

Quantum deformation of the Dirac bracket

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The quantum deformation of the Poisson bracket is the Moyal bracket. We construct quantum deformation of the Dirac bracket for systems which admit global symplectic basis for constraint functions. Equivalently, it can be considered as an extension of the Moyal bracket to second-class constraints systems and to gauge-invariant systems which become second class when gauge-fixing conditions are imposed.

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I. INTRODUCTION

The association rules between real functions in the phase space of unconstrained classical systems and Hermitian operators in the Hilbert space of the corresponding quantum mechanical systems are discussed since a long time [1–5]. The commonly used association rule proposed by Weyl [1] consists in replacing products of the canonical variables with symmetrized products of operators of the canonical variables. The Wigner function [2] is associated to the density matrix.

To the lowest order in the Planck's constant, the product \mathbf{fg} of operators associated to functions f and g in the phase space corresponds to the pointwise product fg . A deformation of the pointwise product which keeps the association rule to all orders in the Planck's constant is constructed by Groenewold [3]. This product is known as the star-product.

The Moyal bracket [4] appears as a skew-symmetric part of the star-product. It defines the representation of a commutator $-i/\hbar[,]$ in the space of the functions. The Moyal bracket is antisymmetric, coincides with the Poisson bracket to the lowest order in the Planck's constant, satisfies the Jacoby identity, and keeps the association rule. The bracket satisfying these properties is essentially unique [5]. The Moyal bracket governs the quantum evolution of systems in the phase space like the Poisson bracket governs the classical evolution. A survey of the star-product and the Moyal quantization can be found in [6, 7].

The quantum dynamics of unconstrained systems can therefore be formulated in the phase space in terms of the Hamiltonian and Wigner functions, with the pointwise product replaced by the star-product. The average values of quantum observables can be computed by averaging the symbols of the Hermitian operators over the Wigner function.

The specific feature of gauge theories is the occurrence of constraints which restrict the phase space of gauge-invariant systems to a submanifold. A systematic Hamiltonian approach to gauge theories and general constraint systems and the corresponding operator quantization schemes were developed by Dirac [8].

Schemes based on the path integral method have also been proposed and found to be useful for quantization of gauge theories [9, 10] and second-class constraints systems [11, 12].

Gauge systems are quantized by imposing gauge-fixing conditions which convert them into second-class constraints systems. Anomalous gauge theories [13–17], the $O(n)$ non-linear sigma model [18–20], many-body theories involving collective and elementary degrees of freedom [21–23] are second class from the start.

If constraint equations are solved, the system can be restricted to the constraint submanifold and treated accordingly as an unconstrained system. In many cases, however, it is not possible to solve constraint equations. The method proposed by Dirac for classical second-class constraints systems solves the evolution problem by applying the Dirac

bracket to functions of canonical variables in the unconstrained phase space. It allows thereby to avoid typical complications connected to restriction of the systems to constraint submanifolds.

In this paper, we discuss second-class constraints systems from the Moyal quantization perspective. The main problem we focus attention to is the construction of quantum deformation of the Dirac bracket able to govern the evolution of quantum constraint systems in the unconstrained phase space. The guiding idea in development of the quantization scheme is that many functions in the unconstrained phase space correspond to one and the same physical observable. We thus come naturally to the notion of equivalence classes of functions, and also operators. In the next Sect., we specify equivalence classes of real functions in the unconstrained phase space and, in Sect. III, of Hermitian operators in the Hilbert space. An association rule between equivalence classes of operators and functions, based on the Weyl's association rule, is established. In Appendix A, some useful properties are derived for the skew-gradient projection in terms of which the equivalence classes of functions and operators are defined. In Sect. IV, we give a summary and, in Appendix B, proofs of the basic properties of the Weyl's association rule and of the star-product. In Sect. V, the quantum deformation of the Dirac bracket is constructed for systems which admit global symplectic basis for constraint functions. In Appendix C, we provide an explicit form of the lowest order $O(\hbar^2)$ quantum correction to the Dirac bracket. As an application, in Sect. VI, the quantum deformation of the Dirac bracket is used to formulate, in the unconstrained phase space, an evolution equation for the Wigner function of an $n - 1$ -dimensional spherical pendulum which represents a mechanical counterpart of the $O(n)$ non-linear sigma model.

II. CLASSICAL SECOND-CLASS CONSTRAINTS SYSTEMS IN THE PHASE SPACE

Second-class constraints $\mathcal{G}_a = 0$ with $a = 1, \dots, 2m$ in an $2n$ -dimensional unconstrained phase space $\xi^i = (\phi^1, \dots, \phi^n, \pi^1, \dots, \pi^n)$ have the Poisson bracket relations which form a non-degenerate $2m \times 2m$ matrix

$$\det\{\mathcal{G}_a, \mathcal{G}_b\} \neq 0. \quad (\text{II.1})$$

Two sets of the constraint functions are equivalent if they describe the same constraint submanifold. One can make therefore non-degenerate transformations on the constraint functions without changing the dynamics.

For an arbitrary given point of the constraint submanifold, there is a neighborhood where one may find the equivalent constraint functions in terms of which the Poisson bracket relations look like

$$\{\mathcal{G}_a, \mathcal{G}_b\} = \mathcal{I}_{ab} \quad (\text{II.2})$$

where

$$\mathcal{I}_{ab} = \left\| \begin{array}{cc} 0 & E_m \\ -E_m & 0 \end{array} \right\|, \quad (\text{II.3})$$

with E_m being the unity $m \times m$ matrix, $\mathcal{I}_{ab}\mathcal{I}_{bc} = -\delta_{ac}$. The upper and lower indices of vectors are discriminated according to the rules $\mathcal{T}_a = \mathcal{I}_{ab}\mathcal{T}^b$, $\mathcal{T}^a = \mathcal{I}^{ab}\mathcal{T}_b$, $\mathcal{I}^{ab} = -\mathcal{I}_{ab}$, $\{\mathcal{G}^a, \mathcal{G}_b\} = \delta_b^a$, etc. The scalar product $\mathcal{T}^a\mathcal{Y}_a$ is invariant with respect to the group of linear symplectic transformations $Sp(2m)$.

The global symplectic basis (II.2) for constraint functions exists obviously for $m = 1$ and, also, for systems of point particles under second-class holonomic constraints [20] and second-class non-holonomic constraints satisfying the Frobenius' condition [24]. The global existence of the basis (II.2) is proved for systems with one primary constraint [25] and for a broader set of systems using additional assumptions [25, 26]. In general case, the global existence of the symplectic basis for constraint functions is an opened question.

The basis (II.2) always exists locally, i.e., in a finite neighborhood of any point of the constraint submanifold [24–26]. This is sufficient for needs of the perturbation theory. The formalism presented in this work can therefore to be used

to formulate, in the sense of the perturbation theory, the evolution problem of any second-class constraints system in the unconstrained phase space. Eqs.(II.2) follow from the local existence of the standard canonical coordinate system [27] where \mathcal{G}_a with $a = 1, \dots, m$ play the role of the first canonical coordinates and \mathcal{G}_a with $a = m + 1, \dots, 2m$ play the role of the first canonical momenta.

Let us construct skew-gradient projections $\xi_s(\xi)$ of the canonical variables ξ onto the constraint submanifold $\mathcal{G}_a(\xi) = 0$ using phase flows generated by the constraint functions. From equations

$$\{\xi_s(\xi), \mathcal{G}_a(\xi)\} = 0 \quad (\text{II.4})$$

using the symplectic basis (II.2) for the constraints and expanding

$$\xi_s(\xi) = \xi + X^a \mathcal{G}_a + \frac{1}{2} X^{ab} \mathcal{G}_a \mathcal{G}_b + \dots \quad (\text{II.5})$$

in the power series of \mathcal{G}_a , one gets

$$\xi_s(\xi) = \sum_{k=0}^{\infty} \frac{1}{k!} \{ \dots \{ \{ \xi, \mathcal{G}^{a_1} \}, \mathcal{G}^{a_2} \}, \dots \mathcal{G}^{a_k} \} \mathcal{G}_{a_1} \mathcal{G}_{a_2} \dots \mathcal{G}_{a_k}. \quad (\text{II.6})$$

One can show (see Appendix A) that any function $f(\xi)$ projected onto the constraint submanifold

$$f_s(\xi) = \sum_{k=0}^{\infty} \frac{1}{k!} \{ \dots \{ \{ f(\xi), \mathcal{G}^{a_1} \}, \mathcal{G}^{a_2} \}, \dots \mathcal{G}^{a_k} \} \mathcal{G}_{a_1} \mathcal{G}_{a_2} \dots \mathcal{G}_{a_k} \quad (\text{II.7})$$

satisfies

$$f_s(\xi) = f(\xi_s(\xi)). \quad (\text{II.8})$$

Eq.(II.4) tells us that variations of $\xi_s(\xi)$ along the phase flows generated by the constraint functions $\mathcal{G}_a(\xi)$ are zero. It means that $\xi_s(\xi)$ belongs to the constraint submanifold

$$\mathcal{G}_a(\xi_s(\xi)) = 0. \quad (\text{II.9})$$

For any function $f(\xi)$, one gets $\{\mathcal{G}_a(\xi), f(\xi_s(\xi))\} = 0$. The reciprocal statement is also true: The coordinates on the constraint submanifold can be parameterized by ξ_s . The coordinates describing shifts from the constraint submanifold can be parameterized by \mathcal{G}_a . The functions f can be presented by $f = f(\xi_s, \mathcal{G}_a)$. If f is identically in involution with \mathcal{G}_a , it depends on ξ_s only. This can be summarized by

$$\{\mathcal{G}_a, f\} = 0 \leftrightarrow f = f(\xi_s(\xi)). \quad (\text{II.10})$$

Eqs.(II.9) and (II.10) become selfevident if one works in the standard canonical coordinate system.

An average of a function $f(\xi)$ is calculated using the probability density distribution $\rho(\xi)$ and the Liouville measure restricted to the constraint submanifold [28]:

$$\langle f \rangle = \int \frac{d^{2n}\xi}{(2\pi)^n} (2\pi)^m \prod_{a=1}^{2m} \delta(\mathcal{G}_a(\xi)) f(\xi) \rho(\xi). \quad (\text{II.11})$$

On the constraint submanifold $\xi_s(\xi) = \xi$, so $f(\xi)$ and $\rho(\xi)$ can be replaced with $f_s(\xi)$ and $\rho_s(\xi)$.

There exist therefore equivalence classes of functions [29] in the unconstrained phase space:

$$f(\xi) \sim g(\xi) \leftrightarrow f_s(\xi) = g_s(\xi). \quad (\text{II.12})$$

Here, $f(\xi) \sim g(\xi)$ means that the functions are equal in the weak sense, $f(\xi) \approx g(\xi)$, i.e., on the constraint submanifold. We shall see that the symbols \sim and \approx acquire distinct meaning upon quantization. Note that $f(\xi) \sim f_s(\xi)$. Eqs.(II.8) and (II.9) imply $\mathcal{G}_a \sim 0$.

Given the hamiltonian function \mathcal{H} , the evolution of a function f is described using the Dirac bracket

$$\frac{\partial}{\partial t}f = \{f, \mathcal{H}\}_D. \quad (\text{II.13})$$

In the symplectic basis (II.2), the Dirac bracket looks like

$$\{f, g\}_D = \{f, g\} + \{f, \mathcal{G}^a\}\{\mathcal{G}_a, g\}. \quad (\text{II.14})$$

On the constraint submanifold, one has

$$\{f, g\}_D = \{f, g_s\} = \{f_s, g\} = \{f_s, g_s\}. \quad (\text{II.15})$$

Two hamiltonian functions are equivalent if they generate within the constraint submanifold identical phase flows. The components of the hamiltonian phase flow, which belong to a subspace spanned at the constraint submanifold by phase flows of the constraint functions, do not affect the dynamics and could be different. \mathcal{H} and \mathcal{H}_s are thereby equivalent, so Eq.(II.12) characterizes an equivalence class for the hamiltonian functions either. Among functions of this class, \mathcal{H}_s is the one whose phase flow is skew-orthogonal to phase flows of the constraint functions.

Replacing \mathcal{H} with \mathcal{H}_s , one can rewrite the evolution equation in terms of the Poisson bracket (cf. Eq.(II.13)):

$$\frac{\partial}{\partial t}f = \{f, \mathcal{H}_s\}. \quad (\text{II.16})$$

Eq.(II.16) applied to $f(t)$ and $g(t)$ which belong at $t = 0$ to the same equivalence class, provides with the use of Eq.(A.3) and the initial condition $f_s(0) = g_s(0)$, $f_s(t) = g_s(t)$. The equivalence relations (II.12) are therefore preserved during the evolution.

If we denote the equivalence class of a function $f(\xi)$ as \mathcal{E}_f , the sum of two equivalence classes \mathcal{E}_f and \mathcal{E}_g can be defined as $\mathcal{E}_f + \mathcal{E}_g = \mathcal{E}_{f+g}$ on line with $f_s + g_s = (f + g)_s$, the associative product can be identified with \mathcal{E}_{fg} , while the skew-symmetric Dirac bracket can be defined as $\{\mathcal{E}_f, \mathcal{E}_g\}_D = \mathcal{E}_{\{f, g\}_D}$. These operations satisfy the Leibniz' law,

$$\{\mathcal{E}_f \mathcal{E}_g, \mathcal{E}_h\}_D = \mathcal{E}_f \{\mathcal{E}_g, \mathcal{E}_h\}_D + \{\mathcal{E}_f, \mathcal{E}_h\}_D \mathcal{E}_g, \quad (\text{II.17})$$

and the Jacoby identity,

$$\{\{\mathcal{E}_f, \mathcal{E}_g\}_D, \mathcal{E}_h\}_D + \{\{\mathcal{E}_g, \mathcal{E}_h\}_D, \mathcal{E}_f\}_D + \{\{\mathcal{E}_h, \mathcal{E}_f\}_D, \mathcal{E}_g\}_D = 0. \quad (\text{II.18})$$

The associative product does not depend, in virtue of Eq.(A.2), on the choice of representatives of the equivalence classes. The Dirac bracket can be calculated for arbitrary representatives of the equivalence classes either. Indeed, $f \sim g$ implies

$$\{f, h\}_D \sim (\{f, h\}_D)_s = \{f_s, h_s\}_D = \{g_s, h_s\}_D = (\{g, h\}_D)_s \sim \{g, h\}_D,$$

where the use is made of Eq.(A.4).

The physical observables in second-class constraints systems are associated with the equivalence classes of real functions in the unconstrained phase space. The equivalence classes \mathcal{E}_f constitute a vector space \mathcal{O} equipped with two multiplication operations, the associative pointwise product and the skew-symmetric Dirac bracket $\{, \}_D$, which confer \mathcal{O} a Poisson algebra structure.

The one-to-one mapping $\mathcal{E}_f \leftrightarrow f_s$ induces a Poisson algebra structure on the vector space of projected functions. The sum $\mathcal{E}_f + \mathcal{E}_g$ converts to $f_s + g_s$, the associative product $\mathcal{E}_f \mathcal{E}_g$ converts to the pointwise product $f_s g_s$, while the Dirac bracket $\{\mathcal{E}_f, \mathcal{E}_g\}_D$ becomes the Poisson bracket (cf. transition from (II.13) to (II.16)):

$$\{f_s, g_s\}_D = \{f_s, g_s\}. \quad (\text{II.19})$$

These operations satisfy the Leibniz' law and the Jacoby identity and, since $(f_s + g_s)_s = f_s + g_s$, $(f_s g_s)_s = f_s g_s$, and $\{f_s, h_s\}_s = \{f_s, h_s\}$ (cf. Eqs.(A.2) and (A.3)), keep the vector space of projected functions closed.

III. QUANTUM SECOND-CLASS CONSTRAINTS SYSTEMS IN THE HILBERT SPACE

The systems are quantized by the algebra mapping $\xi^i \rightarrow \mathfrak{r}^i$ and $\{, \} \rightarrow -i/\hbar[,]$. To any function $f(\xi)$ in the unconstrained phase space one may associate an operator \mathfrak{f} in the corresponding Hilbert space. In particular the hamiltonian function $\mathcal{H}(\xi)$ and the constraint functions $\mathcal{G}_a(\xi)$ correspond to the operators \mathfrak{H} and \mathfrak{G}_a , respectively. By this mapping the quantal image is called operator. The reverse mapping associates to an operator \mathfrak{f} a symbol $f(\xi)$.

Eqs.(II.2) become

$$[\mathfrak{G}_a, \mathfrak{G}_b] = i\hbar\mathcal{I}_{ab}. \quad (\text{III.1})$$

The Weyl's association rule applied to Eqs.(II.2) yields Eqs.(III.1), provided the quantization is performed in the standard canonical coordinate system.

Any operator in the Hilbert space can be represented as a function of $2n$ operators \mathfrak{r}^i associated to $2n$ canonical variables ξ^i .

Let us construct $2n$ operators \mathfrak{r}_s^i associated to the projected variables (II.6). They commute with the constraints

$$[\mathfrak{r}_s, \mathfrak{G}_a] = 0. \quad (\text{III.2})$$

The analogue of Eq.(II.6) looks like

$$\mathfrak{r}_s = \sum_{k=0}^{\infty} \frac{(-i/\hbar)^k}{k!} [\dots[[\mathfrak{r}, \mathfrak{G}^{a_1}], \mathfrak{G}^{a_2}], \dots \mathfrak{G}^{a_k}] \mathfrak{G}_{a_1} \mathfrak{G}_{a_2} \dots \mathfrak{G}_{a_k}. \quad (\text{III.3})$$

Among the operators acting in the Hilbert space, one expects that, equivalence classes exist. For an arbitrary operator \mathfrak{f} , a projected operator \mathfrak{f}_s can be constructed as follows:

$$\mathfrak{f}_s = \sum_{k=0}^{\infty} \frac{(-i/\hbar)^k}{k!} [\dots[[\mathfrak{f}, \mathfrak{G}^{a_1}], \mathfrak{G}^{a_2}], \dots \mathfrak{G}^{a_k}] \mathfrak{G}_{a_1} \mathfrak{G}_{a_2} \dots \mathfrak{G}_{a_k}. \quad (\text{III.4})$$

One has

$$[\mathfrak{f}_s, \mathfrak{G}_a] = 0 \quad (\text{III.5})$$

and

$$(\mathfrak{f}\mathfrak{g})_s = (\mathfrak{f}_s\mathfrak{g})_s = \mathfrak{f}_s\mathfrak{g}_s. \quad (\text{III.6})$$

Two operators \mathfrak{f} and \mathfrak{g} belong to the same equivalence class provided $\mathfrak{f}_s = \mathfrak{g}_s$, i.e.,

$$\mathfrak{f} \sim \mathfrak{g} \leftrightarrow \mathfrak{f}_s = \mathfrak{g}_s. \quad (\text{III.7})$$

The Dirac's quantization method of second-class constraints systems [8] consists in constructing operators reproducing the Dirac bracket for canonical variables and taking constraints to be operator equations. In the classical limit, the commutators of operators (III.3) satisfy the Dirac bracket relations for canonical variables on the constraint submanifold. Furthermore, $(\mathfrak{G}^a)_s = 0$, and so $\mathfrak{G}^a \sim 0$.

As a consequence of the Jacoby identity and Eq.(III.1), the operator

$$\mathfrak{C}^{a_1 \dots a_k} = [\dots[[, \mathfrak{G}^{a_1}], \mathfrak{G}^{a_2}], \dots \mathfrak{G}^{a_k}] \quad (\text{III.8})$$

entering Eqs.(III.3) and (III.4) is symmetric with respect to permutations of a_1, \dots, a_k . Any contraction of the upper indices with \mathcal{I}_{ab} annihilates (III.8). It follows that

$$[\mathfrak{C}^{a_1 \dots a_i \dots a_k}, \mathfrak{G}_{a_i}] = 0. \quad (\text{III.9})$$

The position of the constraint operators with the lower indices and of the operator (III.8) in Eqs.(III.3) and (III.4) is not important. One can place, e.g., \mathfrak{G}_{a_1} on the first position, $\mathfrak{C}^{a_1 \dots a_k}$ on the second position, etc.

In order to calculate the average value of an operator, one has to construct the quantal image of the delta functions product entering Eq.(II.11). The projection operator can be written as follows

$$\mathfrak{P} = \int \frac{d^{2m} \lambda}{(2\pi\hbar)^m} \prod_{a=1}^{2m} \exp\left(\frac{i}{\hbar} \mathfrak{G}^a \lambda_a\right). \quad (\text{III.10})$$

In the classical limit, one recovers the product of the delta functions.

Let us chose a basis in the Hilbert space in which the first m constraint operators are diagonal,

$$\mathfrak{G}^a |g, g_* \rangle = g^a |g, g_* \rangle, \quad (\text{III.11})$$

for $a = 1, \dots, m$. Congruous to this equation, \mathfrak{G}^a might be taken as momentum operators. The last m constraint operators can be treated as quantal coordinates. The additional $n - m$ eigenvalues are denoted by g_* .

The projection operator \mathfrak{P} acts as follows:

$$\mathfrak{P} |g, g_* \rangle = |0, g_* \rangle. \quad (\text{III.12})$$

To arrive at this equation, we split $\mathfrak{G}^a = (\mathbf{P}^A, -\mathbf{Q}^A)$ and $\lambda^a = (\lambda'_A, \lambda''_A)$ and write

$$\begin{aligned} \mathfrak{P} |g, g_* \rangle &= \int \frac{d^{2m} \lambda}{(2\pi\hbar)^m} \prod_{a=1}^{2m} \exp\left(\frac{i}{\hbar} \mathfrak{G}^a \lambda_a\right) |g, g_* \rangle \\ &= \int d^m \lambda' \frac{d^m \lambda''}{(2\pi\hbar)^m} \prod_{A=1}^m \exp\left(\frac{i}{\hbar} \mathbf{P}^A \lambda'_A\right) \prod_{B=1}^m \exp\left(-\frac{i}{\hbar} \mathbf{Q}^B \lambda''_B\right) |g, g_* \rangle \\ &= \int d^m \lambda' \frac{d^m \lambda''}{(2\pi\hbar)^m} \prod_{A=1}^m \exp\left(\frac{i}{\hbar} \mathbf{P}^A \lambda'_A\right) |g - \lambda'', g_* \rangle \\ &= \int d^m \lambda' \frac{d^m \lambda''}{(2\pi\hbar)^m} \prod_{A=1}^m \exp\left(\frac{i}{\hbar} (g^A - \lambda''_A) \lambda'_A\right) |g - \lambda'', g_* \rangle \\ &= \int d^m \lambda'' \prod_{A=1}^m \delta(g^A - \lambda''_A) |g - \lambda'', g_* \rangle \\ &= |0, g_* \rangle \end{aligned}$$

The average value of an operator \mathfrak{f} ,

$$\langle \mathfrak{f} \rangle = Tr[\mathfrak{P} \mathfrak{f}_s \mathfrak{r}_s] \quad (\text{III.13})$$

where $\mathfrak{r} = \mathfrak{r}^+$ is the density matrix, can be transformed to give

$$\begin{aligned} Tr[\mathfrak{P} \mathfrak{f}_s \mathfrak{r}_s] &= \int \frac{d^m g d^{n-m} g_*}{(2\pi\hbar)^n} \langle g, g_* | \mathfrak{P} \mathfrak{f}_s \mathfrak{r}_s | g, g_* \rangle \\ &= \int \frac{d^m g d^{n-m} g_*}{(2\pi\hbar)^n} \langle g, g_* | \mathfrak{f}_s \mathfrak{r}_s | 0, g_* \rangle = \int \frac{d^{n-m} g_*}{(2\pi\hbar)^{n-m}} \langle 0, g_* | \mathfrak{f}_s \mathfrak{r}_s | 0, g_* \rangle. \end{aligned} \quad (\text{III.14})$$

The average values are determined by the physical subspace of the Hilbert space, spanned by the vectors $|0, g_* \rangle$.

These vectors satisfy the equation

$$\mathfrak{G}^a |0, g_* \rangle = 0 \quad (\text{III.15})$$

which can be recognized as the Dirac's supplementary condition [8] of an equivalent gauge system [20, 24–26], where \mathfrak{G}^a with $a = 1, \dots, m$ are gauge generators and \mathfrak{G}^a with $a = m + 1, \dots, 2m$ are gauge-fixing operators.

All density matrices from a given equivalence class correspond to a single physical state, while operators from the same equivalence class have equal average values.

The quantum evolution equation can be written as an extension of the classical evolution equation (II.16)

$$i\hbar \frac{d}{dt} \mathbf{f} = [\mathbf{f}, \mathfrak{H}_s] \quad (\text{III.16})$$

where \mathfrak{H}_s is the projection (III.4) of the Hamiltonian \mathfrak{H} . The evolution has the property that at any time $\mathbf{f}(t) \sim \mathbf{g}(t)$ if $\mathbf{f}(0) \sim \mathbf{g}(0)$. This is suggested by the equation

$$[\mathbf{f}, \mathbf{g}]_s = [\mathbf{f}_s, \mathbf{g}_s] \quad (\text{III.17})$$

which is evident due to $(\mathbf{f} + \mathbf{g})_s = \mathbf{f}_s + \mathbf{g}_s$ and Eq.(III.6).

IV. WEYL'S ASSOCIATION RULE AND THE STAR-PRODUCT

The Weyl's association rule can be formulated in a comprehensive way in terms of the operator function [30]

$$\tilde{\mathfrak{B}}(\eta) = \exp\left(\frac{i}{\hbar} \eta_k \mathbf{r}^k\right) \quad (\text{IV.1})$$

whose Fourier transform,

$$\mathfrak{B}(\xi) = \int \frac{d^{2n} \eta}{(2\pi\hbar)^n} \exp\left(-\frac{i}{\hbar} \eta_k \xi^k\right) \tilde{\mathfrak{B}}(\eta), \quad (\text{IV.2})$$

has the properties (see Appendix B)

$$\mathfrak{B}(\xi)^+ = \mathfrak{B}(\xi), \quad (\text{IV.3})$$

$$\text{Tr}[\mathfrak{B}(\xi)] = 1, \quad (\text{IV.4})$$

$$\int \frac{d^{2n} \xi}{(2\pi\hbar)^n} \mathfrak{B}(\xi) = \mathbf{1}, \quad (\text{IV.5})$$

$$\int \frac{d^{2n} \xi}{(2\pi\hbar)^n} \mathfrak{B}(\xi) \text{Tr}[\mathfrak{B}(\xi) \mathbf{f}] = \mathbf{f}, \quad (\text{IV.6})$$

$$\text{Tr}[\mathfrak{B}(\xi) \mathfrak{B}(\xi')] = (2\pi\hbar)^n \delta^{2n}(\xi - \xi'), \quad (\text{IV.7})$$

$$\mathfrak{B}(\xi) \exp\left(-\frac{i\hbar}{2} \mathcal{P}_{\xi\xi'}\right) \mathfrak{B}(\xi') = (2\pi\hbar)^n \delta^{2n}(\xi - \xi') \mathfrak{B}(\xi'). \quad (\text{IV.8})$$

Here,

$$\mathcal{P}_{\xi\xi'} = -I^{kl} \overleftarrow{\frac{\partial}{\partial \xi^k}} \overrightarrow{\frac{\partial}{\partial \xi'^l}}$$

is the so-called Poisson operator. The matrix I^{kl} looks similarly to the matrix (II.3)

$$I^{kl} = \left\| \begin{array}{cc} 0 & -E_n \\ E_n & 0 \end{array} \right\|, \quad (\text{IV.9})$$

with E_n being the $n \times n$ identity matrix.

Equations

$$f(\xi) = \text{Tr}[\mathfrak{B}(\xi) \mathbf{f}], \quad (\text{IV.10})$$

$$\mathbf{f} = \int \frac{d^{2n} \xi}{(2\pi\hbar)^n} f(\xi) \mathfrak{B}(\xi). \quad (\text{IV.11})$$

define the Weyl's association rule.

The canonical variables appear as symbols of operators of the coordinates and momenta: $\xi^i = Tr[\mathfrak{B}(\xi)\mathfrak{t}^i]$.

The phase space of quantum systems is equipped with the Groenewold star-product [3]. Given two functions

$$\begin{aligned} f(\xi) &= Tr[\mathfrak{B}(\xi)\mathfrak{f}], \\ g(\xi) &= Tr[\mathfrak{B}(\xi)\mathfrak{g}], \end{aligned}$$

one can construct a third function

$$f(\xi) \star g(\xi) = Tr[\mathfrak{B}(\xi)\mathfrak{f}\mathfrak{g}]. \quad (\text{IV.12})$$

The star-product is an associative operation. It splits into symmetric and antisymmetric parts:

$$f \star g = f \circ g + \frac{i\hbar}{2} f \wedge g. \quad (\text{IV.13})$$

The explicit form of the star-product can be derived from Eqs.(IV.3) - (IV.8):

$$f(\xi) \star g(\xi) = f(\xi) \exp\left(\frac{i\hbar}{2}\mathcal{P}\right)g(\xi), \quad (\text{IV.14})$$

where $\mathcal{P} = \mathcal{P}_{\xi\xi}$ and therefore

$$f(\xi) \circ g(\xi) = f(\xi) \cos\left(\frac{\hbar}{2}\mathcal{P}\right)g(\xi), \quad (\text{IV.15})$$

$$f(\xi) \wedge g(\xi) = f(\xi) \frac{2}{\hbar} \sin\left(\frac{\hbar}{2}\mathcal{P}\right)g(\xi). \quad (\text{IV.16})$$

The Planck's constant \hbar appears as a quantum deformation parameter. The antisymmetric product $f(\xi) \wedge g(\xi)$ is known under the name of Moyal bracket. The classical limit of the Moyal bracket is the Poisson bracket:

$$\lim_{\hbar \rightarrow 0} f(\xi) \wedge g(\xi) = \{f(\xi), g(\xi)\}. \quad (\text{IV.17})$$

The star-product and the Moyal bracket obey the Leibniz' law

$$f \wedge (g \star h) = (f \wedge g) \star h + g \star (f \wedge h). \quad (\text{IV.18})$$

This equation is valid separately for symmetric and antisymmetric parts of the star-product. In the last case, Eq.(IV.18) provides the Jacoby identity.

V. QUANTUM SECOND-CLASS CONSTRAINTS SYSTEMS IN THE PHASE SPACE

The phase space of quantum systems is endowed with the star-product. The corresponding Hamiltonian $H(\xi)$ and constraint functions $G_a(\xi)$ appear as symbols of the operators \mathfrak{H} and \mathfrak{G}_a , respectively, as prescribed by Eq.(IV.10). If the quantization is done in the standard canonical coordinate system, then $G_a(\xi) = \mathcal{G}_a(\xi)$. In general case, the equality holds in the classical limit, i.e.,

$$\lim_{\hbar \rightarrow 0} G_a(\xi) = \mathcal{G}_a(\xi). \quad (\text{V.1})$$

Classical systems represent a limiting case of quantum systems, not vice versa, so many quantum systems with the same classical limit usually exist. Some ambiguities arise due to the so-called "operator ordering problem". There exist, as a consequence, several association rules the most popular of which is the Weyl's rule. The explicit forms of distinct association rules are provided by Mehta [5]. The second-class constraints systems have additional ambiguities

connected to the choice of constraint functions $G_a(\xi)$. Here, we require only that $H(\xi)$ and $G_a(\xi)$ exist and tend, respectively, to $\mathcal{H}(\xi)$ and $\mathcal{G}_a(\xi)$ as $\hbar \rightarrow 0$.

The analogue of Eqs.(II.2) and (III.1), which specifies the symplectic basis for the constraint functions G_a , has the form

$$G_a(\xi) \wedge G_b(\xi) = \mathcal{I}_{ab}. \quad (\text{V.2})$$

These equations are automatically fulfilled provided the constraint operators \mathfrak{G}_a obey Eqs.(III.1).

The skew-gradient projections, analogous to Eqs.(II.4) and (III.2), are defined by

$$\xi_t(\xi) \wedge G_a(\xi) = 0. \quad (\text{V.3})$$

The projected canonical variables have the form

$$\xi_t(\xi) = \sum_{k=0}^{\infty} \frac{1}{k!} (\dots ((\xi \wedge G^{a_1}) \wedge G^{a_2}) \dots \wedge G^{a_k}) \circ G_{a_1} \circ G_{a_2} \dots \circ G_{a_k} \quad (\text{V.4})$$

(cf. Eqs.(II.6) and (III.3)). The analogue of Eqs.(II.7) and (III.4) is

$$f_t(\xi) = \sum_{k=0}^{\infty} \frac{1}{k!} (\dots ((f(\xi) \wedge G^{a_1}) \wedge G^{a_2}) \dots \wedge G^{a_k}) \circ G_{a_1} \circ G_{a_2} \dots \circ G_{a_k}. \quad (\text{V.5})$$

The operation \circ is not associative in general. The order in which it acts is, however, not important due to commutativity of the operators in expressions (III.3) and (III.4). In the classical limit one has

$$\lim_{\hbar \rightarrow 0} f_t = f_s. \quad (\text{V.6})$$

The terms $\dots \circ G_a$ entering Eq.(V.4) involve the derivatives and therefore do not vanish when $G_a = 0$, so the variables $\xi_t(\xi)$ are at variances $O(\hbar^2)$ with the variables ξ on the constraint submanifold. Respectively, $f_t(\xi)$ and $f(\xi)$ are at a variance $O(\hbar^2)$ on the constraint submanifold also.

The lowest order quantum corrections in Eq.(V.5) can originate from the Moyal bracket inside of the largest group of terms inside of the round brackets. In such a case, the symmetrized \circ -product can be replaced with the desired accuracy by the pointwise product. On the constraint submanifold, those terms do not contribute to the result and, respectively, the Moyal bracket in Eq.(V.5) can be replaced with the Poisson bracket. The second possible source of the quantum corrections is the operation \circ itself, as defined by Eq.(IV.15). To the lowest order in the Planck's constant, the fourth-order differential operator \mathcal{P}^2 appears. The multiple product $(\dots) \circ G_{a_1} \circ G_{a_2} \dots \circ G_{a_k}$ can be calculated as $(\dots(((\dots) \circ G_{a_1}) \circ G_{a_2}) \dots) \circ G_{a_k}$. The zeroth and second powers of \hbar show up in the first three terms of the k -series only. One gets, on the constraint submanifold,

$$\begin{aligned} f_t(\xi) = & f(\xi) - \frac{\hbar^2}{8} \{f(\xi), G^{a_1}\} \mathcal{P}^2 G_{a_1} - \frac{\hbar^2}{8} \frac{1}{2!} \{\{f(\xi), G^{a_1}\}, G^{a_2}\} G_{a_1} \mathcal{P}^2 G_{a_2} \\ & - \frac{\hbar^2}{8} \frac{1}{3!} \{\{\{f(\xi), G^{a_1}\}, G^{a_2}\}, G^{a_3}\} G_{a_1} G_{a_2} \mathcal{P}^2 G_{a_3} + O(\hbar^4). \end{aligned} \quad (\text{V.7})$$

The operator \mathcal{P} acts on the left and on the right as prescribed by Eq.(IV.15). The terms G_a involving no \mathcal{P} action vanish on the constraint submanifold and are therefore omitted. The projected functions differ in general from the original ones. The constraint functions are, however, an exception, i.e., $(G_a)_t = 0$ on the constraint submanifold and, furthermore, in the unconstrained phase space.

The equivalence relations between the operators Eq.(III.7) lead, under the Weyl's association rule, to the equivalence relations between functions in the phase space:

$$f(\xi) \sim g(\xi) \leftrightarrow f_t(\xi) = g_t(\xi) \quad (\text{V.8})$$

The physical observables are mapped through the quantization procedure onto the equivalence classes of functions. Since $f(\xi) \sim f_t(\xi)$ and $f(\xi) \neq f_t(\xi)$ on the constraint submanifold, the symbols \sim and \approx acquire distinct meaning.

The average value of a function $f(\xi)$ is defined as

$$\langle f \rangle = \int \frac{d^{2n}\xi}{(2\pi\hbar)^n} P(\xi) \star f_t(\xi) \star W_t(\xi) \quad (\text{V.9})$$

where $P(\xi)$ is the symbol of the projection operator \mathfrak{P} and $W(\xi)$ is the Wigner function, i.e., the symbol of the density matrix \mathfrak{r} . In the classical limit

$$\lim_{\hbar \rightarrow 0} \frac{P(\xi)}{(2\pi\hbar)^m} = \prod_{a=1}^{2m} \delta(\mathcal{G}_a(\xi)). \quad (\text{V.10})$$

If the quantization is done in the standard canonical coordinate system, $P(\xi)/(2\pi\hbar)^m$ turns to the product of the delta functions to all orders in \hbar .

The evolution equation which is analogue of Eqs.(II.16) and (III.16) takes the form

$$\frac{\partial}{\partial t} f(\xi) = f(\xi) \wedge H_t(\xi). \quad (\text{V.11})$$

The quantum deformation of the Dirac bracket represents a skew-symmetric multiplication operation on the set of equivalence classes of functions in the phase space of quantum systems.

Like in the classical case, the sum of two equivalence classes E_f and E_g is defined by $E_f + E_g = E_{f+g}$, the associative product $E_f E_g$ is defined for quantum systems using the star-product as $E_{f_t \star g_t}$, and the skew-symmetric Dirac bracket is defined as $\{E_f, E_g\}_D = E_{f_t \wedge g_t}$. These operations satisfy the Leibniz' law and the Jacoby identity. The associative product and the Dirac bracket do not depend on the choice of representatives of the equivalence classes. Since $(G^a)_t = 0$, the deformed Dirac bracket vanishes for any two equivalence classes the one of which contains a constraint function.

By quantization, the Dirac bracket is associated to the commutator of two projected operators. The Moyal bracket for projected functions on the constraint submanifold appears as a Lie bracket in the vector space \mathcal{O} of equivalence classes E_f of functions in the phase space of quantum systems. The associative star-product and the skew-symmetric Moyal bracket gifts \mathcal{O} with a Poisson algebra structure.

The mapping $E_f \leftrightarrow f_t$ induces further a Poisson algebra structure on the vector space of projected functions. The sum $E_f + E_g$ converts to $f_t + g_t$, the associative product $E_f E_g$ converts to $f_t \star g_t$, whereas the Dirac bracket $\{E_f, E_g\}_D$ becomes the Moyal bracket $f_t \wedge g_t$. It is clear that these operations satisfy the Leibniz' law and the Jacoby identity and, furthermore, keep the vector space of projected functions closed, as $(f_t + g_t)_t = f_t + g_t$, $(f_t \star g_t)_t = f_t \star g_t$, and $(f_t \wedge h_t)_t = f_t \wedge h_t$.

The skew-gradient projection of Eq.(V.11) gives an evolution equation in the form involving projected functions only. Eq.(V.11) is, however, more convenient for applications, since it is valid for any representative $f(\xi)$ of an equivalence class and does not presuppose restrictions for $f(\xi)$.

In terms of the projected functions, the classical limit appears according to equations

$$\lim_{\hbar \rightarrow 0} f_t \star g_t = f_s g_s, \quad (\text{V.12})$$

$$\lim_{\hbar \rightarrow 0} f_t \wedge g_t = \{f_s, g_s\} = \{f, g\}_D, \quad (\text{V.13})$$

where the use is made of Eqs.(II.15), (IV.17) and (V.6). It is worthwhile to notice that Eq.(II.15) is valid on the constraint submanifold, so the operation $f_t \wedge g_t$ defined for all functions in the unconstrained phase space converts to the Dirac bracket on the constraint submanifold only. Away from the constraint submanifold, functions and, accordingly, the Dirac bracket do not make any physical sense. The equivalence classes and the projected functions are the only objects associated to physical observables.

In terms of the equivalence classes, the classical limit appears as

$$\lim_{\hbar \rightarrow 0} E_f E_g = \mathcal{E}_f \mathcal{E}_g, \quad (\text{V.14})$$

$$\lim_{\hbar \rightarrow 0} \{E_f, E_g\}_D = \{\mathcal{E}_f, \mathcal{E}_g\}_D. \quad (\text{V.15})$$

The equivalence classes E_f of the quantum systems represent the quantum deformation of the equivalence classes \mathcal{E}_f of the corresponding classical systems. At $\hbar \rightarrow 0$, the star-product quantization recovers the classical constrained dynamics.

To illustrate calculation of the quantum corrections to the Dirac bracket, in Appendix C we derive the analytical expression for the \hbar^2 correction.

VI. QUANTUM SPHERICAL PENDULUM IN THE PHASE SPACE

As an application we consider, in the phase space, the evolution of the Wigner function of a mathematical pendulum on an S^{n-1} sphere of a unit radius in an n -dimensional Euclidean space with the coordinates ϕ^α .

At the classical level, there exists, within the generalized Hamiltonian framework, two constraint functions

$$\mathcal{G}^1 = \ln \phi, \quad \mathcal{G}^2 = \pi^\alpha \phi^\alpha \quad (\text{VI.1})$$

where $\phi = (\phi^\alpha \phi^\alpha)^{1/2}$. The first constraint $\mathcal{G}^1 = 0$ implies that the particle stays on the sphere $\phi = 1$, the second constraint $\mathcal{G}^2 = 0$ suggests that the radial component of the momenta vanishes. The constraint functions constitute a canonical pair (II.2). One can check that the projected canonical variables $\xi_s(\xi) = (\phi_s^\alpha, \pi_s^\alpha)$, with

$$\phi_s^\alpha = \phi^\alpha / \phi, \quad (\text{VI.2})$$

$$\pi_s^\alpha = \phi \pi^\alpha - \phi^\alpha \phi \pi / \phi, \quad (\text{VI.3})$$

and the Hamiltonian projected onto the constraint submanifold (see [24], Sect. 6),

$$\mathcal{H}_s = \frac{1}{2} (\phi^2 \delta^{\alpha\beta} - \phi^\alpha \phi^\beta) \pi^\alpha \pi^\beta, \quad (\text{VI.4})$$

are identically in involution with \mathcal{G}^a , which is on a par with Eqs.(II.4) and (II.10).

Since \mathcal{G}^2 is a second order polynomial with respect to the canonical variables, the infinite power series in the Poisson operator in Eq.(IV.16) is truncated at $O(\hbar^0)$. Indeed, all the higher order terms vanish, so that $\mathcal{G}^1 \wedge \mathcal{G}^2 = \{\mathcal{G}^1, \mathcal{G}^2\} = 1$. Assuming $G^a = \mathcal{G}^a$, Eqs.(V.2) hold.

Similarly, Eqs.(V.3) hold for $\xi_t(\xi) = \xi_s(\xi)$: \mathcal{G}^1 and ϕ_s^α depend on ϕ^α , so $\phi_s^\alpha \wedge \mathcal{G}^1 = 0$. π_s^α is a first degree polynomial with respect to the canonical momenta. In the expression $\pi_s^\alpha \wedge \mathcal{G}^1$, the operators \mathcal{P}^{2k+1} act to the right on the coordinates only, and to the left on the momenta. The power series terms entering Eq.(IV.16) vanish starting with $O(\hbar^2)$, so $\pi_s^\alpha \wedge \mathcal{G}^1 = \{\pi_s^\alpha, \mathcal{G}^1\} = 0$ holds in virtue of Eqs.(II.4). \mathcal{G}^2 is a second degree polynomial with respect to the canonical variables. The infinite power series in Eq.(IV.16) is truncated at $O(\hbar^0)$ again. Due to Eqs.(II.4), we have $\mathcal{G}^2 \wedge \phi_s^\alpha = \{\mathcal{G}^2, \phi_s^\alpha\} = 0$ and $\mathcal{G}^2 \wedge \pi_s^\alpha = \{\mathcal{G}^2, \pi_s^\alpha\} = 0$.

Finally, one has to check that $\mathcal{G}^a \wedge \mathcal{H}_s = 0$. The first equation, for $a = 1$, is valid since \mathcal{G}^1 depends on ϕ^α , while \mathcal{H}_s depends on π^α quadratically. The infinite power series in Eq.(IV.16) is truncated at $O(\hbar^0)$, so the Moyal bracket can be replaced by the Poisson bracket. Since $\{\mathcal{G}^1, \mathcal{H}_s\} = 0$, equation $\mathcal{G}^1 \wedge \mathcal{H}_s = 0$ holds. The second equation, for $a = 2$, is valid since \mathcal{G}^2 is a second degree polynomial. Consequently, $\mathcal{G}^2 \wedge \mathcal{H}_s = \{\mathcal{G}^2, \mathcal{H}_s\} = 0$.

The projected canonical variables $\xi_s(\xi)$, the constraint functions \mathcal{G}^a , and the classical Hamiltonian \mathcal{H}_s coincide with the projected canonical variables $\xi_t(\xi)$, the constraint functions G^a , and the Hamiltonian H_t , respectively.

TABLE I: Skew-symmetric multiplication operations are listed for functions (second column) and equivalence classes of functions (third column) in the phase space of classical systems (first row) and in the phase space of quantum systems (second row).

Systems:	unconstrained	constrained
classical	$\{f, g\}$	$\{f, g\}_D$
quantum	$f \wedge g$	$f_t \wedge g_t$

The evolution equation for the Wigner function has the form (V.11) where the sign of the right side should be changed. The power series expansion of the Moyal bracket over the Poisson operator is truncated at $O(\hbar^2)$, since the Hamiltonian (VI.4) is a fourth degree polynomial of the canonical variables, so we obtain

$$\frac{\partial}{\partial t}W = -\{W, \mathcal{H}_s\} + \frac{\hbar^2}{8} \left(\frac{\partial^3 W}{\partial \phi^\alpha \partial \phi^\beta \partial \pi^\gamma} (2\delta^{\alpha\beta} \phi^\gamma - \delta^{\alpha\gamma} \phi^\beta - \delta^{\beta\gamma} \phi^\alpha) - \frac{\partial^3 W}{\partial \pi^\alpha \partial \pi^\beta \partial \phi^\gamma} (2\delta^{\alpha\beta} \pi^\gamma - \delta^{\alpha\gamma} \pi^\beta - \delta^{\beta\gamma} \pi^\alpha) \right). \quad (\text{VI.5})$$

The first term in the right side is of the classical origin, while the second term represents a quantum correction to the classical Liouville equation and there are no other quantum corrections. Given $W(\xi, 0)$ in the unconstrained phase space, $W(\xi, t)$ can be found by solving the partial differential equation (VI.5).

VII. CONCLUSION

Real functions in the unconstrained phase space of second-class constraints systems split into equivalence classes corresponding to different physical observables. The quantum observables are described by equivalence classes of operators acting on the Hilbert space of states. The Weyl's association rule extends the equivalence relations to functions in the phase space of quantum systems.

The Dirac bracket can be calculated as the Poisson bracket between functions projected onto the constraint submanifold using the phase flows generated by the constraint functions. The quantum deformation of the Dirac bracket is the Moyal bracket calculated for functions projected onto the constraint submanifold of the phase space.

The skew-symmetric multiplication operations are synthesized in Table I.

The operation $f_t \wedge g_t$ designates the quantum deformation of the Dirac bracket consistent with the canonical quantization of constraint systems.

As an application of the general formalism, in Sect. VI we derived with the use of the operation $f_t \wedge g_t$ an evolution equation for the Wigner function of an $n-1$ -dimensional spherical pendulum, which represents a mechanical counterpart of the $O(n)$ non-linear sigma model.

Our final conclusion states that quantum dynamics for constraint systems can be formulated in the unconstrained phase space provided symplectic basis for constraint functions exists globally.

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APPENDIX A: PROPERTIES OF THE SKEW-GRADIENT PROJECTION

Let us apply Eq.(II.7) to a function $f(\xi)$ expanded in a power series in ξ :

$$\begin{aligned}
f_s(\xi) &= \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{\{\dots\{f(\xi), \mathcal{G}^{a_1}\}, \mathcal{G}^{a_2}\}, \dots, \mathcal{G}^{a_k}\}}_k \mathcal{G}_{a_1} \mathcal{G}_{a_2} \dots \mathcal{G}_{a_k} \\
&\stackrel{2}{=} \sum_{p=0}^{\infty} \frac{1}{p!} \frac{\partial^p f(0)}{\partial \xi^{i_1} \dots \partial \xi^{i_p}} \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{\{\dots\{\xi^{i_1} \dots \xi^{i_p}, \mathcal{G}^{a_1}\}, \mathcal{G}^{a_2}\}, \dots, \mathcal{G}^{a_k}\}}_k \mathcal{G}_{a_1} \mathcal{G}_{a_2} \dots \mathcal{G}_{a_k} \\
&\stackrel{3}{=} \sum_{p=0}^{\infty} \frac{1}{p!} \frac{\partial^p f(0)}{\partial \xi^{i_1} \dots \partial \xi^{i_p}} \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{k_1 + \dots + k_p = k} \frac{k!}{k_1! \dots k_p!} \underbrace{\{\dots\{\xi^{i_1}, \mathcal{G}^{a_1}\}, \mathcal{G}^{a_2}\}, \dots, \mathcal{G}^{a_{k_1}}\}}_{k_1} \\
&\quad \dots \underbrace{\{\dots\{\xi^{i_p}, \mathcal{G}^{a_{k-k_p+1}}\}, \mathcal{G}^{a_{k-k_p+2}}\}, \dots, \mathcal{G}^{a_k}\}}_{k_p} \mathcal{G}_{a_1} \mathcal{G}_{a_2} \dots \mathcal{G}_{a_k} \\
&\stackrel{4}{=} \sum_{p=0}^{\infty} \frac{1}{p!} \frac{\partial^p f(0)}{\partial \xi^{i_1} \dots \partial \xi^{i_p}} \sum_{k_1 \geq 0, \dots, k_p \geq 0} \frac{1}{k_1! \dots k_p!} \underbrace{\{\dots\{\xi^{i_1}, \mathcal{G}^{a_1}\}, \mathcal{G}^{a_2}\}, \dots, \mathcal{G}^{a_{k_1}}\}}_{k_1} \\
&\quad \dots \underbrace{\{\dots\{\xi^{i_p}, \mathcal{G}^{a_{k-k_p+1}}\}, \mathcal{G}^{a_{k-k_p+2}}\}, \dots, \mathcal{G}^{a_k}\}}_{k_p} \mathcal{G}_{a_1} \mathcal{G}_{a_2} \dots \mathcal{G}_{a_k} \\
&\stackrel{5}{=} \sum_{p=0}^{\infty} \frac{1}{p!} \frac{\partial^p f(0)}{\partial \xi^{i_1} \dots \partial \xi^{i_p}} \xi_s^{i_1}(\xi) \dots \xi_s^{i_p}(\xi) \\
&= f(\xi_s(\xi)). \tag{A.1}
\end{aligned}$$

In this expression, the Taylor expansion is made first around $\xi = 0$. To reach the step 3 of Eq.(A.1), the multibinomial formula for calculating the Poisson brackets of a product of p functions is used. Going from 3 to 4, the restriction $k_1 + \dots + k_p = k$ is removed. The summation over k_1, \dots, k_p becomes thereby independent. To achieve the step 5 of Eq.(A.1), the summations over k_1, \dots, k_p are performed, which turns ξ into $\xi_s(\xi)$. The final result is given by Eq.(II.8).

The classical counterpart of Eq.(III.6) follows straightforwardly from Eq.(II.8):

$$(f_1 \dots f_p)_s(\xi) = f_{1s}(\xi) \dots f_{ps}(\xi). \tag{A.2}$$

In general, operators do not obey this property. However, under the circumstances specified by Eq.(III.6), this is still valid. Eq.(A.2) implies that 2^p products $\tilde{f}_1(\xi) \dots \tilde{f}_p(\xi)$ where $\tilde{f}_i(\xi) = f_i(\xi)$ or $\tilde{f}_i(\xi) = f_{is}(\xi)$ for $1 \leq i \leq p$ belong to the same equivalence class. Eq.(A.2) shows that class of the projected functions is closed under the pointwise product.

Note other useful relation

$$\{f, g_s\}_s = \{f_s, g_s\}. \tag{A.3}$$

To check it, we apply Eq.(II.7) for $\{f, g_s\}$ and use the Jacoby identity. Since g_s is identically in involution with \mathcal{G}^a , one gets

$$\begin{aligned}
\{f, g_s\}_s &= \sum_{k=0}^{\infty} \frac{1}{k!} \{\dots\{\{f, \mathcal{G}^{a_1}\}, \mathcal{G}^{a_2}\}, \dots, \mathcal{G}^{a_k}\}, g_s\} \mathcal{G}_{a_1} \mathcal{G}_{a_2} \dots \mathcal{G}_{a_k} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \{\dots\{\{f, \mathcal{G}^{a_1}\}, \mathcal{G}^{a_2}\}, \dots, \mathcal{G}^{a_k}\} \mathcal{G}_{a_1} \mathcal{G}_{a_2} \dots \mathcal{G}_{a_k}, g_s\} \\
&= \{f_s, g_s\}.
\end{aligned}$$

In general, however, $\{f, g_s\}_s \neq \{f_s, g_s\}$. To see this, one can set $f = \mathcal{G}^a$ and $g = \mathcal{G}^b$.

The classical analogue of Eq.(III.17) is given by

$$(\{f, g\}_D)_s = \{f_s, g_s\}_D. \quad (\text{A.4})$$

This equation is a consequence of Eqs.(II.15) and, in particular, of the fact that the Leibniz' law applies to the Dirac bracket,

$$\{\{f, g\}_D, \mathcal{G}^a\} = \{\{f, \mathcal{G}^a\}, g\}_D + \{f, \{g, \mathcal{G}^a\}\}_D. \quad (\text{A.5})$$

Using (III.14), we get

$$\begin{aligned} (\{f, g\}_D)_s &= \sum_{k=0}^{\infty} \frac{1}{k!} \{ \dots \{ \{ \{f, g\}_D, \mathcal{G}^{a_1}, \mathcal{G}^{a_2}, \dots, \mathcal{G}^{a_k} \} \mathcal{G}_{a_1} \mathcal{G}_{a_2} \dots \mathcal{G}_{a_k} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{k_1+k_2=k} \frac{k!}{k_1!k_2!} \underbrace{\{ \dots \{ \{f, \mathcal{G}^{a_1}, \mathcal{G}^{a_2}, \dots, \mathcal{G}^{a_{k_1}} \} \}_{k_1}}_{k_1} \underbrace{\{ \dots \{ \{g, \mathcal{G}^{a_{k_1+1}}, \mathcal{G}^{a_{k_1+2}}, \dots, \mathcal{G}^{a_{k_2}} \} \}_{k_2}}_{k_2} \}_{k_1+k_2} \mathcal{G}_{a_1} \mathcal{G}_{a_2} \dots \mathcal{G}_{a_k}. \end{aligned}$$

Taking into account $\{\mathcal{G}^a\}_D = 0$, one can place the last k constraint functions inside of the Dirac bracket. The summation over k can be removed and we obtain

$$\begin{aligned} &\sum_{k_1 \geq 0, k_2 \geq 0} \frac{1}{k_1!k_2!} \underbrace{\{ \dots \{ \{f, \mathcal{G}^{a_1}, \mathcal{G}^{a_2}, \dots, \mathcal{G}^{a_{k_1}} \} \}_{k_1}}_{k_1} \underbrace{\{ \dots \{ \{g, \mathcal{G}^{a_{k_1+1}}, \mathcal{G}^{a_{k_1+2}}, \dots, \mathcal{G}^{a_{k_2}} \} \}_{k_2}}_{k_2} \}_{k_1+k_2} \mathcal{G}_{a_1} \mathcal{G}_{a_2} \dots \mathcal{G}_{a_{k_1+k_2}} \}_{k_1+k_2} \\ &= \{f_s, g_s\}_D. \end{aligned}$$

On the constraint submanifold, according to Eq.(II.15), the subscript can be omitted. In the unconstrained phase space, according to Eqs.(II.14) and the fact that f_s and g_s are identically in involution with the constraint functions, the subscript can be omitted also. Eq.(A.4) shows that class of the projected functions is closed under the Dirac bracket.

APPENDIX B: PROPERTIES OF THE OPERATOR FUNCTION $\tilde{\mathfrak{B}}(\eta)$

In order to check Eqs.(IV.3) - (IV.8), it is useful to derive first the similar properties for $\tilde{\mathfrak{B}}(\eta)$:

$$\tilde{\mathfrak{B}}(\eta)^+ = \tilde{\mathfrak{B}}(-\eta), \quad (\text{B.1})$$

$$Tr[\tilde{\mathfrak{B}}(\eta)] = (2\pi\hbar)^n \delta^{2n}(\eta), \quad (\text{B.2})$$

$$\tilde{\mathfrak{B}}(0) = \mathbf{1}, \quad (\text{B.3})$$

$$\int \frac{d^{2n}\eta}{(2\pi\hbar)^n} \tilde{\mathfrak{B}}(\eta) Tr[\tilde{\mathfrak{B}}(-\eta)f] = f, \quad (\text{B.4})$$

$$Tr[\tilde{\mathfrak{B}}(\eta)\tilde{\mathfrak{B}}(-\eta')] = (2\pi\hbar)^n \delta^{2n}(\eta - \eta'), \quad (\text{B.5})$$

$$\tilde{\mathfrak{B}}(\eta)\tilde{\mathfrak{B}}(-\eta') = \tilde{\mathfrak{B}}(\eta - \eta') \exp\left(-\frac{i}{2\hbar} \eta_k \eta'_l I^{kl}\right). \quad (\text{B.6})$$

Eqs.(B.1) and (B.3) are conspicuous. Eq.(B.6) can be obtained using the identity $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ that holds for operators A and B whose commutator is a c-number. Eqs.(B.2) and (B.5) can be derived by taking into account the explicit form of the matrix elements of $\tilde{\mathfrak{B}}(\eta)$,

$$\langle \phi_1 | \tilde{\mathfrak{B}}(\eta) | \phi_2 \rangle = \delta^n(\phi_1^\gamma - \phi_2^\gamma + \eta_{m+\gamma}) \exp\left(\frac{i}{2\hbar} \sum_{\alpha=1}^n \eta_\alpha (\phi_1^\alpha + \phi_2^\alpha)\right), \quad (\text{B.7})$$

which can be obtained with the help of equation

$$\exp\left(\frac{i}{\hbar} \sum_{\alpha=1}^n \eta_{m+\alpha} x^{n+\alpha}\right) |\phi^\gamma \rangle = |\phi^\gamma - \eta_{m+\gamma} \rangle.$$

Using Eq.(B.7) one gets

$$\int \frac{d^{2n}\eta}{(2\pi\hbar)^n} \langle \phi_1 | \tilde{\mathfrak{B}}(\eta) | \phi_2 \rangle \langle \phi_3 | \tilde{\mathfrak{B}}(-\eta) | \phi_4 \rangle = \delta^n(\phi_1^\alpha - \phi_4^\alpha) \delta^n(\phi_2^\alpha - \phi_3^\alpha). \quad (\text{B.8})$$

Since Eq.(B.4) is valid for any operator, Eqs.(B.4) and (B.8) are equivalent.

Applying the Fourier transform to $\tilde{\mathfrak{B}}(\eta)$, entering Eqs.(B.1) - (B.6), one gets Eqs.(IV.3) - (IV.8).

Let us multiply Eq.(IV.5) by $\mathfrak{f}\mathfrak{g}$, take the trace, and use Eq.(IV.12). As a consequence, we obtain

$$\text{Tr}[\mathfrak{f}\mathfrak{g}] = \int \frac{d^{2n}\xi}{(2\pi\hbar)^n} f(\xi) \circ g(\xi). \quad (\text{B.9})$$

The symmetrized star-product can be replaced with the pointwise product. Indeed,

$$\begin{aligned} f(\xi) \circ g(\xi) &= \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p)!} \left(\frac{\hbar}{2}\right)^{2p} I^{i_1 j_1} \dots I^{i_{2p} j_{2p}} \frac{\partial^{2p} f(\xi)}{\partial \xi^{i_1} \dots \partial \xi^{i_{2p}}} \frac{\partial^{2p} g(\xi)}{\partial \xi^{j_1} \dots \partial \xi^{j_{2p}}} \\ &= f(\xi)g(\xi) + \sum_{p=1}^{\infty} \frac{(-1)^p}{(2p)!} \left(\frac{\hbar}{2}\right)^{2p} I^{i_1 j_1} \dots I^{i_{2p} j_{2p}} \frac{\partial^{2p}}{\partial \xi^{i_1} \dots \partial \xi^{i_{2p}}} \left(f(\xi) \frac{\partial^{2p} g(\xi)}{\partial \xi^{j_1} \dots \partial \xi^{j_{2p}}} \right). \end{aligned} \quad (\text{B.10})$$

The quantum corrections $O(\hbar^{2p})$ which could make the variance disappear, since they represent full derivatives and contribute to the surface integral only. By multiplying Eq.(IV.6) with \mathfrak{g} and taking the trace, one arrives at the same conclusion.

APPENDIX C: LOWEST ORDER QUANTUM CORRECTION TO THE DIRAC BRACKET

The series expansion over the Planck's constant is straightforward, so we restrict ourselves with the lowest order correction $O(\hbar^2)$.

The quantum deformation of the Dirac bracket is associated to the operation

$$g_t(\xi) \wedge f_t(\xi) = (g(\xi) \wedge f_t(\xi))_t. \quad (\text{C.1})$$

Using Eq.(V.7), one can write

$$\begin{aligned} g_t(\xi) \wedge f_t(\xi) &= g(\xi) \wedge f_t(\xi) - \frac{\hbar^2}{8} \{ \{g(\xi), f_s(\xi)\}, G^{a_1} \} \mathcal{P}^2 G_{a_1} \\ &\quad - \frac{\hbar^2}{8} \frac{1}{2!} \{ \{ \{g(\xi), f_s(\xi)\}, G^{a_1} \}, G^{a_2} \} G_{a_1} \mathcal{P}^2 G_{a_2} \\ &\quad - \frac{\hbar^2}{8} \frac{1}{3!} \{ \{ \{ \{g(\xi), f_s(\xi)\}, G^{a_1} \}, G^{a_2} \}, G^{a_3} \} G_{a_1} G_{a_2} \mathcal{P}^2 G_{a_3} + O(\hbar^4). \end{aligned} \quad (\text{C.2})$$

In terms $O(\hbar^2)$, the Moyal bracket is replaced with the Poisson bracket and $f_t(\xi)$ is replaced with $f_s(\xi)$. The arguments similar to that used for the series expansion (V.7) allow to truncate the series expansion over G_a , entering f_s :

$$\begin{aligned} g_t(\xi) \wedge f_t(\xi) &= g(\xi) \wedge f_t(\xi) \\ &\quad - \frac{\hbar^2}{8} \{ \{g(\xi), G^{a_1}\}, f(\xi) + \{f(\xi), G^{a_2}\} G_{a_2} + \frac{1}{2!} \{ \{f(\xi), G^{a_2}\}, G^{a_3} \} G_{a_2} G_{a_3} \\ &\quad + \frac{1}{3!} \{ \{ \{f(\xi), G^{a_2}\}, G^{a_3} \} G^{a_4} \} G_{a_2} G_{a_3} G_{a_4} \} \mathcal{P}^2 G_{a_1} \\ &\quad - \frac{\hbar^2}{8} \frac{1}{2!} \{ \{ \{g(\xi), G^{a_1}\}, G^{a_2} \}, f(\xi) + \{f(\xi), G^{a_3}\} G_{a_3} + \frac{1}{2!} \{ \{f(\xi), G^{a_3}\}, G^{a_4} \} G_{a_3} G_{a_4} \} G_{a_1} \mathcal{P}^2 G_{a_2} \\ &\quad - \frac{\hbar^2}{8} \frac{1}{3!} \{ \{ \{ \{g(\xi), G^{a_1}\}, G^{a_2} \}, G^{a_3} \}, f(\xi) + \{f(\xi), G^{a_4}\} G_{a_4} \} G_{a_1} G_{a_2} \mathcal{P}^2 G_{a_3} + O(\hbar^4). \end{aligned} \quad (\text{C.3})$$

Let us turn to the first term of Eq.(C.3). We have

$$\begin{aligned}
g(\xi) \wedge f_t(\xi) &= (g(\xi) \wedge f(\xi))_t \\
&+ \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{l=1}^k (\dots((\dots((f(\xi) \wedge G^{a_1}) \wedge G^{a_2}) \dots \wedge G^{a_{l-1}}) \wedge (g(\xi) \wedge G^{a_l})) \wedge G^{a_{l+1}}) \dots \wedge G^{a_k}) \\
&\circ G_{a_1} \circ G_{a_2} \dots \circ G_{a_k} + \\
&+ \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{l=1}^k (\dots((f(\xi) \wedge G^{a_1}) \wedge G^{a_2}) \dots \wedge G^{a_k}) \\
&\circ G_{a_1} \circ G_{a_2} \dots \circ G_{a_{l-1}} \circ (g(\xi) \wedge G_{a_l}) \circ G_{a_{l+1}} \dots \circ G_{a_k}.
\end{aligned} \tag{C.4}$$

The first term can be expanded using Eq.(V.7). In the second term, the Moyal bracket can be replaced with the Poisson bracket. The symmetrized star-product \circ is treated as in Eq.(V.7). The series over k to order $O(\hbar^2)$ is truncated at $k = 3$ again. The third term is truncated at $k = 4$. The Dirac bracket originates, to the zeroth order in \hbar , from the first term with $k = 0$ and the third term with $k = 1$ in Eq.(C.4). The result takes the form

$$\begin{aligned}
g(\xi) \wedge f_t(\xi) &= \{g(\xi), f(\xi)\}_D \\
&- \frac{\hbar^2}{24} g(\xi) \mathcal{P}^3 f(\xi) \\
&- \frac{\hbar^2}{8} \{\{f(\xi), g(\xi)\}, G^{a_1}\} \mathcal{P}^2 G_{a_1} - \frac{\hbar^2}{8} \frac{1}{2!} \{\{\{f(\xi), g(\xi)\}, G^{a_1}\}, G^{a_2}\} G_{a_1} \mathcal{P}^2 G_{a_2} \\
&- \frac{\hbar^2}{8} \frac{1}{3!} \{\{\{\{f(\xi), g(\xi)\}, G^{a_1}\}, G^{a_2}\}, G^{a_3}\} G_{a_1} G_{a_2} \mathcal{P}^2 G_{a_3} \\
&- \frac{\hbar^2}{24} (f(\xi) \mathcal{P}^3 G^{a_1}) \{g(\xi), G_{a_1}\} - \frac{\hbar^2}{24} \{f(\xi), G^{a_1}\} (g(\xi) \mathcal{P}^3 G_{a_1}) - \frac{\hbar^2}{8} \{f(\xi), G^{a_1}\} \mathcal{P}^2 \{g(\xi), G_{a_1}\} \\
&- \frac{\hbar^2}{8} \{f(\xi), \{g(\xi), G^{a_1}\}\} \mathcal{P}^2 G_{a_1} \\
&- \frac{\hbar^2}{8} \frac{1}{2!} \{\{f(\xi), \{g(\xi), G^{a_1}\}\}, G^{a_2}\} G_{a_1} \mathcal{P}^2 G_{a_2} \\
&- \frac{\hbar^2}{8} \frac{1}{2!} \{\{f(\xi), G^{a_1}\}, \{g(\xi), G^{a_2}\}\} G_{a_1} \mathcal{P}^2 G_{a_2} \\
&- \frac{\hbar^2}{8} \frac{1}{3!} \{\{\{f(\xi), \{g(\xi), G^{a_1}\}\}, G^{a_2}\}, G^{a_3}\} G_{a_1} G_{a_2} \mathcal{P}^2 G_{a_3} \\
&- \frac{\hbar^2}{8} \frac{1}{3!} \{\{\{f(\xi), G^{a_1}\}, \{g(\xi), G^{a_2}\}\}, G^{a_3}\} G_{a_1} G_{a_2} \mathcal{P}^2 G_{a_3} \\
&- \frac{\hbar^2}{8} \frac{1}{3!} \{\{\{f(\xi), G^{a_1}\}, G^{a_2}\}, \{g(\xi), G^{a_3}\}\} G_{a_1} G_{a_2} \mathcal{P}^2 G_{a_3} \\
&- \frac{\hbar^2}{8} \frac{1}{2!} (\{\{f(\xi), G^{a_1}\}, G^{a_2}\} \mathcal{P}^2 G_{a_1}) \{g(\xi), G_{a_2}\} \\
&- \frac{\hbar^2}{8} \frac{1}{2!} \{\{f(\xi), G^{a_1}\}, G^{a_2}\} G_{a_1} \mathcal{P}^2 \{g(\xi), G_{a_2}\} \\
&- \frac{\hbar^2}{8} \frac{1}{2!} \{\{f(\xi), G^{a_1}\}, G^{a_2}\} \{g(\xi), G_{a_1}\} \mathcal{P}^2 G_{a_2} \\
&- \frac{\hbar^2}{8} \frac{1}{3!} (\{\{\{f(\xi), G^{a_1}\}, G^{a_2}\}, G^{a_3}\} G_{a_1} \mathcal{P}^2 G_{a_2}) \{g(\xi), G_{a_3}\} \\
&- \frac{\hbar^2}{8} \frac{1}{3!} \{\{\{f(\xi), G^{a_1}\}, G^{a_2}\}, G^{a_3}\} G_{a_1} G_{a_2} \mathcal{P}^2 \{g(\xi), G_{a_3}\} \\
&- \frac{\hbar^2}{8} \frac{2}{3!} \{\{\{f(\xi), G^{a_1}\}, G^{a_2}\}, G^{a_3}\} \{g(\xi), G_{a_1}\} G_{a_2} \mathcal{P}^2 G_{a_3} \\
&- \frac{\hbar^2}{8} \frac{4}{4!} \{\{\{\{f(\xi), G^{a_1}\}, G^{a_2}\}, G^{a_3}\}, G^{a_4}\} \{g(\xi), G_{a_1}\} G_{a_2} G_{a_3} \mathcal{P}^2 G_{a_4} + O(\hbar^4)
\end{aligned} \tag{C.5}$$

In the first line, we recover the Dirac bracket (II.14). The second term is the quantum correction to the Poisson

bracket. The third and fourth lines originate from the expansion of the first term in Eq.(C.4) as prescribed by Eq.(V.7). The fifth line comes from the expansion of the $k = 1$ component of the third term in Eq.(C.4). The lines 6 - 11 appear from the expansion of the second term in Eq.(C.4). The rest appears from the components $k > 1$ of the third term in Eq.(C.4). The round brackets unify terms within the action of the operation \mathcal{P}^2 . If the round brackets are suppressed (in the most cases), \mathcal{P}^2 acts on all terms.

Combining Eqs.(C.3) and (C.5), we get the Dirac bracket on the constraint submanifold to the order $O(\hbar^2)$.

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