Yang-Mills theory in Landau gauge as a liquid crystal

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Using a spin–charge separation of the gluon field in the Landau gauge we show that the $SU(2)$ Yang-Mills theory in the low-temperature phase can be considered as a nematic liquid crystal. The ground state of the nematic crystal is characterized by the A^2 condensate of the gluon field. The liquid crystal possesses various topological defects (instantons, monopoles and vortices) which are suggested to play a role in non-perturbative features of the theory.

Separation of degrees of freedom is a useful analytical tool which is widely used in various physical applications. For example, the spin-charge decomposition (often referred to as the slave-boson formalism [1]) of the strongly correlated electrons is a popular technique invoked to describe a low-temperature physics of the high-T_c cuprate superconductors [2]. According to the slave-boson formalism the electron creation operator $c_{i\sigma}^\dagger$ is represented as the product of two operators,

$$c_{i\sigma}^\dagger = f_{i\sigma}^\dagger b_i \,, \tag{1}$$

where i is the lattice site and $\sigma = \uparrow, \downarrow$ is the spin index. The operator $f_{i\sigma}^\dagger$ creates a chargeless spin-1/2 fermion state ("spinon") while the operator b_i annihilates a charged spin-0 boson state ("holon"). Physically, the electron is represented as a composite of the spinon particle (which carries information about the spin of the electron) and the holon particle (which knows about the electron charge). The decomposition conserves the total number of the degrees of freedom because of the constraint $f_{i\uparrow}^\dagger f_{i\uparrow} + f_{i\downarrow}^\dagger f_{i\downarrow} + b_i^\dagger b_i = 1$. In Eq. (1) the states with double occupancy are disregarded for simplicity.

The local nature of the spin-charge decomposition (1) leads to an emergence of an internal *compact* $U(1)$ gauge symmetry realized in the form of the gauge transformations

$$f_{i\sigma} \to e^{i\varphi_i} f_{i\sigma} \,, \quad b_i \to e^{i\varphi_i} b_i \,. \tag{2}$$

Certain properties of the high-T_c superconductors can be described [2, 3] by $U(1)$ gauge models which are utilizing the mentioned internal gauge symmetry. These gauge models are treatable within the mean field approach which predicts a rich phase structure. In particular, the d-wave high–T_c superconductor is suggested [3] to be realized as a phase in which the spinon pairing, $\Delta_{ij} \equiv \langle f_{i\uparrow}^\dagger f_{j\downarrow} - f_{i\downarrow}^\dagger f_{j\uparrow} \rangle \neq 0$, is accompanied with a spontaneous breaking of the internal $U(1)$ symmetry by the holon condensate:

$$b \equiv \langle b_i \rangle \neq 0 \,. \tag{3}$$

The presence and the subsequent spontaneous breaking of the internal gauge symmetry may have important physical consequences if even this symmetry is not realized in the original formulation of the theory.

Quantum Chromodynamics is another example of a strongly interacting system in which the breaking of the internal symmetry may play an essential role. Long time ago it was

suggested [4] that the confinement of quarks into hadrons may happen due to a condensation of special gluonic configurations called Abelian monopoles. In this approach – referred to as the dual superconductivity scenario – a condensate of the monopoles breaks spontaneously an internal (or, "dual") $U(1)$ gauge symmetry. According to the dual superconductor idea, the breaking of the dual symmetry gives rise naturally to the dual Meissner effect, which insures a formation of a QCD string, which in turn leads to the confinement the quarks into hadronic bound states.

The problem of an explicit realization of the dual superconductivity in QCD in terms of the original (gluon) fields is not solved yet. Moreover, the dual superconductivity is shown numerically to be realized [5] only in a special Maximal Abelian gauge which explicitly selects predefined direction(s) in the color gauge group. In this gauge the gluons from the diagonal (Cartan) subgroup are likely to be responsible for the infrared phenomena such as the quark confinement [5]. The off-diagonal gluons were shown to be short–ranged and are largely inessential for the infrared physics [6].

Recently, it was proposed [7] to split the gluons in a manner of the spin-charge separation used in the high-T_c superconductivity models. The splitting is based on the field decomposition [8] which is applied to the off-diagonal gluons while leaving the diagonal gluons intact. In the SU(2) Yang-Mills (YM) theory the splitting of the off-diagonal gluons [7, 8],

$$A_\mu^1 + iA_\mu^2 = \psi_1 \, \mathbf{e}_\mu + \psi_2^* \, \mathbf{e}_\mu^*, \quad \mathbf{e}_\mu \mathbf{e}_\mu = 0, \quad \mathbf{e}_\mu \mathbf{e}_\mu^* = 1, \tag{4}$$

leads to appearance of two electrically charged (with respect to the Cartan subgroup of the color gauge group) Abelian scalar fields $\psi_{1,2}$ and the electrically neutral field $\mathbf{e}_\mu$ which is a complex vector. There are also other popular gluon field decompositions [9], some of which were suggested to be related to the monopole condensation.

In this paper we propose a novel generalization of the spin-charge decomposition of the high–T_c superconductors (1) to the $SU(2)$ Yang–Mills (YM) theory. This decomposition splits the $SU(2)$ gluon field into spin and color degrees of freedom treating all color components equally:

$$A_\mu^a(x) = \Phi^{ai}(x) \, e_\mu^i(x) \,. \tag{5}$$

Here $\Phi^{ai}(x)$ is the 3×3 matrix, and $a_\mu^i(x)$ are the three vectors forming an (incomplete) orthonormal basis in the four dimensional space-time, $e_\mu^i(x)e_\mu^j(x) = \delta^{ij}$. The elements of $\Phi^{ai}(x)$ and $a_\mu^i(x)$ are real functions labeled by the color $(a = 1, 2, 3)$, internal $(i = 1, 2, 3)$ and Euclidean vector $(\mu = 1, \ldots, 4)$ indices. Obviously, Eq. (4) is a color-symmetric generalization of Eq. (4). In order to avoid cluttering of notations with lower and upper indices we prefer to work in the Euclidean space–time.

The splitting (5) of the gluon fields can obviously be written in any SU(2) gauge. However, under the local $SU(2)$ color transformations, $A_\mu(x) \to A_\mu^\Omega(x) = \Omega(A_\mu + ig \, \partial_\mu)\Omega^\dagger$, the fields Φ^{ai} and e_μ^i mix with each other in a complicated way. Here $A_\mu \equiv A_\mu^a \sigma^a/2$ is the gauge field, σ^a are the Pauli matrices, and g is the gauge coupling.

In order to make the splitting (5) well–defined we fix the Landau gauge (6) which minimizes the gauge fixing functional,

$$\min_\Omega F[A^\Omega] \,, \qquad F[A] = \int \mathrm{d}^4 x \left[A_\mu^a(x) \right]^2 \,, \tag{6}$$

over the gauge transformations. This gauge condition fixes the $SU(2)$ color gauge freedom up to the $SU(2)$ *global* freedom (which is usually disregarded): $A_\mu^a(x) \to \Omega_{\mathrm{gl}}^{ab} A_\mu^b(x)$, where $\Omega_{\mathrm{gl}}^{ab} =$

$\text{Tr}\,(\sigma^a\Omega\sigma^b\Omega^\dagger)/2$ is the coordinate-independent matrix belonging to the adjoint representation of the color $SU(2)$ group.

The transformation rules for the components of the gauge field (5) are:

$$\Phi(x) \to \Omega_{\text{gl}}\, \Phi(x)\, \Lambda^T(x)\,, \quad e_\mu(x) \to \Lambda(x)\, e_\nu(x)\, \xi_{\mu\nu}\,, \tag{7}$$

or, explicitly, $\Phi^{ai}(x) \to \Omega_{\text{gl}}^{ab}\, \Phi^{bj}(x)\, \Lambda^{ij}(x)$ and $e_\mu^i(x) \to \Lambda^{ij}(x)\, e_\nu^j(x)\, \xi_{\mu\nu}$. Here Ω_{gl} is the matrix of the SU(2) color global transformations, $\xi_{\mu\nu}$ is the SO(4) element of the rotations in the Euclidean space-time and Λ is the matrix of the internal SO(3) transformations ($\Lambda^T\Lambda = \mathbb{1}$):

$$\Omega_{\text{gl}} \in SO(3)_{\text{global}}^{\text{color}}\,, \qquad \xi \in SO(4)_{\text{global}}^{\text{spin}}\,, \qquad \Lambda(x) \in SO(3)_{\text{local}}^{\text{internal}}\,. \tag{8}$$

Equation (5) can be interpreted as a spin-color separation of the gluon field since the color gauge transformations Ω_{gl} are acting only on the matrix field Φ while spin transformations ξ affect only the field e_μ. Note that the color and the space-time rotations are the symmetries of the original $SU(2)$ gauge theory while the internal symmetry group originates from the form of the splitting (5) in analogy to the compact $U(1)$ internal symmetry (2) of the high-T_c superconductors. For the sake of brevity we call below the group of the internal gauge transformations as $SO(3)_{\text{int}}$.

The proposed splitting (5) is self-consistent from the point of view of counting of the degrees of freedom (d.o.f.). The original field A_μ^a is described by $3 \times 4 = 12$ real functions[1]. The field A_μ^a is now rewritten (5) as the product of the matrix Φ^{ai} ($3 \times 3 = 9$ d.o.f.) and the three vector fields e_μ^i ($3 \times 4 = 12$ d.o.f.) subjected to the orthonormality constraints (-6 d.o.f.). The group $SO(3)_{\text{int}}$ of the internal gauge transformations has 3 generators (-3 d.o.f.). Thus, the number of the d.o.f. in the field A_μ^a (which is 12) is the same as the total number of d.o.f. in the product of the fields Φ^{ai} and e_μ^i: (which is $9 + 12 - 6 - 3 = 12$).

It is instructive to rewrite the SU(2) gauge model in an explicitly $SO(3)_{\text{int}}$ invariant form. To this end one may introduce two composite gauge fields:

$$\Gamma_\mu^{ij} = \frac{1}{2}(e_\nu^i \partial_\mu e_\nu^j - e_\nu^j \partial_\mu e_\nu^i)\,, \quad \vartheta_\mu^{ij} = \frac{1}{2}\Big((\Phi^{-1})^{ia}\partial_\mu \Phi^{aj} - (\Phi^{-1})^{ja}\partial_\mu \Phi^{ai}\Big)\,, \tag{9}$$

and two composite matter fields,

$$\chi^{ij} = \Phi^{ai}\Phi^{aj}\,, \qquad z_\mu^{ij} = \frac{1}{2}\Big((\Phi^{-1})^{ia}\partial_\mu \Phi^{aj} + (\Phi^{-1})^{ja}\partial_\mu \Phi^{ai}\Big)\,, \tag{10}$$

which transform under the internal gauge transformations as

$$\Gamma_\mu \to \Lambda(\Gamma_\mu + \partial_\mu)\Lambda^T\,, \quad \vartheta_\mu \to \Lambda(\vartheta_\mu + \partial_\mu)\Lambda^T\,, \quad \chi \to \Lambda\chi\Lambda^T\,, \quad z_\mu \to \Lambda z_\mu \Lambda^T\,. \tag{11}$$

The $SO(3)_{\text{int}}$ gauge fields Γ_μ and ϑ_μ are asymmetric with respect to permutations of the internal indices while the scalar matter field χ and the vector matter field z_μ are symmetric under these permutations. The matter fields transform in the adjoint representation of the $SO(3)_{\text{int}}$ gauge group. Note that it is impossible to construct composite matter fields from

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[1] While counting the degrees of freedom we do not take into account the pure color gauge degrees of freedom (-3 d.o.f.) and do not impose the Gauss constraint (-3 d.o.f.) to select the physical states because these restrictions equally affect both sides of Eq. (5).

the "spin" field e_μ in a manner of Eq. (10) due to the orthonormality constraints imposed on e_μ.

The Landau gauge functional (6) can be expressed in terms of the matter field χ

$$F[A] \equiv \mathcal{F}[\chi] = \int \mathrm{d}^4 x \, \mathrm{Tr} \, \chi \,. \tag{12}$$

Note that this functional still invariant under all global and local transformations (8).

Technically, the existence of the two gauge fields (9) and one adjoint vector field (10) allows us to define an arbitrary number of covariant derivatives, $D_\mu^{ij}(\gamma) = \partial_\mu \delta^{ij} + \gamma_\mu^{ij}$ where the vector field γ_μ stands for any linear combination of the Γ_μ, ϑ_μ and z_μ fields which transforms as a $SO(3)_{\mathrm{int}}$ gauge field. Then, the derivative of the gauge field A_μ^a can be represented in an explicitly $SO(3)_{\mathrm{int}}$ invariant form, $\partial_\mu A_\nu^a = (\hat{\Phi}^a, D_\mu \hat{e}_\nu) + (D_\mu \hat{\Phi}^a, \hat{e}_\nu)$. The local (differential) condition of the Landau gauge, $\partial_\mu A_\mu^a = 0$, can be rewritten as a constraint

$$(\hat{\Phi}^a, D_\mu \hat{e}_\mu) + (D_\mu \hat{\Phi}^a, \hat{e}_\mu) = 0 \,. \tag{13}$$

Here vectors $\hat{\Phi}^a \equiv (\Phi^{a1}, \Phi^{a2}, \Phi^{a3})^T$ are the columns of the matrix Φ^{ai}, $\hat{e}_\mu = (e_\mu^1, e_\mu^2, e_\mu^3)^T$, and $(a, b) = a^i b^i$ is the scalar product in the internal $SO(3)_{\mathrm{int}}$ space. Below we make the choice $\gamma_\mu^{ij} = \Gamma_\mu^{ij}$ for convenience.

It is also convenient to introduce the vector $e_\mu^4 = \varepsilon_{\mu\nu\alpha\beta} e_\nu^1 e_\alpha^2 e_\beta^3$. The four vectors $e_\mu^{\bar{i}}$, $\bar{i} = 1, \ldots, 4$ form a complete orthonormal basis in the $4D$ space-time, $e_\mu^{\bar{i}} e_\mu^{\bar{j}} = \delta^{\bar{i}\bar{j}}$. The internal $SO(3)_{\mathrm{int}}$ transformations act in the subspace spanned onto vectors e_μ^k with $k = 1, 2, 3$ while leaving the vector e_μ^4 intact.

The YM Lagrangian be divided into the three parts

$$L_{SU(2)}[A] \equiv \frac{1}{4} \big[G_{\mu\nu}^a(A) \big]^2 = L_0[\Phi, \chi, \Gamma] + L_1[\chi, \Gamma, \vartheta] + L_2[\chi] + L_{\mathrm{gf}} \,, \tag{14}$$

where $G_{\mu\nu}^a(A) = \partial_{[\mu} A_{\nu]}^a + g \, \varepsilon^{abc} A_\mu^b A_\nu^c$ is the $SU(2)$ field strength tensor and the term L_n is proportional to the nth power of the $SU(2)$ coupling constant g. For a moment we disregard the term L_{gf} coming from the Landau gauge fixing. Using an appropriate multiplication by the vectors $e_\mu^{\bar{k}}$ to convert the Euclidean indices into the internal $SO(3)_{\mathrm{int}}$ basis we rewrite the YM Lagrangian (14) as follows:

$$L_0[\Phi, \chi, \Gamma] = \frac{1}{2} \Big(\mathcal{D}_{\bar{k}}(\Gamma) \hat{\Phi}^a \Big)^2 + \frac{1}{2} \Big(\Sigma(\Gamma), \chi \Sigma(\Gamma) \Big), \tag{15}$$

$$L_1[\chi, \Gamma, \vartheta] = 2g \sqrt{\det \chi} \cdot (\Gamma_k^{ij} - \vartheta_k^{ij}) \, \varepsilon_{ijk} \,, \tag{16}$$

$$L_2[\chi] = \frac{g^2}{4} \Big[(\mathrm{Tr} \, \chi)^2 - \mathrm{Tr} \, \chi^2 \Big] \,. \tag{17}$$

where $\mathcal{D}_{\bar{k}}(\Gamma) \equiv e_\mu^{\bar{k}} D_\mu(\Gamma)$ is the covariant derivative acting on the internal $SO(3)_{\mathrm{int}}$ indices. Note that the spin field $\hat{e}_\mu$ enters the Lagrangian (14) only in the form of the connection $\Gamma_{\bar{k}}^{ij}$.

In order to simplify the L_0 part of the YM Lagrangian (15) we used the differential Landau gauge condition and neglected a full-derivative surface term. The first term in L_0 is the kinetic term for the "color" component of the gluon field $\hat{\Phi}^a$ in the background of the $SO(3)_{\mathrm{int}}$ gauge field Γ. The second term in L_0 can be interpreted as a "dielectric" energy

density associated with the (space-dependent) "dielectric susceptibility" χ and the (dynamical) $SO(3)_{\text{int}}$ "electric field" $\Sigma^i(x) = \Lambda_{\mathcal{E}}^{ij}(x)\mathcal{E}_j(x)$. Here the $SO(3)_{\text{int}}$ gauge transformation $\Lambda_{\mathcal{E}}^{ij}$ diagonalizes the matrix $\Gamma_{\bar{k}}^{4i}\Gamma_{\bar{k}}^{4j} = [\Lambda_{\mathcal{E}}\,\text{diag}(\mathcal{E}_1^2, \mathcal{E}_2^2, \mathcal{E}_3^2)\,\Lambda_{\mathcal{E}}^T]^{ij}$ with $\mathcal{E}_i(x) \geqslant 0$.

The second part (16) of the Lagrangian represents the interaction between the gauge fields Γ and ϑ with the effective coupling $g\det^{1/2}\chi \equiv g\det\Phi$. The third part (17) is a local potential $V(\chi)$ on the "dielectric susceptibility" field χ.

The analogy of the spin-color separation of the gluon in YM theory (5) with the spin-charge separation of the electron in the high-T_c superconductor models [2] manifests itself also in the absence of the kinetic terms for the composite gauge fields Γ_μ and ϑ_μ. This fact is natural since the local construction of each of the composite gauge fields (9) involves already a single derivative while canonical local Lagrangians (*i.e.*, the YM Lagrangian) contain terms with at most two derivatives. The only explicitly propagating field in formulation (14) is $\hat{\Phi}^a$.

Besides the remarkable analogy of the spin–color separation in the YM theory with the spin-charge separation in the high–T_c superconductivity, the YM theory has another interesting analogue in the condensed matter physics. Namely, the YM Lagrangian (14-17) can be interpreted as the free energy density of a nematic liquid crystal.

The ordinary nematic crystals [10] consist of rod-like molecules which tend to align parallel to a direction $\mathbf{n}(\mathbf{x}, t)$. The molecule is invariant under reflections with respect to a plane perpendicular to the molecule axis. The unit vector $\mathbf{n}$ – called the Frank director – is chosen spontaneously in the absence of external electric or magnetic fields. The molecules in liquid crystals do not have a positional order contrary to solid crystals characterized by lattice-like structures. The energetically favored ground state of the nematic crystal is realized at low temperatures and is characterized by a constant director field, $\mathbf{n}(\mathbf{x}, t) = \mathbf{n}_0$. As temperature increases the system undergoes a transition from the nematic phase to the ordinary (isotropic) phase.

Due to the symmetries of the nematic molecule the symmetry group of the ordinary nematic is $G = SO(3)/\mathbb{Z}_2$. Therefore, the order parameter in a nematic may be a unit vector but without associated direction [10] (*i.e.*, a vector without arrowhead). However, it is more convenient to define the order parameter to be diadic in n_i similarly to the diamagnetic (or, dielectric) susceptibility $\tilde{\chi}_{\alpha\beta}$. The excellent candidate for the order parameter which discriminates between the nematic and isotropic phases [10] is the amount of disorder in $\tilde{\chi}_{\alpha\beta}$:

$$\tilde{Q}_{\alpha\beta} = \tilde{\chi}_{\alpha\beta} - \frac{1}{3}\delta_{\alpha\beta}\,\tilde{\chi}_{\gamma\gamma} = \Delta\tilde{\chi}\sum_s\left(n_\alpha^{(s)}n_\beta^{(s)} - \frac{1}{3}\delta_{\alpha\beta}\right), \tag{18}$$

where the last equality is written for the molecules with exact axial symmetry. In Eq. (18) the summation is going over all molecules in a small but macroscopic volume, $\mathbf{n}^{(s)}$ is the direction of the axis of the sth molecule, and $\Delta\tilde{\chi} = \tilde{\chi}_\parallel - \tilde{\chi}_\perp$ is the anisotropy in the diamagnetic (dielectric) susceptibility along and perpendicular to the molecule axis. The quantity $\tilde{Q}_{\alpha\beta}$ is non-zero in the nematic phase while it vanishes in the isotropic phase. Below we refer to $\tilde{\chi}$ as to the dielectric susceptibility.

The dependence of the free energy on the order parameter (18) is usually given by an effective Landau–Lifshitz (LL) potential [10],

$$F_{LL}(\tilde{Q}) = F_0 + \int \text{d}^3\mathbf{x}\sum_{n\geqslant 2}\alpha_n\text{Tr}\,\tilde{Q}^n \tag{19}$$

where α_n are functions of temperature T. The dependence of the free energy on the isotropic factor $\mathrm{Tr}\,\tilde{\chi}$ may be included into the free energy of the normal state, F_0.

The deviations of the Frank director $\mathbf{n}$ from the ground state $\mathbf{n}_0$ are typically described by the Oseen–Zöcher–Frank (OZF) free energy,

$$F_{OZF}[\mathbf{n}] = \frac{1}{2}\int \mathrm{d}^3\mathbf{x}\left[K_1(\nabla \cdot \mathbf{n})^2 + K_2(\mathbf{n}\cdot \nabla \times \mathbf{n})^2 + K_3(\mathbf{n}\times \nabla \times \mathbf{n})^2\right], \qquad (20)$$

where the first three terms describe the free energy associated with the splay, twist and bend distortions. The total free energy of the nematic crystal is $F[\tilde{Q}, \mathbf{n}] = F_{LL}(\tilde{Q}) + F_{OZF}[\mathbf{n}]$. Note that relation (18) makes it possible to rewrite the OZF free energy as a more complicated (compared to (20)) expression in terms of the order parameter $\tilde{Q}$.

The YM theory (14-17) can be associated with a nematic crystal in which the "molecules" are directed in the internal $SO(3)_{\mathrm{int}}$ space. There are three species of equivalent molecules in each space-time point (the number of species equals to the number of the gluons, $N_c = 3$). Consequently, the direction of the local color field in the YM theory, $(\hat{\Phi}^a)^i(x)/|\hat{\Phi}^a(x)|$, is associated with the direction $n_i^{(a)}(x)$ of the ath molecule species in the point x. Then the adjoint matter field $\chi^{ij} = \sum_a \Phi^{ai}\Phi^{aj}$ can be associated with the dielectric susceptibility, $\tilde{\chi}_{\alpha\beta} = \Delta\tilde{\chi}\sum_s n_\alpha^{(s)}n_\beta^{(s)}$. Note that YM "dielectric susceptibility" χ is diadic in the fields $\hat{\Phi}^a$ similarly to the dielectric susceptibility $\tilde{\chi}$ of the nematic.

The proposed association is largely based on the form of the YM term $L_2(\chi)$, Eq. (17), which plays a role of the LL potential (19) for the YM "dielectric" field χ. This term can be rewritten via the isotropic factor $\mathrm{Tr}\,\chi$ and the traceless symmetric matrix Q^{ij}, constructed from the "susceptibility" χ^{ij} similarly to the nematic case (18): $L_2[\chi] = \frac{g^2}{6}(\mathrm{Tr}\,\chi)^2 - \frac{g^2}{4}\mathrm{Tr}\,Q^2$. The negative sign in front of the second term leads to the instability to develop a disorder in the "dielectric susceptibility" χ^{ij}.

The L_0 term, Eq. (15), is a covariant generalization of the kinetic part of the OZF free energy (20) corresponding to the liquid crystal whose splay, twist and bend distortion constants are equal, $K_1 = K_2 = K_3 = 1$. Indeed, in this case the first three terms in Eq. (20) are reduced to $\frac{1}{2}\sum_{i,j=1}^3(\nabla_i n_j)$. Then, we get the L_0 term in the YM Lagrangian by (i) imposing the natural requirement of the $SO(3)_{\mathrm{int}}$ covariance, $\nabla_\mu \to D_{\bar{k}}(\Gamma)$, and (ii) taking into account all molecule species, $\mathbf{n} \to \hat{\Phi}^a(x)$.

As for the L_1 term, Eq. (16), it can be interpreted as an energy density associated with a mutual non–alignment of the directions of the different molecule species $(\hat{\Phi}^a)^i(x)/|\hat{\Phi}^a(x)|$.

Let us find the ground state of the nematic associated with the YM theory (15,16,17). In terms of the eigenvalues of the matrix $\chi = \mathrm{diag}(\chi_1, \chi_2, \chi_3)$, the ground state $\chi = \chi^{(0)}$ is defined by the relations:

$$\sum_{\substack{i,j=1 \\ i>j}}^3 \chi_i^{(0)}\chi_j^{(0)} = 0\,, \quad \sum_{i=1}^3 \chi_i^{(0)} \geqslant 0\,, \quad \prod_{i=1}^3 \chi_i^{(0)} \geqslant 0\,, \qquad (21)$$

where the first relation comes from the condition $\mathrm{Tr}\,\chi^2 = (\mathrm{Tr}\,\chi)^2$ corresponding the global minimum of the Ginzburg–Landau potential (17). The last two relations in Eq. (21) come from the specific definition of the χ–field (10) implying that $\mathrm{Tr}\,\chi \equiv \sum_{ai}(\Phi^{ai})^2 \geqslant 0$ and $\det\chi \equiv (\det\Phi)^2 \geqslant 0$, respectively. Equations (21) imply that at least two eigenvalues of χ must be zero. Without loss of generality we take $\chi_1^{(0)} = \chi_2^{(0)} = 0$, and therefore the ground state is $\chi^{(0)} = \mathrm{diag}(0, 0, \chi_0)$, where $\chi_0 \geqslant 0$ is not fixed.

The perturbative vacuum (in terms of the original gluon fields A_μ^a) corresponds to $\chi_0 = 0$, *i.e.* to the isotropic liquid state. What makes the YM field similar to the nematic liquid is the non–perturbative part of χ_0, which is fixed by the minimum of the Landau gauge functional (12). This minimum is nothing but the A^2–condensate [11], $\operatorname{Tr} \chi = \langle A_\mu^2 \rangle$, evaluated in the Landau gauge. Thus, the isotropic liquid state is broken to the nematic crystal state by the A^2 condensate. This spontaneous symmetry breaking of the isotropic $SO(3)_{\mathrm{int}}$ is similar to the breaking of the compact gauge group by the holon condensate (3).

Technically, a particular non-zero value of the A^2-condensate emerges due to the presence of the gauge–fixing term L_{gf} in Eq. (14) which also contributes to the free energy of the nematic liquid and which was disregarded till now. According to the numerical calculations of the A^2 condensate [12], $g^2 \chi_0 \approx (3\,\mathrm{GeV})^2$.

The non-perturbative vacuum state, $\chi^{(0)} = \operatorname{diag}(0, 0, \chi_0)$ with $\chi_0 > 0$, is still invariant under the (unbroken) group of rotations about the third axis in the internal space, $H = SO(2)_{\mathrm{int}}$. Due to the fact that the $SO(3)_{\mathrm{int}}$ gauge field Γ is non-propagating, the partial spontaneous breaking of the original internal symmetry does not lead to a massless vector field.

The interesting question is a possible existence of topological defects which are generally characterized by non-trivial homotopic groups $\pi_n(G/H)$ of the vacuum manifold G/H of the model. The vacuum manifold of the YM theory with Lagrangian written in the form (14-17) is similar to the vacuum manifold of an ordinary nematic [13] with $G/H = SO(3)/(\mathbb{Z}_2 \times SO(2))$. In particular, the nematic state contains the Z_2 vortices since $\pi_1(G/H) = \mathbb{Z}_2$. This feature may make the physics of the YM nematic state similar to the center vortex picture of the quark confinement in the YM theory [14].

Moreover, the nematic crystal contains monopole-like defects characterized by non-negative integers since $\pi_2(G/H) = \mathbb{Z}/\mathbb{Z}_2 \equiv Z_+ = 0, 1, 2, \ldots$. The monopoles have the hedgehog–like structure constructed from the arrowless "molecules" (the last fact leads to an identification of the monopoles with anti-monopoles). The presence of the monopoles may provide a relation between the nematic liquid crystal and the dual superconductor in the YM theory [4]. A signature of this relation may already be found in Ref. [16] by observing the dual Meissner effect in the Landau gauge. Finally, the third homotopy group of the vacuum manifold is also nontrivial, $\pi_3(G/H) = \mathbb{Z}$, which may have a link to the instanton physics.

The disorder, caused by the presence of the described topological defects in the Landau gauge may lead to the non–trivial consequences for the non–perturbative physics of the YM theory similarly to the effects caused by the center vortex percolation [14] and by the Abelian monopole condensation [4].

Finally, we note that lattice simulations [15] indicate that the A^2 condensate drops by amount of 92% at the finite-temperature phase transition, $T = T_c$. Therefore one may expect that in the deconfinement phase, $T > T_c$, the $4D$ nematic state may transform to a $3D$ nematic state characterized by much lower value of the "nematic dielectric susceptibility" χ. Since the spatial dynamics of the gluon fields remains non–perturbative in the deconfinement phase, one may expect that the nematic crystal splits into two modes: the temporal components of the gluon fields form an ordinary "isotropic liquid" while the spatial components are still in a nematic state.

Summarizing, the spin-charge separation idea – originally invented to describe properties of the high-T_c superconductors – may also be applied to the YM theory in the form of the spin-color separation. This approach allows to identify the ground state of the low-

temperature phase YM theory in the Landau gauge with a nematic liquid crystal. The perturbative isotropic liquid state is broken down to the nematic liquid crystal state by the A^2 condensate. The nematic crystal contains various topological defects which may play a role in explaining of non-perturbative features of the YM theory.

Acknowledgments

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