j \int \frac{d^4 k}{(2\pi)^4} \ln [k^2 - M_j^2(\rho_c)] , \quad (3.18)$$

where the negative sign in Eq. (3.18) stands for boson fields, while the positive one is for fermion (and ghost) fields.  $g_j$  labels the number of degrees of freedom for the particle species coupled to the scalar Higgs field and  $M_j(\rho_c)$  their mass spectrum. The momentum integrals in Eq. (3.18), when working in the Matsubara formalism of finite temperature field theory (see e.g. [2, 4, 5]), are expressed as

$$\int \frac{d^4 k}{(2\pi)^4} = i \frac{1}{\beta} \sum_{\omega_n} \int \frac{d^3 k}{(2\pi)^3} ,$$

and the four-momentum  $k_\mu = (\mathbf{k}, i\omega_n)$ , where  $\omega_n = 2\pi nT$ ,  $n = 0, \pm 1, \dots$ , represent the Matsubara frequencies for bosons, while for fermions we have  $\omega_n = (2n + 1)\pi T$ .

Using Eq. (3.18) and from Eq. (3.17), we obtain quantum correction coming from the  $\rho'$ ,  $W_{\mu\nu}$ ,  $B_\mu$ ,  $\bar{\eta}$ ,  $\eta$  fields. At the one-loop level, we then obtain the effective potential for the dual Abelian Higgs model,

$$\begin{aligned} V_{\text{eff}}(\rho_c) &= \frac{m_\phi^2}{2} \rho_c^2 + \frac{\lambda}{4!} \rho_c^4 - \frac{1}{2}i \int \frac{d^4 k}{(2\pi)^4} \ln \det [iD^{-1}(k)]_{\rho'} - \frac{1}{2}i \int \frac{d^4 k}{(2\pi)^4} \ln \det [iD^{-1}(k)]_{B_\mu, W_{\mu\nu}} \\ &\quad - 3i \int \frac{d^4 k}{(2\pi)^4} \ln \rho_c + (\text{terms independent of } \rho_c) , \end{aligned} \quad (3.19)$$

where  $[iD^{-1}(k)]_{\rho'}$  comes from the quadratic term in  $\rho'$  of the Lagrangian density, given in momentum space by

$$[iD^{-1}(k)]_{\rho'} = k^2 + m_\phi^2 - \lambda \rho_c^2 / 2 , \quad (3.20)$$

while  $[iD^{-1}(k)]_{B_\mu, W_{\mu\nu}}$  is the matrix of quadratic terms in the gauge field  $B_\mu$  and antisymmetric field  $W_{\mu\nu}$ ,

$$[iD^{-1}(k)]_{B_\mu, W_{\alpha\beta}} = \begin{pmatrix} -g^{\mu\nu} k^2 + (1 - 1/\xi) k^\mu k^\nu & -i (\mu_W - \frac{u}{\theta}) k^\lambda g^{\mu\rho} \\ i (\mu_W - \frac{u}{\theta}) k^\alpha g^{\beta\nu} & \mu_W^2 \left( \frac{k^2}{e^2 \rho_c^2} - 1 \right) G^{\alpha\lambda\beta\rho} + \left( \frac{1}{\theta} - \frac{\mu_W^2}{e^2 \rho_c^2} \right) K^{\alpha\lambda\beta\rho} \end{pmatrix} . \quad (3.21)$$

where we have used the notation

$$G^{\alpha\lambda\beta\rho} = \frac{1}{4} (g^{\alpha\lambda} g^{\beta\rho} - g^{\alpha\rho} g^{\beta\lambda}) , \quad (3.22)$$

and

$$K^{\alpha\lambda\beta\rho} = \frac{1}{2} (k^\alpha k^\lambda g^{\beta\rho} - k^\alpha k^\rho g^{\beta\lambda}) . \quad (3.23)$$

The explicit computation of (3.19) is a tedious one, but it can be shown that all gauge dependence factorize from (3.19) as terms independent of the background field and consequently can be dropped out. For the generating function (3.17) this has been shown by the authors of the first reference in [34]. For the computation of the effective potential this is most easily shown in the case of the original model. As we have emphasized before, the model described by Eq.

(3.17) is just the dual of the Abelian Higgs model in the covariant gauge in the polar representation for the complex Higgs field. As such, they are physically equivalent and the effective potential for the shifted action in (3.17) must lead to the same effective potential as that obtained from the original Abelian Higgs model in the covariant gauge. This is easily seen from Eq. (3.5), where, by taking a covariant gauge fixing term, one has the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\rho)^2 + \frac{1}{2}\rho^2(\partial_\mu\chi - eA_\mu)^2 + \frac{m_\phi^2}{2}\rho^2 - \frac{\lambda}{4!}\rho^4 - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \bar{\eta}\rho\eta + \bar{c}\partial^2c, \quad (3.24)$$

where  $\bar{\eta}, \eta$  are the ghost fields for the Jacobian factor in Eq. (3.4) and  $\bar{c}, c$  are the ghosts due to the gauge-fixing term. The effective potential for a constant field background  $\rho_c$  is defined in the usual way, as said above. We obtain, for instance, the one-loop effective potential,

$$\begin{aligned} V_{\text{eff}}(\rho_c) &= \frac{m_\phi^2}{2}\rho_c^2 + \frac{\lambda}{4!}\rho_c^4 - \frac{1}{2}i \int \frac{d^4k}{(2\pi)^4} \ln \det [iD^{-1}(k)]_{\rho'} - \frac{1}{2}i \int \frac{d^4k}{(2\pi)^4} \ln \det [iD^{-1}(k)]_{\chi, A_\mu} \\ &+ i \int \frac{d^4k}{(2\pi)^4} \ln \rho_c + i \int \frac{d^4k}{(2\pi)^4} \ln k^2, \end{aligned} \quad (3.25)$$

where the last two terms in (3.25) come from the functional integration over the ghost terms of (3.24).  $[iD^{-1}(k)]_{\rho'}$  is the same as before, given by Eq. (3.20), while  $[iD^{-1}(k)]_{\chi, A_\mu}$  is the matrix of quadratic terms (in momentum space) for the  $\chi$  and  $A_\mu$  fields,

$$[iD^{-1}(k)]_{\chi, A_\mu} = \begin{pmatrix} \rho_c^2 k^2 & ie\rho_c^2 k^\nu \\ -ie\rho_c^2 k^\mu & -g^{\mu\nu}(k^2 - e^2\rho_c^2) + (1 - 1/\xi)k^\mu k^\nu \end{pmatrix}. \quad (3.26)$$

Substituting (3.20) and (3.26) in (3.25), we obtain the result

$$\begin{aligned} V_{\text{eff}}(\rho_c) &= \frac{m_\phi^2}{2}\rho_c^2 + \frac{\lambda}{4!}\rho_c^4 - \frac{1}{2}i \int \frac{d^4k}{(2\pi)^4} \ln(k^2 - M_H^2) - \frac{1}{2}i \int \frac{d^4k}{(2\pi)^4} \ln \left[ -\frac{1}{\xi}(k^2 - M_A^2)^{3/2} k^4 \rho_c^2 \right] \\ &+ i \int \frac{d^4k}{(2\pi)^4} \ln \rho_c + i \int \frac{d^4k}{(2\pi)^4} \ln k^2, \end{aligned} \quad (3.27)$$

where  $M_H^2 = -m_\phi^2 + \lambda\rho_c^2/2$  and  $M_A^2 = e^2\rho_c^2$  are the Higgs and gauge field (squared) masses as usual. From Eq. (3.27) we readily see that the contributions from the ghost fields, including the divergent contribution due to the Jacobian coming from the radial parametrization for the scalar field  $\phi$ , cancel with identical terms coming from the gauge and scalar phase field matrix quadratic term, Eq. (3.26). These same cancellations happens when working with the analogous expression for the effective potential, Eq. (3.19), in terms of the dual  $B_\mu$  and  $W_{\mu\nu}$  fields, including again the cancellation of the divergent Jacobian due to an analogous contribution appearing in the  $W_{\mu\nu}$  field quadratic term, as seen from the matrix of quadratic terms, Eq. (3.21). All gauge dependence (on  $\xi$ ) can be separated from (3.27) as a background independent term that can be dropped out. The emerging result is identical to the effective potential obtained, e.g., in Ref. [36].

Once the equivalence of the original and the dual model is checked and the gauge-fixing peculiarities of the dual model can be dealt with conveniently, we can move on to consider the contribution of singular field-configurations with non-trivial vorticity to the effective potential.

#### IV. THE EFFECTIVE POTENTIAL IN THE PRESENCE OF VORTEX-STRING VACUUM CONFIGURATIONS

Let us now reinstate the contribution due to nontrivial singular structures of the Higgs phase in the calculations of the one-loop effective potential. This is given by the last term in Eq. (3.13), for the coupling of the antisymmetric field  $W_{\mu\nu}$  with the vorticity term due to the singular phase of the Higgs field. As we saw, it is associated to the existence of vortex-like solutions for the equations of motion of the action (3.2) [7]. These can be associated to string-like topological defect configurations that are either infinite in length or forming finite-size closed loops. By open configurations we mean the existence of magnetic monopoles at the end points [12] and we will not consider these kind of structures here since we restrict our study only to the Abelian theory. Also, we will only consider here field

configurations which generate closed magnetic vortex lines in the three spatial Euclidean dimensions, since these are more suitable to the field theoretical analysis we will adopt in the following and are also expected to be the dominant topology for strings close to the transition point [12].

The coupling term of the antisymmetric field with the vorticity source  $\omega_{\mu\nu}$ , defined in Eq. (3.7), is non-vanishing for the singular term  $\chi_{\text{sing}}$  of the Higgs field phase and hence this interaction term will contribute to the action, along with the world sheet of the string. In the zero temperature case, the source  $\omega_{\mu\nu}$  is associated to the surface element of a (tube-like) world sheet of a closed vortex-string [19, 29, 30]. Following the Dirac construction [37], it is given by

$$\omega_{\mu\nu}(x) = n \int_S d\sigma_{\mu\nu}(x) \delta^4[x - y(\xi)] , \quad (4.1)$$

where  $n$  is a topological quantum number, the winding number, which we here restrict to the lowest values,  $n = \pm 1$ , corresponding to the energetically dominant configurations. The element of area on the world sheet swept by the string is given by

$$d\sigma_{\mu\nu}(x) = \left( \frac{\partial x_\mu}{\partial \xi^0} \frac{\partial x_\nu}{\partial \xi^1} - \frac{\partial x_\mu}{\partial \xi^1} \frac{\partial x_\nu}{\partial \xi^0} \right) d^2\xi \quad (4.2)$$

and  $y_\mu(\xi)$  represents a point on the world sheet  $S$  of the vortex-string, with internal coordinates  $\xi^0$  and  $\xi^1$ . As usual, we consider that  $\xi^1$  is a periodic variable, since we work with closed strings, whereas  $\xi^0$  will be proportional to the time variable (at zero temperature), in such a way that  $\xi^1$  parameterizes a closed string at a given instant  $\xi^0$ . Using (4.1), the interaction of the string with the antisymmetric field in the action becomes

$$\int d^4x i\pi \frac{\mu_W}{e} W_{\mu\nu}(x) \omega_{\mu\nu}(x) = \frac{i}{2} \int_S d\sigma^{\mu\nu}(y) \frac{2\pi\mu_W}{e} W_{\mu\nu}(y). \quad (4.3)$$

To proceed further with the evaluation of the string contribution to the partition function we will now introduce a (nonlocal) field associated to the string. For this we take the standard Marshall–Ramond procedure [38, 39] of quantizing the vortex-strings as nonlocal objects and associate to them a wave function  $\Psi[C]$ , a functional field, where  $C$  is the closed vortex-string curve in Euclidean space-time. In the second-quantized form this means that the quanta associated to the field  $\Psi$  are the vortex-strings in the system. In introducing the vortex-string field, we first note that the interaction term Eq. (4.3) is in the form of a current coupled to the antisymmetric field. Second, the coupling of the field  $\Psi[C]$  with  $W_{\mu\nu}$  should respect the gauge symmetries of the model, in particular the hypergauge one, Eq. (3.14). This is fulfilled by defining the following covariant derivative term, as proposed by Nambu [38],

$$D_{\sigma^{\mu\nu}}(x) = \frac{\delta}{\delta \sigma^{\mu\nu}(x)} - i \frac{2\pi\mu_W}{e} W_{\mu\nu}(x) . \quad (4.4)$$

Here  $\delta \sigma^{\mu\nu}(x)$  is to be considered as an infinitesimal rectangular deformation of area  $\delta A$  of the original curve  $A$  at a point  $x$  and so the functional derivative of the string field can be defined as the difference between  $\Psi[C + \delta\sigma]$  and the original configuration  $\Psi[C]$ , divided by the infinitesimal area, taking the limit  $\delta A \rightarrow 0$  (see for instance, Refs. [20, 40, 41]). The hypergauge transformation (3.14) is now supplemented by the vortex-string field transformation

$$\Psi[C] \rightarrow \exp \left[ -i \frac{2\pi\mu_W}{e} \oint dx^\mu \xi_\mu(x) \right] \Psi[C] . \quad (4.5)$$

This gives sense to Eq. (4.4) as a covariant derivative, since it commutes with the above phase change of  $\Psi[C]$ .

From the definition of the covariant derivative (4.4) the invariant action for the string under the combined transformations (3.14) and (4.5) becomes (see the Appendix for more details)

$$S_{\text{string}}(\Psi[C], W_{\mu\nu}) = \oint_C dx_\nu [ |D_{\sigma^{\mu\nu}} \Psi[C]|^2 - M_0^2 |\Psi[C]|^2 ] , \quad (4.6)$$

whose explicit form and derivation has been given originally by the authors of Refs. [19, 20] when considering the existence of  $N$  connected vortex world surfaces in Euclidean space-time. The mass term for the string field in (4.6) is given by Eq. (4.9) below. It is also possible to write an action over local fields by defining a functional

$$\hat{\psi}_C \equiv 4 \left( \frac{2\pi}{e} \right)^2 \sum_{C_{x,t}} \frac{1}{a^3 l} |\Psi[C]|^2, \quad (4.7)$$

where  $l$  is the length of a curve  $C$ , and  $C_{x,t}$  represents a curve passing through a point  $x$  in a fixed direction  $t$ ; also, the parameter  $a$  is to be considered as a small quantity (the lattice spacing in Ref. [19]), which we choose to be proportional to  $\Lambda^{-1}$ . The vacuum expectation value of  $\hat{\psi}_C$  is denoted by  $\psi_C$ , which represents the sum of existence probabilities of vortices in  $C_{x,t}$ . In terms of  $\hat{\psi}_C$ , it can be shown that the contribution of the vortices to the quantum partition function, indicated by the last term in Eq. (3.13) and involved with the integration over  $\chi_{\text{sing}}$ , can be written as [19]

$$\int \mathcal{D}\Psi[C] \mathcal{D}\Psi^*[C] \exp \left\{ - \int d^4x \left[ \frac{1}{4} \left( \frac{e}{2\pi} \right)^2 M_0^4 \hat{\psi}_C + \frac{\mu_W^2}{4} W_{\mu\nu}^2 \hat{\psi}_C \right] \right\}, \quad (4.8)$$

where

$$M_0^4 \equiv \frac{1}{a^4} \left( e^{\tau_s a^2} - 6 \right) \quad (4.9)$$

and  $\tau_s$  is the string tension (the total energy per unit length of the vortex-string) [20, 41]. In terms of the parameters of the Abelian Higgs model the string tension is given by [42]  $\tau_s = \pi \rho_c^2 \epsilon(\lambda/e^2)$ , where  $\epsilon(\lambda/e^2)$  is a function that increases monotonically with the ratio of coupling constants. We should also note that the factor  $a^4$  in Eq. (4.9) does not have a direct relation with a four-dimensional space-time. Thus, the relation between  $M_0$  and  $\Lambda$  ( $\sim a^{-1}$ ) is still expected at finite temperature.

Eq. (4.8) implies, together with Eq. (3.13), that an immediate consequence of  $\psi_C \neq 0$  is the increase of the  $W_{\mu\nu}$  mass. This is directly associated with a shift in the mass of the original gauge field in the broken phase,  $M_A = e\rho_c$ , as

$$M_A^2 \rightarrow M_A^2 (1 + \psi_C). \quad (4.10)$$

Since the field  $\psi_C$ , defined by Eq. (4.7), works just like a local field for the vortex-strings, we are allowed to define an effective potential for its vacuum expectation value  $\psi_C$  in just the same way as we do for a constant Higgs field. Since this vortex-string field only couples directly to  $W_{\mu\nu}$ , at the one-loop level the effective potential for  $\psi_C$  will only involve internal propagators of the antisymmetric tensor field. This effective potential, at one-loop order and at  $T = 0$ , was actually computed in Ref. [19] in the Landau gauge for the antisymmetric tensor field propagator and it is given by (in Euclidean momentum space and at finite temperatures)

$$V_{\text{eff}}^{1\text{-loop}}(\psi_C) = \left( \frac{e}{2\pi} \right)^2 M_0^4 \psi_C + \frac{3}{2} \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^3k}{(2\pi)^3} \ln \left[ \frac{\omega_n^2 + \mathbf{k}^2 + M_A^2 (1 + \psi_C)}{\omega_n^2 + \mathbf{k}^2 + M_A^2} \right]. \quad (4.11)$$

By performing the sum over the Matsubara frequencies in (4.11), we obtain the finite-temperature expression for  $V_{\text{eff}}^{1\text{-loop}}(\psi_C)$ . This is a standard calculation that gives

$$V_{\text{eff}}^{(\beta)}(\psi_C) = \left( \frac{e}{2\pi} \right)^2 M_0^4 \psi_C + \frac{3}{2} \int \frac{d^3k}{(2\pi)^3} \omega_{\psi_C}(\mathbf{k}) + 3 \frac{1}{\beta} \int \frac{d^3k}{(2\pi)^3} \ln \{ 1 - \exp[-\beta \omega_{\psi_C}(\mathbf{k})] \}, \quad (4.12)$$

where

$$\omega_{\psi_C}^2(\mathbf{k}) = \mathbf{k}^2 + M_A^2 (1 + \psi_C), \quad (4.13)$$

and in Eq. (4.12) we have neglected the terms independent of  $\psi_C$ . Eq. (4.12) can now be used to estimate the critical temperature for which vortex-strings condense exactly like when we take the effective potential for a constant scalar field to determine the critical temperature of phase transition [2]. By expanding  $V_{\text{eff}}^{(\beta)}$  in the high-temperature limit

$M_A \sqrt{1 + \psi_C} / T \ll 1$  (this entails expanding the temperature-dependent term in (4.12) just the same way we expand the corresponding term in the usual effective potential for a constant scalar field [2, 4]), we obtain

$$V_{\text{eff,string}}^{(\beta)}(\psi_C) = \left(\frac{e}{2\pi}\right)^2 M_0^4 \psi_C + \frac{3}{2} \int \frac{d^3 k}{(2\pi)^3} \omega_{\psi_C}(\mathbf{k}) - \frac{\pi^2}{30\beta^4} + \frac{M_A^2(1 + \psi_C)}{8\beta^2} - \frac{1}{4\pi\beta} M_A^3(1 + \psi_C)^{3/2} - \frac{3M_A^4(1 + \psi_C)^2}{64\pi^2} \ln[\beta^2 M_A^2(1 + \psi_C)] + \frac{3c}{64\pi^2} M_A^4(1 + \psi_C)^2 + \mathcal{O}[M_A^6(1 + \psi_C)^3 \beta^2], \quad (4.14)$$

where  $c \simeq 5.4076$ . The momentum integral appearing in the right-hand side of Eq. (4.14) represents the temperature-independent part of the effective potential, and it can be done directly. Using the cutoff  $\Lambda$  we obtain for that term result

$$\begin{aligned} \frac{3}{2} \int \frac{d^3 k}{(2\pi)^3} \omega_{\psi_C}(\mathbf{k}) &= \frac{3\Lambda}{16\pi^2} [\Lambda^2 + M_A^2(1 + \psi_C)]^{3/2} - \frac{3\Lambda}{32\pi^2} M_A^2(1 + \psi_C) [\Lambda^2 + M_A^2(1 + \psi_C)]^{1/2} \\ &\quad - \frac{3M_A^4}{32\pi^2} (1 + \psi_C)^2 \ln \left\{ \frac{\Lambda + [\Lambda^2 + M_A^2(1 + \psi_C)]^{1/2}}{M_A(1 + \psi_C)^{1/2}} \right\}. \end{aligned} \quad (4.15)$$

Before entering in the analysis of Eq. (4.14) it is useful to recall that the Abelian Higgs model can support either second order or first order phase transitions. The ratio of the coupling constants  $\alpha = e^2/\lambda$ , that measure the relative intensity of the gauge coupling  $e$  and the fourth power of the Higgs potential  $\lambda$ , controls these two regimes. Thus, for  $\alpha \ll 1$  the gauge coupling is quite small and the phase diagram is dominated by the second order phase transition of the pure Higgs model. On the other hand, as  $\alpha$  gets bigger, the gauge field fluctuations are more relevant opening the possibility of inducing a first order transition. This is evident from the result (4.14), where the gauge field contribution to the effective potential generates already at one-loop order a cubic term in the Higgs background field, which in the usual effective potential for the Higgs field is the term that leads to a first order phase transition in the model.

The discussion above is also in parallel with the phenomenology of the Landau-Ginzburg theory for superconductors, where the parameter  $\kappa \sim 1/\alpha^{1/2}$  (also called the Ginzburg parameter), measuring the ratio of the penetration depth and the coherent length, controls the regimes called Type II and Type I superconductors. In the former  $\alpha \ll 1$  (or  $\kappa > 1$ ), the metal-superconductor transition is second order and the gauge fluctuations are not important, while in the latter  $\alpha \gtrsim 1$ , the gauge fluctuations could turn the transition first order via a Coleman-Weinberg mechanism [35]. In our case, the coherent length is governed by  $a \sim 1/M_H$ , where  $M_H$  is here the temperature dependent Higgs mass, while the penetration depth is proportional to  $1/M_A$ , where  $M_A$  is the (temperature dependent) gauge field mass. Although this effect, of the emergence of a first order phase transition, is so weak that it is not observable in superconductors, it could play an important role in relativistic quantum field theory (for a pedagogic discussion of this issues see, for instance the first volume of Kleinert's books in Ref. [13]).

We turn back now to the analysis of Eq. (4.14). The lattice spacing  $a = 1/\Lambda$  can be taken as the distance between strings [43]. Therefore, we can consider that close to the critical point for condensation, determined by some temperature  $T_s$ ,  $a$  can approximately be given by the string typical radius. Then, since we are interested in the determination of a critical point, we can write (see for example also Ref. [15])

$$1/a \sim m_\phi \left(1 - \frac{T^2}{T_c^2}\right)^{1/2}. \quad (4.16)$$

If we also use that  $\rho_c$  (the Higgs vacuum expectation value) can be expressed as

$$\rho_c \simeq \sqrt{\frac{6m_\phi^2}{\lambda}} \left(1 - \frac{T^2}{T_c^2}\right)^{1/2}, \quad (4.17)$$

we see that, in the deep second order regime, where  $\alpha = e^2/\lambda \ll 1$ , we have  $\Lambda^2 \gg M_A^2(1 + \psi_C)$  and we can expand Eq. (4.15) accordingly. Substituting this expansion back in Eq. (4.14) and using Eq. (4.9), we obtain the result (neglecting  $\psi_C$ -independent terms and higher order terms)

$$V_{\text{eff,string}}^{(\beta)}(\psi_C) \simeq \left[ \frac{e^2}{4\pi^2 a^4} (e^{\tau_s a^2} - 6) + \frac{3e^2 \rho_c^2}{16\pi^2 a^2} + \frac{e^2 \rho_c^2}{8} T^2 \right] \psi_C - \frac{e^3 \rho_c^3}{4\pi} (1 + \psi_C)^{3/2} T - \frac{3e^4 \rho_c^4 \ln(2\Lambda/T)}{32\pi^2} \psi_C^2, \quad (4.18)$$

With  $a$  and  $\rho_c$  given by Eqs. (4.16) and (4.17), we can then see that the quantum and thermal corrections in the effective potential for strings, Eq. (4.18), are naturally ordered in powers of  $\alpha$ . Therefore, in the regime  $\alpha \ll 1$  the leading order correction to the tree-level potential in Eq. (4.18) is linear in  $\psi_C$ , while the second and the third correction terms are  $\mathcal{O}(\alpha^{3/2})$  and  $\mathcal{O}(\alpha^2)$ , respectively. Thus, the linear term in  $\psi_C$  controls the transition in the deep second order regime since the other terms are all subleading in  $\alpha$ . Thus, near criticality, determined by some temperature  $T_s$  where the linear term in Eq. (4.18) vanishes,  $V_{\text{eff,string}}^{(\beta)}(\psi_C) \sim 0$  in the  $\alpha \ll 1$  regime.

The phase transition temperature  $T_s$ , which is interpreted as the temperature of transition from the normal vacuum to the state of condensed strings, is then determined by the temperature where the linear term in  $\psi_C$  in Eq. (4.18) vanishes and it is found to be

$$T_s = \frac{\sqrt{2}}{\pi a^2 \rho_c} \left( 6 - e^{\tau_s a^2} - \frac{3a^2 \rho_c^2}{4} \right)^{1/2}, \quad (4.19)$$

where the rhs of Eq. (4.19) is evaluated at  $T = T_s$ . We can now compare the result obtained for  $T_s$ , given by the solution of Eq. (4.19), with the usual mean-field critical temperature  $T_c = \sqrt{12m_\phi^2/(3e^2 + 2\lambda/3)}$  [2], for which the effective mass term of the Higgs field, obtained from  $V_{\text{eff}}^{(\beta)}(\rho_c)$ , vanishes. Using again Eqs. (4.16) and (4.17), with the result  $\tau_s a^2 \sim \mathcal{O}(1/\lambda)$  and in the perturbative regime  $e^2 \ll \lambda \ll 1$ , it follows from Eq. (4.19) that

$$\frac{T_c - T_s}{T_c} \sim \mathcal{O}\left(\frac{e^{-1/\lambda}}{\lambda^2}\right) [1 + \mathcal{O}(\alpha)], \quad (4.20)$$

with next order corrections to the critical temperatures difference being of order  $\mathcal{O}(\alpha)$ . This result for  $T_s$  allows us to identify it with the Ginzburg temperature  $T_G$  for which the contribution of the gauge field fluctuations become important. These results are also found to be in agreement with the calculations done by the authors in Ref. [15], who analyzed an analogous problem using the partition function for strings configurations, in the same regime of deep second order transition.

Also, in the regime where gauge fluctuations are stronger,  $\alpha = e^2/\lambda \gtrsim 1$ , the second term in Eq. (4.18) of order  $\alpha^{3/2}$ , induces a cubic term  $\rho_c^3$  to the effective potential, favoring the appearance of a first order phase transition instead of a second order one. This mechanism of changing a second order phase transition into a first order one by means of gauge fluctuations is usually referred to as the Coleman-Weinberg mechanism [35]. Coleman and Weinberg analyzed this effect in the context of a fourth dimensional Ginzburg-Landau theory, while a similar effect in a three dimensional theory was subsequently studied in Ref. [44].

In our context, we see that the non-trivial vacuum  $\psi_c \neq 0$  above the critical temperature  $T_s$  enhance the first order phase transition by an amount  $(1 + \psi_c)^{3/2}$ . Here, since  $T_s \sim T_c$ , we see that the driven mechanism of the first order transition is a melting of topological defects. This mechanism is very well known in condensed matter physics (see for instance the first reference in [13]) and always leads to a first order phase transition (except in two dimensions).

## V. CONCLUSIONS

In this paper we have considered the evaluation of the partition function for the finite temperature Abelian Higgs model in the context of a dualized model realization. The advantage of adopting this procedure is that in the dual version of the model we explicitly identify the contribution of topological defects in the action. This way we can identify the coupling of a topological current with the matter fields, which in the dual field model, refers to a two-form, antisymmetric gauge field that emerges from the dualization procedure. We also have discussed the issue of gauge invariance in the context of the dual model and computed all gauge fixing and required ghost terms.

The importance of the procedure we here have adopted is that now we can take into account in the functional path integration the contribution of not only constant vacuum field fluctuations but also those nontrivial, inhomogeneous vacuum excitations that must emerge whenever in a theory that exhibits spontaneous symmetry breaking the associated homotopy group differs from the identity, which then points out to the existence (in the broken phase) of stable topological excitations. In this paper we have considered the case of vortex-string topological excitations of the  $U(1)$  complex Higgs field gauged model.

By considering closed magnetic fluxes in  $3 + 1$  dimensions, we have been able to define a local order parameter associated to the quantal vortex-string field, making then possible to define and calculate the effective potential associated to this vortex-string field order parameter. Evaluating the effective potential at one-loop order and at finite temperatures we have presented an explicit formula for the condensation temperature for vortex-strings in the

system, which then characterizes a transition point that we have shown to lie *below* the mean-field critical temperature obtained just from the contributions of the constant scalar Higgs field vacuum expectation value to the partition function.

We have shown that in the deep second order regime  $e^2/\lambda \ll 1$ , the critical temperature for vortex condensation can be associated with the Ginzburg temperature where the gauge fluctuations become important, in agreement with similar results, but obtained by a different method, by the authors in Ref. [15]. Further, we have been able to show a manifestation of the Coleman-Weinberg mechanism, by means of which the second order phase transition can turn into a first order one through the effect due to gauge field fluctuation contributions in the effective potential. The vortex condensation above  $T_s$  is seen to enhance the transition. Usually, it is possible to estimate the latent heat from the cubic term in the effective potential. However, in the high  $\alpha \equiv e^2/\lambda$  regime, where this term is important it is not simple to calculate a reliable value for the vortex condensation  $|\psi_C|$  since we have disregarded in our model vortex interactions.

The fact that  $T_s < T_c$  tempts us to interpret this transition in two steps. As we reach the temperature  $T_s$  from below, we have a vortex condensation, but without completely restoring the broken symmetry, obtaining in this way an intermediate phase at temperatures  $T < T_c$ , since we still have a nonvanishing value for the Higgs background field  $\rho_c$ . As we continue rising the temperature, we have the final melting at  $T_c$ . This is usually known in the condensed matter community as a premelting process. The possibility of having this type of mechanism is very interesting in the context of relativistic quantum field theory, specially related with inflationary scenarios. However, we need to be very careful with this interpretation. The actual window  $T_s < T < T_c$  is very difficult to estimate, and is certainly very tiny in the regime  $\alpha \ll 1$  as discussed above and seen from the result Eq. (4.20). A better interpretation of the problem may be possible if both  $\psi_C$  and  $\rho_c$ , the vortex-string expectation value and the Higgs vacuum expectation value, respectively, are considered as two independent variables in the complete effective potential  $V_{\text{eff}}^{(\beta)}(\rho_c, \psi_C)$  and study the problem as a coupled two-field system. However, for greater  $\alpha$ , where this mechanism is more suitable to be realized, it is not possible to disregard higher order terms in the effective potential. In particular, we have not considered in our model vortex interactions and they could be very important in this regime, possibly changing this scenario. Nevertheless, this premelting mechanism is a very interesting possibility signaled by our one-loop calculation and we believe it should deserved further attention in future works.

We also hope that the method we have employed in this paper will be useful for further investigations, in an analytical way, of the importance of topological excitations to phase transitions in general, not only in the case of the Abelian gauge Higgs model studied here, but also for non-Abelian gauge Higgs models as well, where, e.g. magnetic monopole like excitations can also be studied in the same context.

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## APPENDIX A: THE DUAL FORMALISM FOR TOPOLOGICAL FIELD CONFIGURATIONS

In the formalism developed in Refs. [19, 20, 41], the torus-like world sheets of a closed string contribute to the partition function as a sum over the number and shapes of such world sheets. The formalism is easier to understand when one considers first the corresponding monopole problem, which involves a topological object of one dimension less than the string problem, and one may proceed by analogy.

In the monopole case, one deals with a sum over the number and shapes of closed loops. The monopole is taken as a relativistic particle in interaction with an electromagnetic potential, for which we write the action

$$S[x_\mu(\tau)] = mn^2 \int ds \frac{4\pi n}{e} \oint A_\mu(x) \frac{dx^\mu}{d\tau} d\tau, \quad (\text{A1})$$

where  $m$ ,  $e$  are the mass and charge of the monopole and  $n$  its topological number. For  $N$  monopoles, each with its own topological number, we have the functional integration

$$\begin{aligned}
& \sum_{N=0}^{\infty} \frac{1}{N!} \int \prod_{i=1}^N \mathcal{D}y_{\mu}^{(i)} \sum_{\{n^{(i)}\}} \exp \left\{ i \sum_{i=1}^N \left[ -M \left( n^{(i)} \right)^2 \oint ds + \frac{4\pi n}{e} \oint dy_{\mu}^{(i)} A^{\mu}(y^{(i)}) \right] \right\} \\
&= \exp \left[ \int \mathcal{D}y \sum_n e^{i \left( -Mn^2 \oint ds + \frac{4\pi n}{e} \oint dy_{\mu} A^{\mu}(y) \right)} \right], \tag{A2}
\end{aligned}$$

and because of this exponentiation one needs to consider only the action of a single monopole. From now on, we put  $n = 1$  for the most favorable case. The functional integral measure is defined through the introduction of a hypercubic space-time lattice, with lattice spacing  $a$ . In this way, the integral measure is reduced to the sum over all closed paths  $C$ . The first term in the action is just the total length of a path; if there are  $L$  steps of size  $a$  on the lattice for the entire path, then its total length is  $aL$ . The second term, the line integral of the field potential, is a Wilson loop over the closed path. Defining as usual a link variable  $A_{\ell}$  for each step  $\ell$  on the path, we may write a lattice partition function

$$\sum_C e^{-MaL(C) + i \sum_{\ell \in C} \frac{4\pi}{e} A_{\ell}} = \sum_{L=0}^{\infty} \frac{1}{L} \sum_n K(n, n; L), \tag{A3}$$

where we have introduced the kernel

$$K(n, m; L) = \sum_{C(n \rightarrow m; L)} e^{-MaL + i \sum_{\ell \in C} \frac{4\pi}{e} a A_{\ell}}, \tag{A4}$$

for which it is understood that the sum is carried out over all paths that go from site  $n$  to site  $m$  in  $L$  steps. The  $1/L$  factor on the right-hand side of (A3) is included in order to avoid double counting.

In an analogous manner, one may construct an expression for the sum over the number and shapes of the closed world sheets in the string problem [19, 20, 41]. One starts with the Nambu–Goto action, together with an interaction of the string with an antisymmetric (Kalb–Ramond) field,

$$S[x^{\mu}(\xi^0, \xi^1)] = -\tau_s \oint d^2\xi \sqrt{-g} + i \frac{\pi}{e} mn \oint d^2\xi \sqrt{-g} \epsilon^{ab} \partial_a x^{\mu} \partial_b x^{\nu} W_{\mu\nu}(x). \tag{A5}$$

Here  $x^{\mu}$  is a point on the world sheet described by the string as it propagates through space-time and  $g$  is the determinant of the sheet metric tensor, given by  $g_{ab} = \frac{\partial x^{\mu}}{\partial \xi^a} \frac{\partial x^{\nu}}{\partial \xi^b}$ ,  $a, b = 0, 1$ , with  $\xi^0$  a time-like coordinate variable on the world sheet and  $\xi^1$  a space-like one. The factor  $\tau_s$  in Eq. (A5) is identified with the string tension. We follow Kawai [20], differently from Seo and Sugamoto [19], and keep the string dynamics term in our computations.

As in the monopole case, the sum over all numbers of world sheets also exponentiate, so that we may write

$$e^Z = \exp \left\{ \int \mathcal{D}x e^{iS[x^{\mu}(\xi)]} \right\}. \tag{A6}$$

In the present case, the integration measure is again defined through the use of a space-time lattice of spacing  $a$  in all directions. The partition function on the lattice then reads

$$Z = \sum_{\text{all closed torus-like surfaces } S} e^{-\tau_s a^2 A(S) + i \frac{2\pi m}{e} \sum_{p \in S} a^2 W_{p,n}}, \tag{A7}$$

where  $a^2$  is the area of an elementary lattice plaquette and  $A(S)$  is the number of plaquettes on the surface  $S$ ;  $W_{p,n}$  is the gauge (Kalb–Ramond) field relative to the plaquette  $p$  at site  $n$ .

Proceeding with the analogy with the monopole case, we now have a kernel relative to the tube-like surface of  $A$  plaquettes, with the curves  $C_1$  and  $C_2$  as boundaries of  $S$ ,

$$K(C_1, C_2; A) = \sum_{S(C_1, C_2; A)} e^{-\tau_s a^2 A + i \frac{2\pi m}{e} \sum_{p \in S} a^2 W_p}, \tag{A8}$$

so that

$$Z = \sum_{A=0}^{\infty} \frac{1}{A} \sum_C K(C, C; A). \quad (\text{A9})$$

Both the monopole and string kernels satisfy a recurrence equation, as they should be seen respectively as the transition probability for the monopole at site  $m$  to go to site  $n$  in  $L$  steps or the string to evolve from curve  $C_1$  to curve  $C_2$  sweeping a surface with  $A$  plaquettes. In the monopole case, the recurrence is established by stating that the probability for the monopole to arrive at site  $n$  in  $L$  steps is in fact the product of the probability for it to arrive at some nearest-neighbor site of  $n$  in  $L - 1$  steps and the probability of the last step. Therefore,

$$K(n, m; L) = \sum_{\pm\mu} K(n - a\hat{\mu}, m; L - 1) e^{-Ma + i\frac{4\pi}{e}aA_{n-a\hat{\mu},\mu}}, \quad (\text{A10})$$

where  $\ell = (n - a\hat{\mu}, \mu)$  is the last link, on which we have the gauge field  $A_{n-a\hat{\mu},\mu}$ . Likewise, in the string case,

$$K(C_1, C_2; A) = \sum_{\pm\mu, \mu \neq t} K(C_{1,n,\mu}, C_2; A - 1) e^{-\tau_s a^2 + i\frac{2\pi m}{e}a^2 W_{n-a\hat{\mu},t\mu}}, \quad (\text{A11})$$

where  $C_{1,n,\mu}$  is a deformation of the curve  $C_1$  in which one eliminates the link  $n, n + a\hat{t}$  for inclusion or deletion of a plaquette of area  $a^2$ . Also, the sum is taken over all directions  $\mu$  perpendicular to the curve ( $t$  is a variable on the curve).

By going to the continuum limit ( $a \rightarrow 0$ ), both kernels satisfy a diffusion-like equation similar to that found by Stone and Thomas [27],

$$\frac{\partial}{\partial \bar{L}} K(x, y; \bar{L}) = \left[ \left( \partial_\mu^x + i\frac{4\pi}{e}A_\mu(x) \right)^2 - m^2 \right] K(x, y; \bar{L}), \quad (\text{A12})$$

with  $\bar{L} = a^2 L e^{-Ma}$ ,  $m^2 = \frac{1}{a^2}(e^{Ma} - 8)$ , and [19]

$$\frac{\partial}{\partial \bar{A}} K(C_1, C_2; \bar{A}) = \left[ \left( \frac{\delta}{\delta \sigma_{\mu t}} + i\frac{2\pi m}{e}W_{\mu t}(x) \right)^2 - M^2 \right] K(C_1, C_2; \bar{A}), \quad (\text{A13})$$

where  $\bar{A} = a^4 A e^{-\tau_s a^2}$  and  $M^2 = \frac{1}{a^4}(e^{\tau_s a^2} - 6)$ . In fact, the differential operators on the right-hand side of both Eqs. (A12) and (A13) have the form of a squared covariant derivative. In the first case the operator is

$$D_\mu = \partial_\mu + i\frac{4\pi}{e}A_\mu(x). \quad (\text{A14})$$

On the lattice, acting on a scalar field  $\phi(x)$ , it is written as [45]

$$D_\mu \phi(x) = \frac{1}{a} \left( U_{x,x+a\hat{\mu}}^{-1} \phi(x + a\hat{\mu}) - \phi(x) \right), \quad (\text{A15})$$

with  $U_{x,x+a\hat{\mu}} = \exp \left[ ia\frac{4\pi}{e}A_\mu(x) \right]$  being the gauge field link variable. Its square then reads

$$D_\mu D_\mu \phi(x) = \frac{1}{a^2} \left[ \sum_\mu \left( U_{x,x+a\hat{\mu}} \phi(x + a\hat{\mu}) + U_{x,x-a\hat{\mu}}^{-1} \phi(x - a\hat{\mu}) \right) - 8\phi(x) \right], \quad (\text{A16})$$

so that when acting on the first argument of the kernel, we have

$$\left( D_\mu D_\mu \phi(x) + \frac{8}{a^2} \right) K(x, y; L - 1) = \frac{1}{a^2} \sum_{\pm\mu} K(x - a\hat{\mu}, y; L - 1) e^{ia\frac{4\pi}{e}A_\mu(x-a\hat{\mu})}, \quad (\text{A17})$$

and, therefore,

$$\frac{K(n, m, L) - K(n, m, L - 1)}{e^{-Ma}a^2} = \left[ D_\mu D_\mu - \frac{1}{a^2} (e^{Ma} - 8) \right] K(n, m; L - 1), \quad (\text{A18})$$

from which follows the given continuum equation

$$\frac{\partial}{\partial \bar{L}} K(n, m; \bar{L}) = (D_\mu^2 - m^2) K(n, m; \bar{L}), \quad (\text{A19})$$

for  $\bar{L} = e^{-Ma}a^2L$  and  $m^2 = \frac{1}{a^2}(e^{Ma} - 8)$ , as stated.

A similar reasoning in one dimension less shows the string recurrence relation (A11) appearing as a discretized form of the second diffusion equation (A13).

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