A Large N expansion for Gravity

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Abstract

A Large N expansion for gravity is proposed. The scheme is based on the splitting of the Einstein-Hilbert action into the BF topological action plus a constraint. The method also allows to include matter fields. The relation between matter and non orientable fat graphs in the expansion is stressed; the special role of scalars is shortly discussed. The connections with the Holographic Principle and higher spin fields are analyzed.

Key words: Large N expansion, Einstein-Hilbert action, Holographic Principle. *PACS:* 11.15.Pg, 04.90.+e, 04.50.+h, 11.25.-w.

1 Introduction

Large N-expansion for a SU(N) Gauge Theory, introduced by 't Hooft in [22] [23], is indeed one of the most powerful non perturbative techniques available to investigate non linear gauge theories. In particular, even if large N SU(N)Gauge Theory has not been solved yet, the large N- expansion provides the issues of confinement, chiral symmetry breaking and the relation with string theory with a rather detailed understanding. The Veneziano limit [26], which corresponds to a 't Hooft limit in which the ratio N/N_f is kept fixed (N_f being the number of quarks flavours), shed further light on non perturbative features of large N SU(N) Gauge Theory clarifying several features of quarks and mesons dynamics; while a clear analysis of the role of Baryons at large N (which, in many respects, behave as solitons) has been provided in [27] (see, for two detailed pedagogical reviews, [17]). It is also worth to mention that (several types of) large N-expansions played a prominent role to provide the

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Holographic Principle with further supports and practical realizations (see, for a detailed review, [1]). For the above reasons, it would be very important to have at one's disposal similar techniques to investigate the non perturbative features of Einstein-Hilbert action. Indeed, *General Relativity* bears a strong resemblance with Gauge Theory because of the several common geometrical structures (such as the curvature tensor "measuring" the deviation from flatness, the "minimal coupling rule" which allows to rewrite Lorentz invariant equations in the presence of a non trivial gravitational field by substituting ordinary derivatives with covariant derivatives and so on). On the other hand, there are many striking differences which make General Relativity and Gauge Theory very unlike theories: first of all, while Gauge Theory is a theory defined on a fixed background space-time, General Relativity determines the dynamics of the background spacetime itself; the actions of General Relativity and Gauge Theory are different and General Relativity is not perturbatively renormalizable in the ordinary field theoretical sense. There are many others technical differences, here it is worth to mention that in Gauge Theory there is a clear separation between internal and space-time symmetries (and this allows to consider the large N-expansion by increasing the internal symmetry group while keeping fixed the space-time structure), for the above reasons in General Relativity this is not the case and it is not obvious what a large **N**-expansion in gravity could be.

An interesting attempt to obtain a similar expansion in gravity has been performed in [24] and [21] and further refined in [3]: in these papers the authors argued that the role of the internal index \mathbf{N} on the Gauge Theory's side could be played on the General Relativity's side by the number of space-time dimensions: in other words, the authors proposed an expansion in which the small parameter is 1/D where D is the number of space-time dimensions. The result of their analysis is that in a large D expansion of the Einstein-Hilbert action, unlike the large **N**-expansion of Gauge Theory in which all the planar diagrams are on equal footing, a subclass of the full set of planar diagrams dominates. Unfortunately, their analysis is affected by some technical problems which are absent in the Gauge Theory case. In particular, in a large Dexpansion, the power of D in a given diagram of the expansion is determined not only by the topology of the two dimensional surface on which such a diagram can be drawn (as it happens in the Gauge Theory case): a prominent role is also played by space-time and/or momentum integrals which also depend on D and a complete analysis of the contribution of the above integrals to the D-dependence of any diagram is a rather hopeless task [21] [3]. Another technical problem related to the previous one is the fact that, in a large D expansion, it is necessary to consider gravitational fluctuations around the trivial flat solution: the reason is that it is not possible to disentangle the space-time dependence due to space-time and/or momentum integrals from the "internal indices" dependence of any diagram since these two different kinds of structures overlap in the General Relativity case. If it would be possible to single out suitable "internal indices" in General Relativity then one could achieve an expansion whose topological character can be analyzed independently on the background: a non trivial background would only affect the space-time and/or momentum integrals of the diagrams while the internal index structure (which plays the prominent role in the large \mathbf{N} limit) would not change.

Here a scheme is proposed to overcome the above difficulties of the large D expansion which is based on the BF-theoretical formulation of Einstein-Hilbert action in which the General Relativity action is splitted into a topological term plus a constraint (the analysis of the relations between General Relativity and BF theory has a long history: see, for example, [7] [8] [12] [19] [9] and references therein): this way of writing the General Relativity action allows to distinguish internal index structure from space-time index structure and is strictly related to the connection formulation(s) of General Relativity which played a prominent role in the formulation of Loop Quantum Gravity (for a detailed review see, for example, [2]). Moreover, it is possible to make an interesting comparison between the BF formulation of General Relativity and Yang-Mills theory introduced in [15].

The paper is organized as follows: in the second section the formulation of General Relativity as a constrained BF theory is shortly described. In the third section a suitable "large \mathbf{N} 't Hooft expansion" on the internal indices is introduced. In the fourth section the inclusion of matter and a "Veneziano-like" limit are discussed. In the fifth section, the General Relativity and the Gauge Theory expansions are compared and the connections with the Holographic Principle and higher spin fields are analyzed. Eventually, the conclusions are drawn.

2 BF-theory and gravity

In this section the formulation of General Relativity as a constrained BF theory will be shortly described along the lines of [7] [8] [12] [19] [9].

BF theories are topological theories, that is, they exhibit the following properties: firstly, they are defined without any reference to a background metric and, moreover, they have no local degrees of freedom. Here, only the four dimensional case will be considered. The BF theory in four dimensions is defined

by the following action

$$S[A,B] = \int_{M} B^{IJ} \wedge F_{IJ}(A) = \frac{1}{4} \int_{M} \varepsilon^{\alpha\beta\gamma\delta} B^{IJ}_{\alpha\beta} F_{\gamma\delta IJ} d^{4}x \tag{1}$$

$$B^{IJ} = \frac{1}{2} B^{IJ}_{\alpha\beta} dx^{\alpha} \wedge dx^{\beta}, \quad F_{IJ} = \frac{1}{2} F_{\alpha\beta IJ} dx^{\alpha} \wedge dx^{\beta}$$
$$F_{\alpha\beta IJ} = (\partial_{\alpha} A_{\beta} - \partial_{\alpha} A_{\beta})_{IJ} + A^{L}_{\alpha I} A_{\beta LJ} - A^{L}_{\beta I} A_{\alpha LJ}, \quad (2)$$

where M is the four-dimensional space-time, the greek letters denote spacetimes indices, $\varepsilon^{\alpha\beta\gamma\delta}$ is the totally skew-symmetric Levi-Civita symbol in fourdimensional space-times, I, J and K are Lorentz (internal) indices which are raised and lowered with the Minkowski metric η_{IJ} : I, J = 1, ..., D. Thus, the basic fields are a so(D - 1, 1)-valued two form B_{IJ} and a so(D - 1, 1)connection one form $A_{\alpha LJ}$, the internal gauge group being SO(D - 1, 1). Also the Riemannian theory can be considered in which the internal gauge group is SO(D) and the internal indices are raised and lowered with the euclidean metric δ_{IJ} ; in any case, both B_{IJ} and $A_{\alpha LJ}$ are in the adjoint representation of the (algebra of the) internal gauge group: this simple observation will be important in order to develop a 't Hooft like large **N** expansion. The equations of motion are

$$F = 0, \quad \nabla_A B = 0 \tag{3}$$

where ∇_A is the covariant derivative with respect to the connection $A_{\alpha LJ}$. The above equations tell that $A_{\alpha LJ}$ is, locally, a pure gauge and B^{IJ} is covariantly constant. Obviously, the *BF* action does not describe the dynamics of general relativity which, indeed, has local degrees of freedom. On the other hand, if B^{IJ} would have this form

$$B^{IJ} = \frac{1}{2} \varepsilon^{IJ}_{KL} e^K \wedge e^L \tag{4}$$

then the action (1) would be nothing but the Palatini form of the generalized Einstein-Hilbert action (obviously, standard General Relativity is recovered when D = 4). It turns out that eq. (4) can be enforced by adding to the action (1) a suitable constraint; thus, the basic action in the BF formalism is

$$\kappa S_{GR} = S\left[A,B\right] - \int_{M} \left(\phi_{IJKL}B^{IJ} \wedge B^{KL} + \mu H\left(\phi\right)\right) \tag{5}$$

where κ is the gravitational coupling constant, μ is a four-form and $H(\phi)$ is a scalar which can have the following three expressions:

$$H_1 = \phi_{IJ}^{IJ}, \quad H_2 = \phi_{IJKL} \varepsilon^{IJKL}, \quad H_3 = a_1 H_1 + a_2 H_2,$$
 (6)

where a_i are real constants. It is worth to note here that the scalar ϕ takes value in the tensor product of the adjoint representation of so(D-1,1) (or of so(D) with itself. The form (5) of the Einstein-Hilbert action will be the starting point of the "gravitational" large **N** expansion.

3 A gravitational large N expansion

In this section, a 't Hooft like expansion based on the form (5) of the Einstein-Hilbert action is proposed: this scheme allows an interesting comparison with the BF formulation of Yang-Mills theory introduced in [15].

The first question which should be answered in order to set in a suitable framework for a gravitational large **N** expansion is: who is **N** in the gravitational case? The suggestion of [21] [3] is that **N** should be identified with the number of space-time dimensions D. Indeed, the basic variables of the BF-like formulation (and, in general, of any connection formulation of General Relativity) have the great merit to manifest a natural separation between space-time and internal indices. It becomes possible to consider a limit in which the space-time dimensionality is fixed while the dimension of the internal symmetry group approaches to infinity. In the Gauge Theory case, the gluonic fields are in the adjoint of $\mathbf{U}(\mathbf{N})$ (which can be thought as the tensor product of the fundamental and the anti-fundamental representations) while the quarks in the fundamental. It is then possible to introduce the celebrated 't Hooft double line notation in which it is easy to show that, in a generic diagrams, to every closed color line corresponds a factor of \mathbf{N} which is nothing but the dimension of the fundamental representation of $\mathbf{U}(\mathbf{N})$.

In the General Relativity case, two of the basic fields (A and B) are p-forms taking values in the adjoint representation of so(D-1,1): in other words, in the internal space, A and B are real $D \times D$ skew-symmetric matrices carrying, therefore, two vectorial indices running from 1 to D. While the Lagrangian multiplier ϕ carries four vectorial indices running from 1 to D (this fact will play a role in the following). The above considerations clarify that, in the gravitational case, \mathbf{N}^{-1} should be identified with 1/D:

$$\mathbf{N}^{-1} = D^{-1}; \tag{7}$$

however, in order to avoid confusion with the notation of [21] [3] and with the letter which is often used to denote the number of space-time dimensions (which, in fact, in this approach is kept fixed), the left hand side of eq. (7) will be used to denote the expansion parameter. An important benefit of the connection formulation is that the double line notation can be fairly adopted: the only difference with respect to the Gauge Theory case is that, being the fundamental representation of so(N - 1, 1) real, the lines of internal indices carry no arrow 1 .

In order to provide the large \mathbf{N} expansion with the usual topological classification one should write down the Feynman rules for the Einstein-Hilbert action (5) plus the gauge-fixing and ghost terms. As far as a large \mathbf{N} expansion is concerned, the ghost terms are not important since they do not influence the topological character of the expansion and the topological classification of the diagrams (see, for example, [17]). In this case it is convenient to find the Feynman rules along the lines of [18] where the authors considered the case of the *BF* formulation of Yang-Mills theory: in this way it will be easier to make a comparison between the large \mathbf{N} expansion in General Relativity and Gauge Theory. Thus, the starting point is the action

$$S_{GR} = \frac{1}{\kappa} \left[S\left[A, B\right] - \frac{c_2}{2} \int\limits_M \left(\phi_{IJKL} B^{IJ} \wedge B^{KL} + \mu H\left(\phi\right) \right) \right]$$
(8)

where κ is the gravitational coupling constant, the real constant c_2 keeps track of the terms which distinguish *BF*-theory from General Relativity. The natural choice is to consider as the Gaussian part the off-diagonal kinetic term

$$S_0 = \frac{1}{\kappa} \int_M \left(\varepsilon^{\alpha\beta\gamma\delta} B^{IJ}_{\alpha\beta} \partial_\gamma A_{\delta IJ} \right), \tag{9}$$

in such a way that the $A \to B$ propagator (which propagates A_{μ} into $B_{\nu\gamma}$) has the following structure (see fig. (1)):

$$\Delta_{(A,B)\mu\nu\gamma}^{(IJ,KL)} = \delta^{IL} \delta^{JK} F_1(p)_{\mu\nu\gamma} \tag{10}$$

$$F_1(p)_{\mu\nu\gamma} = -\frac{1}{2}\varepsilon_{\mu\nu\gamma\alpha}\frac{p^{\alpha}}{p^2}$$
(11)

where $F_1(p)_{\mu\nu\gamma}$ (which is the space-time-momentum dependent part of the propagator²) is not relevant as far as the large **N** expansion is concerned. The internal index structures of the $A \to A$ propagator and of the $B \to B$

¹ The point is that in the Gauge Theory case one has to distinguish the fundamental and the anti-fundamental representations of (the algebra of) U(N): this is achieved by adding to the color lines incoming or outgoing arrows.

 $^{^2}$ $F_1(p)_{\mu\nu\gamma}$ has a form similar to the $A \to B$ propagator in [18] in the usual Feynman gauge; the procedure to find the explicit expression of $F_1(p)_{\mu\nu\gamma}$ is analogous to the procedure of [18] and [10] but, as far as the purposes of the present paper are concerned, is not relevant since here only the internal index structure is needed.

Fig. 1. As in standard gauge theories, the structure of the propagator implies that internal indices are conserved along the internal lines.

propagator are analogous to the expressions found in [18]:

$$\Delta_{(A,A)\mu\nu}^{(IJ,KL)} = \delta^{IL} \delta^{JK} F_2(p)_{\mu\nu}, \quad \Delta_{(B,B)\mu\nu\gamma\rho}^{(IJ,KL)} = \delta^{IL} \delta^{JK} F_3(p)_{\mu\nu\gamma\rho} \tag{12}$$

$$F_2(p)_{\mu\nu} = \frac{1}{p^2} (\delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}) + \alpha_1 \frac{p^\mu p^\nu}{p^4}$$
(13)

$$F_3(p)_{\mu\nu\gamma\rho} = -\alpha_2 \frac{p^\mu p^\nu}{p^4} \tag{14}$$

where F_2 and F_3^3 (which are the space-time-momentum dependent parts of the propagators) will not be relevant as far as the large **N** expansion is concerned and α_1 and α_2 are real gauge parameters. Actually, even if F_2 and/or F_3 would vanish, the topological classification of the fat graphs and the large **N** expansion would not change: it is only important to note that, in a double-line notation, the internal index is conserved along the "color" lines.

A remark is in order here: in the BF Yang-Mills case the action can be written as follows

$$S_{BFYM} = S[A, B] - g^2 \int_M B^{IJ} \wedge *B_{IJ}$$
(15)

where * is the Hodge dual. The second term on the right hand side can be considered both as a part of the kinetic Lagrangian and as a true vertex. The

^{$\overline{3}$} A procedure to deduce them can be found, for example, in [18] and [10].

first choice gives rise to standard quantization procedure via gauge fixing and ghost terms. The second choice instead would be rather involved because one should gauge fix also the topological symmetry of the BF action and this would require a "ghost of ghost" structure. In fact, in the General Relativity case, there is no choice, the only available possibility is the (more involved) second one: in this case the quantization procedure to follow can be found in [10]. The point is that the quadratic term of the BF Yang-Mills case is replaced by a (rather unusual) cubic interaction term. However, as it has been already remarked, from a large **N** perspective the ghost terms are not relevant ⁴.

Unlike the already mentioned Yang-Mills case, the theory has two vertices:

$$V_1(A^a_\mu, A^b_\nu, B^c_{\alpha\beta}) = \frac{g_3}{3} f^{abc} \varepsilon_{\mu\nu\alpha\beta}, \quad V_2(B^a_{\mu\nu}, B^b_{\alpha\beta}, \phi^{cd}) = \frac{c_2}{2} \delta_{ac} \delta_{bd} \varepsilon^{\mu\nu\alpha\beta}$$
(16)

where a, b, c and d are internal indices in the adjoint representation, f^{abc} are the structure constants, g_3 (which can be assumed to be positive) and c_2 are adimensional coupling constants which keep track of the vertices in the large **N** counting and the reason of the seemingly strange normalization of g_3 and c_2 in the above equation will be clarified in the next subsection (see fig. (2) and fig. (3)). The second vertex is also present in the Gauge Theory case while the first one pertains to General Relativity only. In a large **N** perspective, the more convenient way to look at the Lagrange multiplier field ϕ is to consider it as a propagating field with a very high mass: this point of view allows an interesting interpretation of its physical role as it will be shown in the last section. The last term on the right hand side of eq. (8) (defined in eq. (6)) should not be considered as true vertex: it enforces some restrictions⁵ on the internal index structure of ϕ and will not be relevant as far as the large **N** counting is concerned.

Eventually, the last point to be clarified before to carry on the large N limit is the scaling of the coupling constant κ : in the large N limit, it is natural to assume that the product

$$\gamma_e = \mathbf{N}\kappa,\tag{17}$$

which plays the role of effective coupling constant, as fixed.

 $[\]overline{}^{4}$ The reason is that, from an "internal lines" perspective, they add no new vertices: in other words, the ghost vertices have the same internal index structures and coupling constants of the physical vertices (see, for example, [17]).

⁵ Such restrictions become less and less important at large **N**: to see this, it is enough to note that these restrictions imply that one of the component of ϕ can be expressed in terms of the others. When **N** is large (since the number of components of ϕ grows as **N**⁴) this fact is not relevant.

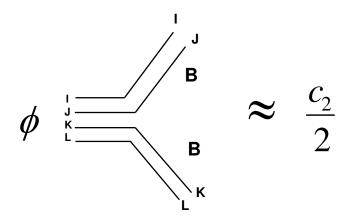


Fig. 2. This vertex (in which a field represented by four internal lines appears) only pertains to General Relativity. Such a field plays a very peculiar role as it will be shown in the following.

3.1 The General Relativity 't Hooft limit

Now it is possible to perform the 't Hooft limit: the main object of interest is the free energy so that only fat graphs with no external legs will be considered.

However, before to proceed, a point needs to be clarified. The vertex V_2 in eq. (16) behaves in some sense (actually, as it will be explained in the last section, there is an important difference with a non trivial physical interpretation) as an effective quadruple vertex for B (see fig. (4)): the point is that the field ϕ is only coupled to B so that, in closed graphs with no external legs the vertices V_2 are always in pairs. Thus, the effective coupling constant g_4 of the quadruple vertex is

$$g_4 = (c_2)^2 \,, \tag{18}$$

while the coupling constant of the cubic vertex is simply g_3 .

At last, it is possible to apply the standard "large **N** counting rules" for fat graphs (see, for two detailed pedagogical reviews, [13]). These counting rules can be deduced by using *matrix models*; it is usually convenient to choice the coupling constant of n-uple vertex by dividing it by n [13]: this is the

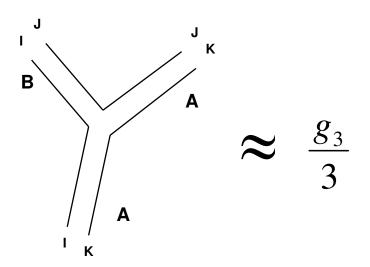


Fig. 3. This vertex is similar to the BF Yang-Mills vertex.

reason behind the normalization in eq. (16). Thus, the usual matrix models techniques [13] tell that a generic fat graph Γ with no external legs will have the following dependence on **N** and on the coupling constants:

$$W_{\Gamma}(E, V, n_p) = \kappa^{E-V} \mathbf{N}^F \prod_p g_p^{n_p}, \quad \sum_p n_p = V$$
(19)

where E is the number of propagators, n_p is the number of p-uple vertices in the graph Γ (in this case the only vertices to be counted are the ones with p = 3; the physical role of the effective quadruple vertex will be analyzed in the fifth section) and F is the number of faces of the fat graph. In this purely gravitational case with no matter fields with one internal index, one has

$$F = h \tag{20}$$

where h is number of closed "color" loops of Γ .

By using eq. (20) and the well known Euler formula

$$2\overline{g} - 2 = E - V - F,\tag{21}$$

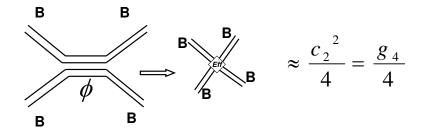


Fig. 4. The vertex with ϕ gives rise to an effective quadruple vertex for *B*. However, being ϕ represented by four internal lines, this quadruple vertex is not on equal footing with standard gauge theoretical vertices. This fact has a holographic interpretation.

where \overline{g} is the least genus⁶ of a Riemann surface on which the fat graph Γ can be drawn without intersecting lines, the weight W_{Γ} of the fat graph turns out to be

$$W_{\Gamma}(E,V,n_p) = \kappa^{2\overline{g}-2} \gamma_e^h \prod_p g_p^{n_p}$$
(22)

where the effective coupling constant γ_e has been introduced in eq. (17).

At a first glance, this result gives rise to the usual topological expansion for the free energy F, similar to the one of the Gauge Theory case, as a sum of the above factor in eq. (22) times a suitable space-time-momentum factor F_{Γ} over the closed connected fat graphs

⁶ There is a subtlety here in the definition of genus \overline{g} (this point will be discussed in the next sections) related to the fact that the fundamental representation of the internal gauge group is real. For this reason the notation \overline{g} (instead of the usual one g) will be adopted.

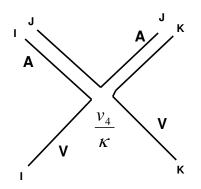


Fig. 5. This is an example of a planar graph with eight internal index loops and twelve triple vertices.

$$F = \sum_{\substack{\Gamma closed\\connected}} W_{\Gamma}(E, V, n_p) F_{\Gamma}$$

in which the leading term in the genus expansion is the planar one and the corrections in the topological expansion are suppressed as powers of $1/N^2$. In fact, there are interesting differences, related both to the gauge group and to the vertices, which will be analyzed in the next sections.

4 The inclusion of matter and the Veneziano limit

In this section the inclusion of matter in the gravitational 't Hooft limit and the Veneziano limit will be discussed.

Once the purely gravitational 't Hooft limit has been introduced, the inclusion of matter is the natural further step. However, the situation is less clear than in the Gauge Theory case. Vectors, in the standard metric formalism, are coupled to gravity via the Levi-Civita covariant derivative which, of course, acts on its vectorial index. Therefore, in this scheme, vectors are represented as scalar particles with an internal index J running from 1 to **N**

$$V_{\mu} \rightarrow V_J.$$

and should be coupled to the gravitational connection A by terms Υ_i as (see

fig. (6) and fig. (7))

$$\Upsilon_4 \sim \frac{v_4}{4\kappa} A^{IJ}_{\mu} V_J A^{\mu}_{IL} V^L, \quad \Upsilon_3 \sim \frac{v_3}{3\kappa} \left(p_{\mu} V^J \right) A^{\mu}_{JL} V^L \tag{23}$$

where v_i are coupling constants normalized in a suitable way to take advantage of the (already mentioned) large **N** counting techniques [13]. The above vertices could come from, for example, a kinetic term of the form

$$\nabla_A V^J \nabla_A V_J$$

where ∇_A is the covariant derivative of the connection A.

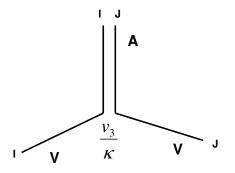


Fig. 6. This is a triple vertex with two matter fields and a gravitational connection.

For spinors the situation is more involved: the covariant derivative ∇_{γ} on a generic spinor Ψ in the standard metric formalism reads

$$i(\gamma^{\mu}(x)\partial_{\mu} - \gamma^{\mu}(x)\Gamma_{\mu})\Psi = \nabla_{\gamma}\Psi,$$

$$\gamma^{\mu}(x) = e_{K}^{\mu}\gamma^{K}$$

where e_K^{μ} are the vierbeins, Γ_{μ} is the spinorial Levi-Civita connection (in which, besides the connection A, also the vierbeins enter: see, for example, [11]) and γ^K are the standard flat Dirac matrices. The problem is that in the BF formulation of General Relativity the vierbeins do not appear directly: the fundamental field in the first order BF formalism is B which, actually, is the exterior product of two vierbeins. Thus, it is not clear how to construct interaction vertices with spinors. For this reason, here it will be considered only the contribution of vectors. It is worth to stress here that, in any case, the coupling terms with spinor should not be very different from Υ_3 and Υ_4 which,

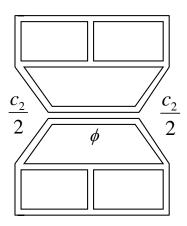


Fig. 7. This is a quadruple vertex with two matter fields and two gravitational connections.

therefore, provide the role of matter at large **N** with a detailed description. This can be argued as follows: a spinor is always accompanied by a flat Dirac matrix and by a vierbein and in a spinor current there are always two spinors. Hence, in a spinor current, there are always two vierbeins which, from an internal index perspective, are similar to B and, by the way, B has the same internal index structure of A. Consequently, as far as a large **N** counting is concerned, a spinorial vertex is well described by the vertex Υ_4 in eq. (23).

However, there is an apparent difficulty in dealing with scalar particles. Ordinary matter couples to the gravitational connection A through a vectorial internal index. On the other hand, at a first glance scalars do not couple to the gravitational connection A since, on them, covariant derivatives coincide with ordinary derivatives. This difficulty is very similar to the difficulty which one encounters in dealing with baryons in large \mathbf{N} SU(\mathbf{N}) (in this case at a first glance, being the fundamental representation of $so(\mathbf{N} - 1, 1)$ real, it is not clear what states could be analogous to mesons): baryons⁷ in many respects behave as soliton in a large \mathbf{N} expansion [27]. In particular, this implies that their (relatively large) masses are of order of an inverse power of the 't Hooft coupling and their interactions are suppressed by powers of $1/\mathbf{N}$. On the gravitational side, this seems to suggest that scalar particles which are not neutral under charge conjugation (such as the Higgs boson) should have relatively large masses compared to vectors and spinors. To provide this suggestive analogy with quantitative supports would require a detailed analysis

⁷ Baryons are color singlets made of particles with the same sign under charge conjugation: therefore, they are not neutral under charge conjugation.

of the space-time-momentum dependent parts of the fat graphs: this is out of the scopes of the present paper. Indeed, this is a direction worth to be investigated which could be rich of phenomenological consequences.

Now, it is possible to include matter fields also in the expansion. In general, when there are vertices with matter fields which, in the 't Hooft notation, are represented by single lines (as it happens in the present case), eq. (20) is modified in this way

$$F = h + L$$

where L is the number of matter loops in the closed connected fat graph. On the other hand, matter loops do not contribute to the (exponent of the) power of **N** of the fat graph since, due to the interactions, the closed matter loops do not correspond to closed internal index loops. Consequently, as one should expect, in this case eq. (22) has to be modified as follows

$$W_{\Gamma}(E, V, n_p, L, n_v) = \kappa^{2\overline{g}-2+L} \gamma_e^h \prod_p g_p^{n_p} \prod_{i=3,4} v_i^{n_i}$$
(24)

where v_i are the coupling constants of the matter vertices in eq. (23) and n_i is the number of matter vertices with coupling constant v_i . Thus, in the gravitational case also "ordinary" matter fields are suppressed in the large **N** expansion.

Here it becomes visible a striking difference between the Gauge Theory and the General Relativity case. In the purely gluonic sector of large \mathbf{N} SU(\mathbf{N}) Yang-Mills theory, in the topological expansion the subleading terms are suppressed by powers of $1/\mathbf{N}^2$ (in fact, matter loops give rise to factors of the order of powers of 1/N: of course, as it was first discovered by 't Hooft, this is due to the Euler formula for the genus of orientable two-dimensional surfaces. In the large N expansion of SU(N) Gauge Theory only orientable surfaces enter because the fundamental representation of $SU(\mathbf{N})$ is not real and the adjoint representation of SU(N) is the tensor product of the fundamental and the anti-fundamental. Graphically, this is expressed by adopting "the arrow" notation [22] in which the gluon is represented by two lines having arrows pointing in opposite directions: this necessarily implies that the fat graph is orientable. In (the BF formulation of) General Relativity the situation is different: the gauge group is SO(N-1,1) and the fundamental representation is real. For this reason, non orientable two-dimensional surfaces cannot be omitted in the topological expansion. For non orientable surfaces also there is an Euler formula which relates the right hand side of eq. (21) to the genus of the non orientable surfaces (which is always a positive integer). Non orientable two-dimensional surfaces can be obtained by cutting n discs from a sphere and then attaching n Mebius strips to the sphere by gluing the boundaries of the

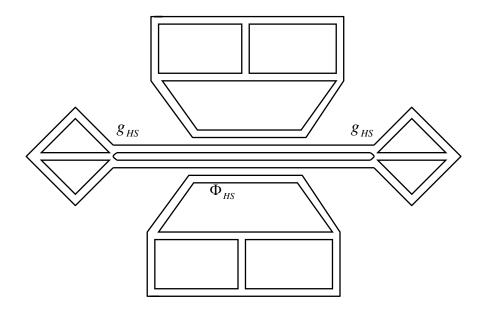


Fig. 8. Non orientable two dimensional surfaces can be constructed by gluing N Mebius strips onto a sphere from which N spherical caps have been removed. Such a surface has genus equal to N.

Mebius strips with the boundaries of the holes of the sphere (see fig. (8)). The surface obtained in this way is a non orientable surface of genus g equal to n. The Euler formula in this case reads (see, for example, [20])

$$g - 2 = E - V - F.$$
 (25)

Consequently, when non orientable surfaces are included, the right hand side of the above equation can be odd as well.

In order to use a unified notation it is more convenient to consider only eq. (21) with the convention that \overline{g} can be both integer (for orientable surfaces) and half-integer (for non orientable surfaces). Thus, unlike the Gauge Theory case, in the purely gravitational large **N** expansion of General Relativity the subleading terms are suppressed by powers of $1/\mathbf{N}$ which are of the same order of matter loops corrections: in a sense, the contributions of non orientable fat graphs are able to "mimic" matter. This point will be discussed in slightly more details in the next section.

Another interesting limit worth to be considered in this scheme is the Veneziano limit. In the Gauge Theory case, the Veneziano limit [26] had an important role in clarifying non trivial features of quarks dynamics which in the 't Hooft

Gluing the boundaries

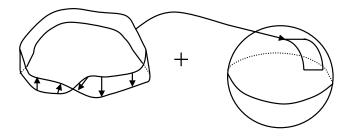


Fig. 9. This is an example of a planar graph with one matter loop (represented by the dashed line), four internal index loops, two triple gravitational vertices and four triple matter vertices.

limit were not manifest because of the further suppression in $1/\mathbf{N}$ due to the matter loops. The idea is to keep fixed, in the large \mathbf{N} limit, the ratio \mathbf{N}_f/\mathbf{N} (where \mathbf{N}_f is the number of flavour) too: in this way the suppression due to the matter loops is compensated by a factor \mathbf{N}_f (of course, we are assuming that the masses of matter fields are the same otherwise flavour symmetry would be explicitly broken). Consequently, the weight factor (24) of the generic closed connected fat graph Γ with L matter loops becomes

$$W_{\Gamma}^{V}(E, V, n_{p}, L, n_{v}) = (\mathbf{N}_{f})^{L} \kappa^{2\overline{g}-2+L} \gamma_{e}^{h} \prod_{p} g_{p}^{n_{p}} \prod_{i=3,4} v_{i}^{n_{i}}, \qquad (26)$$
$$\rho = \mathbf{N}_{f}/\mathbf{N}.$$

In this limit, matter loops are not further suppressed: the technical advantage is that one has at own disposal two natural coupling constants γ_e and ρ which measure respectively the strength of the gravitational and of the matter loops.

Thus, one can write the following formal expression for the free energy F:

$$F = \sum_{\substack{\Gamma closed \\ connected}} W_{\Gamma}^{V}(E, V, n_{p}, L, n_{v}) F_{\Gamma}^{V} =$$
(27)

$$=\sum_{\overline{g},h,L,n_p,n_i} \left(\mathbf{N}^{2-2\overline{g}} \gamma_e^h \rho^L \prod_p g_p^{n_p} \prod_{i=3,4} v_i^{n_i} \right) F_{\Gamma}^V$$
(28)

where, as in the previous section, F_{Γ}^{V} represents the spacetime-momentum dependent part of fat graph Γ in which also matter loops and vertices have been included in the large **N** limit with ρ fixed.

5 Comparing General Relativity and Gauge Theory expansions, Holography and Higher Spins

Here some differences between the General Relativity and Gauge Theory large **N** expansions will be discussed and the relation with the Holographic Principle will be analyzed.

The most evident difference between the two theories manifests itself when there is no matter: in the purely gluonic sector of large N SU(N) the corrections are suppressed by powers of $1/N^2$ while in the purely gravitational sector of large \mathbf{N} (BF formulation of) General Relativity the corrections are of order of powers of 1/N (which are of the same order of matter loops corrections). As it has been already mentioned, this is due to the contribution of non orientable fat graphs. Thus, gravity seems to be able to "imitate" matter: this should not appear really as a surprise. Since the works of Kaluza and Klein, many purely gravitational higher dimensional models have been constructed in which gravity in higher dimensions appears in lower dimensions as gravity plus matter. One could except that for pure gravity in four dimensions the Kaluza Klein idea does not provide with matter-like gravitational solutions. In fact, exact solutions of vacuum four-dimensional Einstein equations which can be interpreted as spin 1 particles (see [6]) and (more surprisingly) as spin 1/2particles (see |14|, for recent results and an updated list of references see |16|) have been constructed. The present results tell that this property of gravity to be able to "look like" matter should survive at a quantum level.

There is another difference which is less evident but, perhaps, more intriguing in a Holographic perspective. As it has been stressed in the previous sections, the Lagrange field ϕ (which, in the double line notation, carries four internal lines) gives rise to an effective quadruple vertex for the field *B*. In fact, this effective quadruple vertex is not completely analogous to a standard quadruple vertex: there is an interesting point missing in this picture. Let us imagine to give a very large but not infinite mass to ϕ (in other words, we are using a very powerful "magnifying glass" to disclose the internal structure of the effective quadruple vertex). It is clear that, to the eves of a gauge theorists something strange is happening: many connected fat graphs with ϕ vertices appear as disconnected fat graphs of some more usual Gauge Theory in which there are not fields represented by four (or more) color lines. In other words, it is not difficult to imagine, for example, some *Matrix Model* which, in its large N expansion, admits these fat graphs: however, this *Matrix Model* (in which only fields carrying two internal lines appear) would consider these fat graphs as disconnected and, therefore, not relevant for computing the free energy. Of course, in the General Relativity case, these graphs are not disconnected and do contribute to the free energy since ϕ is a basic field of the theory. Thus, in the General Relativity case, there are many more fat graphs contributing to the free energy which in a Gauge Theory with fields described by single and double lines would be neglected (see fig. (10)). The physical interpretation of this fact could be related to the Holographic Principle (see, for a detailed review, [5]). The reason is that, quite generically, since there are "many more"⁸ terms contributing to the free energy, the free energy itself is likely to be "higher"⁹.

To provide this last sentence with an analytical proof would require the analysis of the space-time-momentum dependent part of a generic fat graph and is a completely hopeless task. However, there are two quite sound arguments supporting it. Firstly, in order for the free energy in the General Relativity case not to be "higher" (in the sense specified above), there should be many fortuitous cancellations in the sum giving rise to the free energy among terms with different topological weights W_{Γ}^{V} . In other words, quite unlikely, the contribution to the free energy of a given "*GT-disconnected*" fat graph ¹⁰ should be cancelled by the contribution(s) of graph(s) with different genus, different number of "color" and matter loops and a different distribution of vertices. The meaning of this fact is "Holographic" in nature: the free energy can be written as

F = H - TS

where H is the internal energy, T the temperature and S the entropy. A "higher" free energy can be seen as a "lower" entropy and this is precisely what one would expect in a holographic theory: the Holographic Principle

⁸ Here "many more" means "many more with respect to a gauge theory having the same fat graphs in the topological expansion but having, in the 't Hooft notation, only fields represented by single and double lines."

⁹ Here "higher" means "higher than in a gauge theory which has the same fat graphs in the topological expansion but has only field represented by single and double lines".

¹⁰ Which means "Disconnected if interpreted as fat graphs of a Gauge Theory with only single and double line fields, but connected when fields represented by more than two internal lines (such as ϕ) are taken into account."

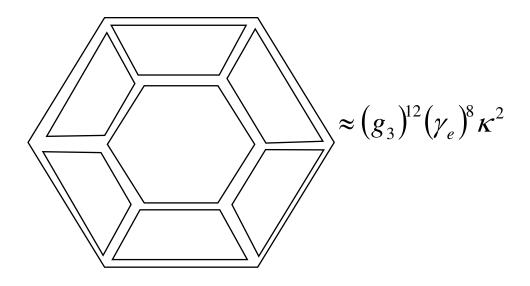


Fig. 10. This planar graph would appear disconnected into two pieces without a basic field represented by four internal lines (ϕ in this case) which, in fact, makes it connected. This implies that in theories in which there are fields represented by more than two internal lines the free energy receives many more contributions.

implies a striking reduction of the degrees of freedom (see, for example, [5]) and, therefore, of the entropy with respect to a local Quantum Field Theory ¹¹. The main role to achieve this decreasing of the entropy has been played by the field ϕ which, in the 't Hooft notation, is represented by four internal lines: obviously, the more internal lines are needed to represent a given field, the more such a field is able to decrease the entropy because of the many "*GT-disconnected*" fat graphs (see fig. (11) in which there is a fat graph with a higher spin fields Φ_{HS} represented by eight internal lines interacting with ordinary fields through the coupling constant g_{HS}).

The second argument supporting this scheme is related to *string theory*: *string theory* is expected to be a holographic theory but, unfortunately, is very far from being solved. However, in *string theory* are predicted an infinite number of higher spin fields which have very interesting geometrical properties (see, for example, [4]). Such fields, in the present notation, would be represented by many internal lines (according to their spin: the higher the spin, the more the

¹¹ In Quantum Field Theory the entropy, when suitably regolarized, is proportional to the volume of the space where the fields live.

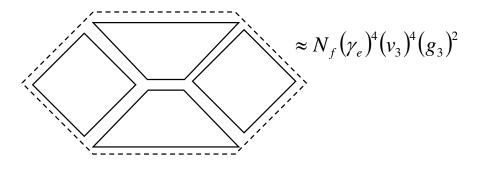


Fig. 11. In the presence of fields represented by more internal lines (eight in this case) there are fat graphs which would appear disconnected (into three pieces in this case) and, in fact, are connected due to the higher spin field. Here there is an example of a theory with a field represented by eight internal lines: similar planar fat graphs give contributions to the free energy which are absent in ordinary gauge theories.

internal lines). Thus, if there are no fortuitous cancellations, the entropy of the theory with these higher spin fields included would be strongly reduced. Hence, higher spin fields could play the main role in making *string theory* Holographic. Up to now, a microscopic mechanism able to explain, at least qualitatively, what kind of interactions could reduce the entropy as required by the Holographic Principle has not been found yet. The present results suggest that such a microscopic mechanism could be related to the interactions of higher spin fields which, being represented in the present notation by multiple internal lines, could give rise to the desired reduction of the entropy. It is interesting to note that this is the first precise microscopical mechanism which could be able to explain the Holographic Principle and it is based on higher spin fields which are very natural objects in *string theory*.

Eventually, it is worth to note the close parallelism between the BF formulation of General Relativity and the *unfolded* formulation of higher spin dynamics due to Vasiliev [25]: in both cases, the dynamics is formulated as a trivial "topological" dynamics plus a constraint which gives a non trivial content to the theory. In the *unfolded* formulation of higher spin dynamics the basic equations can be reduced to (see, for detailed reviews, [4])

$$d\omega = \omega \wedge \omega, \quad \omega = dx^{\nu} \omega_{\nu}^{a} T_{a} \tag{29}$$

$$\nabla_{\omega}\overline{B} = 0, \quad \overline{B} = \overline{B}^{A}T_{A} \tag{30}$$

$$\chi(\overline{B}) = 0 \tag{31}$$

where ω are one forms taking values in some Lie (super)algebra \pounds with generators T_a , \overline{B} are zero forms taking values is some (in general) different representation of \pounds , ∇_{ω} is the covariant derivative associated to ω and $\chi(\overline{B})$ is an algebraic constraint which is invariant under the gauge transformations of the first two equations (29) and (30). If one would neglect eq. (31), then eqs. (29) and (30) would be solved by pure gauge fields. Indeed, eqs. (29) and (30) bear a strong resemblance with the eq. (3) of the BF model, while the few differences appear to be technical in nature. The main suggestion related to such a close parallelism between the BF formulation of General Relativity and the *unfolded* formulation of higher spin dynamics is that the BF formulation of General Relativity could be very useful to find a local Lagrangian for interacting higher spin fields.

6 Conclusions and Perspectives

In this paper a large **N** expansion for General Relativity has been proposed. It is based on the BF formulation of General Relativity in which the Einstein-Hilbert action is splitted into a topological term plus a constraint. The scheme proposed allows to overcome some technical problems present in other proposals - such as the impossibility to evaluate the exact dependence of a given fat graph on the small expansion parameter(s). This method allowed to show that, unlike ordinary Gauge Theory, in the purely gravitational sector of the theory in the large N expansion the subleading terms are of order of powers of 1/N (and not $1/N^2$ as it happens in ordinary Gauge Theory) and so they are of the same order of matter loops corrections. The technical reason is that, being the gauge group SO(N-1,1) whose fundamental representation is real, in the topological expansion non orientable fat graphs cannot be excluded. This can be related to the fact that General Relativity is, in a sense, able to "imitate" matter: besides the well known Kaluza-Klein mechanism, classical exact solutions of vacuum four dimensional Einstein equations describing spin 1/2 and spin 1 particles are available too. The present results tell that such a property should be kept by the theory also at a quantum level. It is also possible to include matter in this scheme: it has been stressed that it is not clear how to include scalars in this picture. At a first glance, it seems that scalars, which have not "SO(N-1,1)-color", could be analogous to baryons in $SU(\mathbf{N})$: this could explain why they are so heavy (so heavy that they have

not been observed yet) and weakly interacting. Another interesting outcome of the analysis is the role of fields represented by more than two internal lines (higher spin fields). The presence of higher spin fields implies that, quite generically (this means "unless fortuitous cancellations occur") the free energy is higher or, equivalently, the entropy is lower than in ordinary Gauge Theory. This could be the microscopical mechanism responsible for the Holographic Principle which implies a striking reduction of the degrees of freedom. Moreover, higher spin fields are very natural objects in string theory. There are many directions worth to be further analyzed. First of all, it would be very important and rich of phenomenological consequences (from particles physics to cosmology) to clarify the nature of scalars in this scheme and, in particular, if they could be considered as a sort of baryons. A deeper understanding of the higher spin fields in a Holographic perspective is also welcome: the dynamics of higher spins, as this method clarifies, is likely to have a very strong influence on the microscopical entropy.

Acknowledgements

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A Large N expansion for Gravity

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Abstract

A Large N expansion for gravity is proposed. The scheme is based on the splitting of the Einstein-Hilbert action into the BF topological action plus a constraint. The method also allows to include matter fields. The relation between matter and non orientable fat graphs in the expansion is stressed; the special role of scalars is shortly discussed. The connections with the Holographic Principle and higher spin fields are analyzed.

Key words: Large N expansion, Einstein-Hilbert action, Holographic Principle. *PACS:* 11.15.Pg, 04.90.+e, 04.50.+h, 11.25.-w.

1 Introduction

Large N-expansion for a SU(N) Gauge Theory, introduced by 't Hooft in [22] [23], is indeed one of the most powerful non perturbative techniques available to investigate non linear gauge theories. In particular, even if large N SU(N)Gauge Theory has not been solved yet, the large N- expansion provides the issues of confinement, chiral symmetry breaking and the relation with string theory with a rather detailed understanding. The Veneziano limit [26], which corresponds to a 't Hooft limit in which the ratio N/N_f is kept fixed (N_f being the number of quarks flavours), shed further light on non perturbative features of large N SU(N) Gauge Theory clarifying several features of quarks and mesons dynamics; while a clear analysis of the role of Baryons at large N (which, in many respects, behave as solitons) has been provided in [27] (see, for two detailed pedagogical reviews, [17]). It is also worth to mention that (several types of) large N-expansions played a prominent role to provide the

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Holographic Principle with further supports and practical realizations (see, for a detailed review, [1]). For the above reasons, it would be very important to have at one's disposal similar techniques to investigate the non perturbative features of Einstein-Hilbert action. Indeed, *General Relativity* bears a strong resemblance with Gauge Theory because of the several common geometrical structures (such as the curvature tensor "measuring" the deviation from flatness, the "minimal coupling rule" which allows to rewrite Lorentz invariant equations in the presence of a non trivial gravitational field by substituting ordinary derivatives with covariant derivatives and so on). On the other hand, there are many striking differences which make General Relativity and Gauge Theory very unlike theories: first of all, while Gauge Theory is a theory defined on a fixed background space-time, General Relativity determines the dynamics of the background spacetime itself; the actions of General Relativity and Gauge Theory are different and General Relativity is not perturbatively renormalizable in the ordinary field theoretical sense. There are many others technical differences, here it is worth to mention that in Gauge Theory there is a clear separation between internal and space-time symmetries (and this allows to consider the large N-expansion by increasing the internal symmetry group while keeping fixed the space-time structure), for the above reasons in General Relativity this is not the case and it is not obvious what a large **N**-expansion in gravity could be.

An interesting attempt to obtain a similar expansion in gravity has been performed in [21] and further refined in [3]: in these papers the authors argued that the role of the internal index \mathbf{N} on the Gauge Theory's side could be played on the General Relativity's side by the number of space-time dimensions: in other words, the authors proposed an expansion in which the small parameter is 1/D where D is the number of space-time dimensions. The result of their analysis is that in a large D expansion of the Einstein-Hilbert action, unlike the large **N**-expansion of Gauge Theory in which all the planar diagrams are on equal footing, a subclass of the full set of planar diagrams dominates. Unfortunately, their analysis is affected by some technical problems which are absent in the Gauge Theory case. In particular, in a large Dexpansion, the power of D in a given diagram of the expansion is determined not only by the topology of the two dimensional surface on which such a diagram can be drawn (as it happens in the Gauge Theory case): a prominent role is also played by space-time and/or momentum integrals which also depend on D and a complete analysis of the contribution of the above integrals to the D-dependence of any diagram is a rather hopeless task [21] [3]. Another technical problem related to the previous one is the fact that, in a large D expansion, it is necessary to consider gravitational fluctuations around the trivial flat solution: the reason is that it is not possible to disentangle the space-time dependence due to space-time and/or momentum integrals from the "internal indices" dependence of any diagram since these two different kinds of structures overlap in the General Relativity case. If it would be possible to single out suitable "internal indices" in General Relativity then one could achieve an expansion whose topological character can be analyzed independently on the background: a non trivial background would only affect the space-time and/or momentum integrals of the diagrams while the internal index structure (which plays the prominent role in the large \mathbf{N} limit) would not change.

Here a scheme is proposed to overcome the above difficulties of the large D expansion which is based on the BF-theoretical formulation of Einstein-Hilbert action in which the General Relativity action is splitted into a topological term plus a constraint (the analysis of the relations between General Relativity and BF theory has a long history: see, for example, [7] [8] [12] [19] [9] and references therein): this way of writing the General Relativity action allows to distinguish internal index structure from space-time index structure and is strictly related to the connection formulation(s) of General Relativity which played a prominent role in the formulation of Loop Quantum Gravity (for a detailed review see, for example, [2]). Moreover, it is possible to make an interesting comparison between the BF formulation of General Relativity and Yang-Mills theory introduced in [15].

The paper is organized as follows: in the second section the formulation of General Relativity as a constrained BF theory is shortly described. In the third section a suitable "large \mathbf{N} 't Hooft expansion" on the internal indices is introduced. In the fourth section the inclusion of matter and a "Veneziano-like" limit are discussed. In the fifth section, the General Relativity and the Gauge Theory expansions are compared and the connections with the Holographic Principle and higher spin fields are analyzed. Eventually, the conclusions are drawn.

2 BF-theory and gravity

In this section the formulation of General Relativity as a constrained BF theory will be shortly described along the lines of [7] [8] [12] [19] [9].

BF theories are topological theories, that is, they exhibit the following properties: firstly, they are defined without any reference to a background metric and, moreover, they have no local degrees of freedom. Here, only the four dimensional case will be considered. The BF theory in four dimensions is defined

by the following action

$$S[A,B] = \int_{M} B^{IJ} \wedge F_{IJ}(A) = \frac{1}{4} \int_{M} \varepsilon^{\alpha\beta\gamma\delta} B^{IJ}_{\alpha\beta} F_{\gamma\delta IJ} d^{4}x \tag{1}$$

$$B^{IJ} = \frac{1}{2} B^{IJ}_{\alpha\beta} dx^{\alpha} \wedge dx^{\beta}, \quad F_{IJ} = \frac{1}{2} F_{\alpha\beta IJ} dx^{\alpha} \wedge dx^{\beta}$$
$$F_{\alpha\beta IJ} = (\partial_{\alpha} A_{\beta} - \partial_{\alpha} A_{\beta})_{IJ} + A^{L}_{\alpha I} A_{\beta LJ} - A^{L}_{\beta I} A_{\alpha LJ}, \quad (2)$$

where M is the four-dimensional space-time, the greek letters denote spacetimes indices, $\varepsilon^{\alpha\beta\gamma\delta}$ is the totally skew-symmetric Levi-Civita symbol in fourdimensional space-times, I, J and K are Lorentz (internal) indices which are raised and lowered with the Minkowski metric η_{IJ} : I, J = 1, ..., D. Thus, the basic fields are a so(D - 1, 1)-valued two form B_{IJ} and a so(D - 1, 1)connection one form $A_{\alpha LJ}$, the internal gauge group being SO(D - 1, 1). Also the Riemannian theory can be considered in which the internal gauge group is SO(D) and the internal indices are raised and lowered with the euclidean metric δ_{IJ} ; in any case, both B_{IJ} and $A_{\alpha LJ}$ are in the adjoint representation of the (algebra of the) internal gauge group: this simple observation will be important in order to develop a 't Hooft like large **N** expansion. The equations of motion are

$$F = 0, \quad \nabla_A B = 0 \tag{3}$$

where ∇_A is the covariant derivative with respect to the connection $A_{\alpha LJ}$. The above equations tell that $A_{\alpha LJ}$ is, locally, a pure gauge and B^{IJ} is covariantly constant. Obviously, the *BF* action does not describe the dynamics of general relativity which, indeed, has local degrees of freedom. On the other hand, if B^{IJ} would have this form

$$B^{IJ} = \frac{1}{2} \varepsilon^{IJ}_{KL} e^K \wedge e^L \tag{4}$$

then the action (1) would be nothing but the Palatini form of the generalized Einstein-Hilbert action (obviously, standard General Relativity is recovered when D = 4). It turns out that eq. (4) can be enforced by adding to the action (1) a suitable constraint; thus, the basic action in the BF formalism is

$$\kappa S_{GR} = S\left[A,B\right] - \int_{M} \left(\phi_{IJKL}B^{IJ} \wedge B^{KL} + \mu H\left(\phi\right)\right) \tag{5}$$

where κ is the gravitational coupling constant, μ is a four-form and $H(\phi)$ is a scalar which can have the following three expressions:

$$H_1 = \phi_{IJ}^{IJ}, \quad H_2 = \phi_{IJKL} \varepsilon^{IJKL}, \quad H_3 = a_1 H_1 + a_2 H_2,$$
 (6)

where a_i are real constants. It is worth to note here that the scalar ϕ takes value in the tensor product of the adjoint representation of so(D-1,1) (or of so(D) with itself. The form (5) of the Einstein-Hilbert action will be the starting point of the "gravitational" large **N** expansion.

3 A gravitational large N expansion

In this section, a 't Hooft like expansion based on the form (5) of the Einstein-Hilbert action is proposed: this scheme allows an interesting comparison with the BF formulation of Yang-Mills theory introduced in [15].

The first question which should be answered in order to set in a suitable framework for a gravitational large **N** expansion is: who is **N** in the gravitational case? The suggestion of [21] [3] is that **N** should be identified with the number of space-time dimensions D. Indeed, the basic variables of the BF-like formulation (and, in general, of any connection formulation of General Relativity) have the great merit to manifest a natural separation between space-time and internal indices. It becomes possible to consider a limit in which the space-time dimensionality is fixed while the dimension of the internal symmetry group approaches to infinity. In the Gauge Theory case, the gluonic fields are in the adjoint of $\mathbf{U}(\mathbf{N})$ (which can be thought as the tensor product of the fundamental and the anti-fundamental representations) while the quarks in the fundamental. It is then possible to introduce the celebrated 't Hooft double line notation in which it is easy to show that, in a generic diagrams, to every closed color line corresponds a factor of \mathbf{N} which is nothing but the dimension of the fundamental representation of $\mathbf{U}(\mathbf{N})$.

In the General Relativity case, two of the basic fields (A and B) are p-forms taking values in the adjoint representation of so(D-1,1): in other words, in the internal space, A and B are real $D \times D$ skew-symmetric matrices carrying, therefore, two vectorial indices running from 1 to D. While the Lagrangian multiplier ϕ carries four vectorial indices running from 1 to D (this fact will play a role in the following). The above considerations clarify that, in the gravitational case, \mathbf{N}^{-1} should be identified with 1/D:

$$\mathbf{N}^{-1} = D^{-1}; \tag{7}$$

however, in order to avoid confusion with the notation of [21] [3] and with the letter which is often used to denote the number of space-time dimensions (which, in fact, in this approach is kept fixed), the left hand side of eq. (7) will be used to denote the expansion parameter. An important benefit of the connection formulation is that the double line notation can be fairly adopted: the only difference with respect to the Gauge Theory case is that, being the fundamental representation of so(N - 1, 1) real, the lines of internal indices carry no arrow 1 .

In order to provide the large \mathbf{N} expansion with the usual topological classification one should write down the Feynman rules for the Einstein-Hilbert action (5) plus the gauge-fixing and ghost terms. As far as a large \mathbf{N} expansion is concerned, the ghost terms are not important since they do not influence the topological character of the expansion and the topological classification of the diagrams (see, for example, [17]). In this case it is convenient to find the Feynman rules along the lines of [18] where the authors considered the case of the *BF* formulation of Yang-Mills theory: in this way it will be easier to make a comparison between the large \mathbf{N} expansion in General Relativity and Gauge Theory. Thus, the starting point is the action

$$S_{GR} = \frac{1}{\kappa} \left[S\left[A, B\right] - \frac{c_2}{2} \int\limits_M \left(\phi_{IJKL} B^{IJ} \wedge B^{KL} + \mu H\left(\phi\right) \right) \right]$$
(8)

where κ is the gravitational coupling constant, the real constant c_2 keeps track of the terms which distinguish *BF*-theory from General Relativity. The natural choice is to consider as the Gaussian part the off-diagonal kinetic term

$$S_0 = \frac{1}{\kappa} \int_M \left(\varepsilon^{\alpha\beta\gamma\delta} B^{IJ}_{\alpha\beta} \partial_\gamma A_{\delta IJ} \right), \tag{9}$$

in such a way that the $A \to B$ propagator (which propagates A_{μ} into $B_{\nu\gamma}$) has the following structure (see fig. (1)):

$$\Delta_{(A,B)\mu\nu\gamma}^{(IJ,KL)} = \delta^{IL} \delta^{JK} F_1(p)_{\mu\nu\gamma} \tag{10}$$

$$F_1(p)_{\mu\nu\gamma} = -\frac{1}{2}\varepsilon_{\mu\nu\gamma\alpha}\frac{p^{\alpha}}{p^2}$$
(11)

where $F_1(p)_{\mu\nu\gamma}$ (which is the space-time-momentum dependent part of the propagator²) is not relevant as far as the large **N** expansion is concerned. The internal index structures of the $A \to A$ propagator and of the $B \to B$

¹ The point is that in the Gauge Theory case one has to distinguish the fundamental and the anti-fundamental representations of (the algebra of) U(N): this is achieved by adding to the color lines incoming or outgoing arrows.

 $^{^2}$ $F_1(p)_{\mu\nu\gamma}$ has a form similar to the $A \to B$ propagator in [18] in the usual Feynman gauge; the procedure to find the explicit expression of $F_1(p)_{\mu\nu\gamma}$ is analogous to the procedure of [18] and [10] but, as far as the purposes of the present paper are concerned, is not relevant since here only the internal index structure is needed.

Fig. 1. As in standard gauge theories, the structure of the propagator implies that internal indices are conserved along the internal lines.

propagator are analogous to the expressions found in [18]:

$$\Delta_{(A,A)\mu\nu}^{(IJ,KL)} = \delta^{IL} \delta^{JK} F_2(p)_{\mu\nu}, \quad \Delta_{(B,B)\mu\nu\gamma\rho}^{(IJ,KL)} = \delta^{IL} \delta^{JK} F_3(p)_{\mu\nu\gamma\rho} \tag{12}$$

$$F_2(p)_{\mu\nu} = \frac{1}{p^2} (\delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}) + \alpha_1 \frac{p^\mu p^\nu}{p^4}$$
(13)

$$F_3(p)_{\mu\nu\gamma\rho} = -\alpha_2 \frac{p^\mu p^\nu}{p^4} \tag{14}$$

where F_2 and F_3^3 (which are the space-time-momentum dependent parts of the propagators) will not be relevant as far as the large **N** expansion is concerned and α_1 and α_2 are real gauge parameters. Actually, even if F_2 and/or F_3 would vanish, the topological classification of the fat graphs and the large **N** expansion would not change: it is only important to note that, in a double-line notation, the internal index is conserved along the "color" lines.

A remark is in order here: in the BF Yang-Mills case the action can be written as follows

$$S_{BFYM} = S[A, B] - g^2 \int_M B^{IJ} \wedge *B_{IJ}$$
(15)

where * is the Hodge dual. The second term on the right hand side can be considered both as a part of the kinetic Lagrangian and as a true vertex. The

^{$\overline{3}$} A procedure to deduce them can be found, for example, in [18] and [10].

first choice gives rise to standard quantization procedure via gauge fixing and ghost terms. The second choice instead would be rather involved because one should gauge fix also the topological symmetry of the BF action and this would require a "ghost of ghost" structure. In fact, in the General Relativity case, there is no choice, the only available possibility is the (more involved) second one: in this case the quantization procedure to follow can be found in [10]. The point is that the quadratic term of the BF Yang-Mills case is replaced by a (rather unusual) cubic interaction term. However, as it has been already remarked, from a large **N** perspective the ghost terms are not relevant ⁴.

Unlike the already mentioned Yang-Mills case, the theory has two vertices:

$$V_1(A^a_\mu, A^b_\nu, B^c_{\alpha\beta}) = \frac{g_3}{3} f^{abc} \varepsilon_{\mu\nu\alpha\beta}, \quad V_2(B^a_{\mu\nu}, B^b_{\alpha\beta}, \phi^{cd}) = \frac{c_2}{2} \delta_{ac} \delta_{bd} \varepsilon^{\mu\nu\alpha\beta}$$
(16)

where a, b, c and d are internal indices in the adjoint representation, f^{abc} are the structure constants, g_3 (which can be assumed to be positive) and c_2 are adimensional coupling constants which keep track of the vertices in the large **N** counting and the reason of the seemingly strange normalization of g_3 and c_2 in the above equation will be clarified in the next subsection (see fig. (2) and fig. (3)). The second vertex is also present in the Gauge Theory case while the first one pertains to General Relativity only. In a large **N** perspective, the more convenient way to look at the Lagrange multiplier field ϕ is to consider it as a propagating field with a very high mass: this point of view allows an interesting interpretation of its physical role as it will be shown in the last section. The last term on the right hand side of eq. (8) (defined in eq. (6)) should not be considered as true vertex: it enforces some restrictions⁵ on the internal index structure of ϕ and will not be relevant as far as the large **N** counting is concerned.

Eventually, the last point to be clarified before to carry on the large N limit is the scaling of the coupling constant κ : in the large N limit, it is natural to assume that the product

$$\gamma_e = \mathbf{N}\kappa,\tag{17}$$

which plays the role of effective coupling constant, as fixed.

 $[\]overline{}^{4}$ The reason is that, from an "internal lines" perspective, they add no new vertices: in other words, the ghost vertices have the same internal index structures and coupling constants of the physical vertices (see, for example, [17]).

⁵ Such restrictions become less and less important at large **N**: to see this, it is enough to note that these restrictions imply that one of the component of ϕ can be expressed in terms of the others. When **N** is large (since the number of components of ϕ grows as **N**⁴) this fact is not relevant.

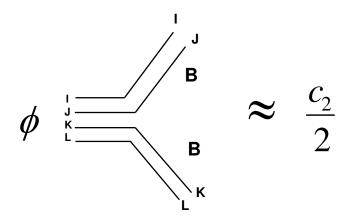


Fig. 2. This vertex (in which a field represented by four internal lines appears) only pertains to General Relativity. Such a field plays a very peculiar role as it will be shown in the following.

3.1 The General Relativity 't Hooft limit

Now it is possible to perform the 't Hooft limit: the main object of interest is the free energy so that only fat graphs with no external legs will be considered.

However, before to proceed, a point needs to be clarified. The vertex V_2 in eq. (16) behaves in some sense (actually, as it will be explained in the last section, there is an important difference with a non trivial physical interpretation) as an effective quadruple vertex for B (see fig. (4)): the point is that the field ϕ is only coupled to B so that, in closed graphs with no external legs the vertices V_2 are always in pairs. Thus, the effective coupling constant g_4 of the quadruple vertex is

$$g_4 = (c_2)^2 \,, \tag{18}$$

while the coupling constant of the cubic vertex is simply g_3 .

At last, it is possible to apply the standard "large **N** counting rules" for fat graphs (see, for two detailed pedagogical reviews, [13]). These counting rules can be deduced by using *matrix models*; it is usually convenient to choice the coupling constant of n-uple vertex by dividing it by n [13]: this is the

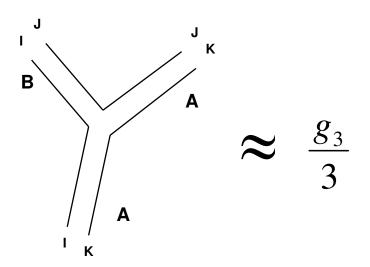


Fig. 3. This vertex is similar to the BF Yang-Mills vertex.

reason behind the normalization in eq. (16). Thus, the usual matrix models techniques [13] tell that a generic fat graph Γ with no external legs will have the following dependence on **N** and on the coupling constants:

$$W_{\Gamma}(E, V, n_p) = \kappa^{E-V} \mathbf{N}^F \prod_p g_p^{n_p}, \quad \sum_p n_p = V$$
(19)

where E is the number of propagators, n_p is the number of p-uple vertices in the graph Γ (in this case the only vertices to be counted are the ones with p = 3; the physical role of the effective quadruple vertex will be analyzed in the fifth section) and F is the number of faces of the fat graph. In this purely gravitational case with no matter fields with one internal index, one has

$$F = h \tag{20}$$

where h is number of closed "color" loops of Γ .

By using eq. (20) and the well known Euler formula

$$2\overline{g} - 2 = E - V - F,\tag{21}$$

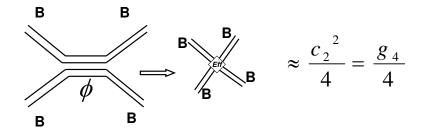


Fig. 4. The vertex with ϕ gives rise to an effective quadruple vertex for *B*. However, being ϕ represented by four internal lines, this quadruple vertex is not on equal footing with standard gauge theoretical vertices. This fact has a holographic interpretation.

where \overline{g} is the least genus⁶ of a Riemann surface on which the fat graph Γ can be drawn without intersecting lines, the weight W_{Γ} of the fat graph turns out to be

$$W_{\Gamma}(E,V,n_p) = \kappa^{2\overline{g}-2} \gamma_e^h \prod_p g_p^{n_p}$$
(22)

where the effective coupling constant γ_e has been introduced in eq. (17).

At a first glance, this result gives rise to the usual topological expansion for the free energy F, similar to the one of the Gauge Theory case, as a sum of the above factor in eq. (22) times a suitable space-time-momentum factor F_{Γ} over the closed connected fat graphs

⁶ There is a subtlety here in the definition of genus \overline{g} (this point will be discussed in the next sections) related to the fact that the fundamental representation of the internal gauge group is real. For this reason the notation \overline{g} (instead of the usual one g) will be adopted.

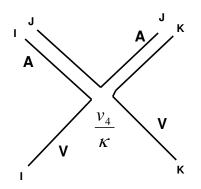


Fig. 5. This is an example of a planar graph with eight internal index loops and twelve triple vertices.

$$F = \sum_{\substack{\Gamma closed\\connected}} W_{\Gamma}(E, V, n_p) F_{\Gamma}$$

in which the leading term in the genus expansion is the planar one and the corrections in the topological expansion are suppressed as powers of $1/N^2$. In fact, there are interesting differences, related both to the gauge group and to the vertices, which will be analyzed in the next sections.

4 The inclusion of matter and the Veneziano limit

In this section the inclusion of matter in the gravitational 't Hooft limit and the Veneziano limit will be discussed.

Once the purely gravitational 't Hooft limit has been introduced, the inclusion of matter is the natural further step. However, the situation is less clear than in the Gauge Theory case. Vectors, in the standard metric formalism, are coupled to gravity via the Levi-Civita covariant derivative which, of course, acts on its vectorial index. Therefore, in this scheme, vectors are represented as scalar particles with an internal index J running from 1 to **N**

$$V_{\mu} \rightarrow V_J.$$

and should be coupled to the gravitational connection A by terms Υ_i as (see

fig. (6) and fig. (7))

$$\Upsilon_4 \sim \frac{v_4}{4\kappa} A^{IJ}_{\mu} V_J A^{\mu}_{IL} V^L, \quad \Upsilon_3 \sim \frac{v_3}{3\kappa} \left(p_{\mu} V^J \right) A^{\mu}_{JL} V^L \tag{23}$$

where v_i are coupling constants normalized in a suitable way to take advantage of the (already mentioned) large **N** counting techniques [13]. The above vertices could come from, for example, a kinetic term of the form

$$\nabla_A V^J \nabla_A V_J$$

where ∇_A is the covariant derivative of the connection A.

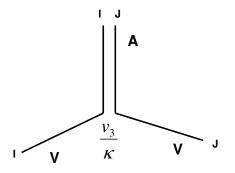


Fig. 6. This is a triple vertex with two matter fields and a gravitational connection.

For spinors the situation is more involved: the covariant derivative ∇_{γ} on a generic spinor Ψ in the standard metric formalism reads

$$i(\gamma^{\mu}(x)\partial_{\mu} - \gamma^{\mu}(x)\Gamma_{\mu})\Psi = \nabla_{\gamma}\Psi,$$

$$\gamma^{\mu}(x) = e_{K}^{\mu}\gamma^{K}$$

where e_K^{μ} are the vierbeins, Γ_{μ} is the spinorial Levi-Civita connection (in which, besides the connection A, also the vierbeins enter: see, for example, [11]) and γ^K are the standard flat Dirac matrices. The problem is that in the BF formulation of General Relativity the vierbeins do not appear directly: the fundamental field in the first order BF formalism is B which, actually, is the exterior product of two vierbeins. Thus, it is not clear how to construct interaction vertices with spinors. For this reason, here it will be considered only the contribution of vectors. It is worth to stress here that, in any case, the coupling terms with spinor should not be very different from Υ_3 and Υ_4 which,

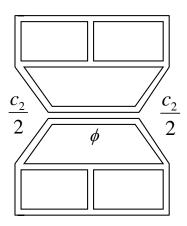


Fig. 7. This is a quadruple vertex with two matter fields and two gravitational connections.

therefore, provide the role of matter at large **N** with a detailed description. This can be argued as follows: a spinor is always accompanied by a flat Dirac matrix and by a vierbein and in a spinor current there are always two spinors. Hence, in a spinor current, there are always two vierbeins which, from an internal index perspective, are similar to B and, by the way, B has the same internal index structure of A. Consequently, as far as a large **N** counting is concerned, a spinorial vertex is well described by the vertex Υ_4 in eq. (23).

However, there is an apparent difficulty in dealing with scalar particles. Ordinary matter couples to the gravitational connection A through a vectorial internal index. On the other hand, at a first glance scalars do not couple to the gravitational connection A since, on them, covariant derivatives coincide with ordinary derivatives. This difficulty is very similar to the difficulty which one encounters in dealing with baryons in large \mathbf{N} SU(\mathbf{N}) (in this case at a first glance, being the fundamental representation of $so(\mathbf{N} - 1, 1)$ real, it is not clear what states could be analogous to mesons): baryons⁷ in many respects behave as soliton in a large \mathbf{N} expansion [27]. In particular, this implies that their (relatively large) masses are of order of an inverse power of the 't Hooft coupling and their interactions are suppressed by powers of $1/\mathbf{N}$. On the gravitational side, this seems to suggest that scalar particles which are not neutral under charge conjugation (such as the Higgs boson) should have relatively large masses compared to vectors and spinors. To provide this suggestive analogy with quantitative supports would require a detailed analysis

⁷ Baryons are color singlets made of particles with the same sign under charge conjugation: therefore, they are not neutral under charge conjugation.

of the space-time-momentum dependent parts of the fat graphs: this is out of the scopes of the present paper. Indeed, this is a direction worth to be investigated which could be rich of phenomenological consequences.

Now, it is possible to include matter fields also in the expansion. In general, when there are vertices with matter fields which, in the 't Hooft notation, are represented by single lines (as it happens in the present case), eq. (20) is modified in this way

$$F = h + L$$

where L is the number of matter loops in the closed connected fat graph. On the other hand, matter loops do not contribute to the (exponent of the) power of **N** of the fat graph since, due to the interactions, the closed matter loops do not correspond to closed internal index loops. Consequently, as one should expect, in this case eq. (22) has to be modified as follows

$$W_{\Gamma}(E, V, n_p, L, n_v) = \kappa^{2\overline{g}-2+L} \gamma_e^h \prod_p g_p^{n_p} \prod_{i=3,4} v_i^{n_i}$$
(24)

where v_i are the coupling constants of the matter vertices in eq. (23) and n_i is the number of matter vertices with coupling constant v_i . Thus, in the gravitational case also "ordinary" matter fields are suppressed in the large **N** expansion.

Here it becomes visible a striking difference between the Gauge Theory and the General Relativity case. In the purely gluonic sector of large \mathbf{N} SU(\mathbf{N}) Yang-Mills theory, in the topological expansion the subleading terms are suppressed by powers of $1/\mathbf{N}^2$ (in fact, matter loops give rise to factors of the order of powers of 1/N: of course, as it was first discovered by 't Hooft, this is due to the Euler formula for the genus of orientable two-dimensional surfaces. In the large N expansion of SU(N) Gauge Theory only orientable surfaces enter because the fundamental representation of $SU(\mathbf{N})$ is not real and the adjoint representation of SU(N) is the tensor product of the fundamental and the anti-fundamental. Graphically, this is expressed by adopting "the arrow" notation [22] in which the gluon is represented by two lines having arrows pointing in opposite directions: this necessarily implies that the fat graph is orientable. In (the BF formulation of) General Relativity the situation is different: the gauge group is SO(N-1,1) and the fundamental representation is real. For this reason, non orientable two-dimensional surfaces cannot be omitted in the topological expansion. For non orientable surfaces also there is an Euler formula which relates the right hand side of eq. (21) to the genus of the non orientable surfaces (which is always a positive integer). Non orientable two-dimensional surfaces can be obtained by cutting n discs from a sphere and then attaching n Mebius strips to the sphere by gluing the boundaries of the

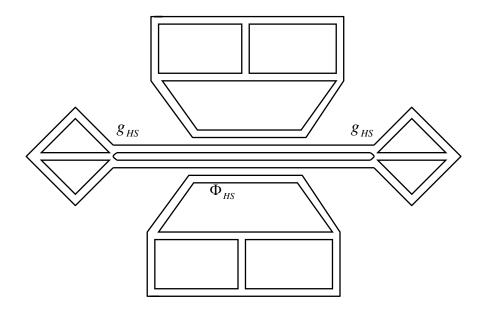


Fig. 8. Non orientable two dimensional surfaces can be constructed by gluing N Mebius strips onto a sphere from which N spherical caps have been removed. Such a surface has genus equal to N.

Mebius strips with the boundaries of the holes of the sphere (see fig. (8)). The surface obtained in this way is a non orientable surface of genus g equal to n. The Euler formula in this case reads (see, for example, [20])

$$g - 2 = E - V - F.$$
 (25)

Consequently, when non orientable surfaces are included, the right hand side of the above equation can be odd as well.

In order to use a unified notation it is more convenient to consider only eq. (21) with the convention that \overline{g} can be both integer (for orientable surfaces) and half-integer (for non orientable surfaces). Thus, unlike the Gauge Theory case, in the purely gravitational large **N** expansion of General Relativity the subleading terms are suppressed by powers of $1/\mathbf{N}$ which are of the same order of matter loops corrections: in a sense, the contributions of non orientable fat graphs are able to "mimic" matter. This point will be discussed in slightly more details in the next section.

Another interesting limit worth to be considered in this scheme is the Veneziano limit. In the Gauge Theory case, the Veneziano limit [26] had an important role in clarifying non trivial features of quarks dynamics which in the 't Hooft

Gluing the boundaries

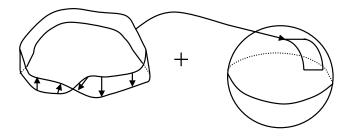


Fig. 9. This is an example of a planar graph with one matter loop (represented by the dashed line), four internal index loops, two triple gravitational vertices and four triple matter vertices.

limit were not manifest because of the further suppression in $1/\mathbf{N}$ due to the matter loops. The idea is to keep fixed, in the large \mathbf{N} limit, the ratio \mathbf{N}_f/\mathbf{N} (where \mathbf{N}_f is the number of flavour) too: in this way the suppression due to the matter loops is compensated by a factor \mathbf{N}_f (of course, we are assuming that the masses of matter fields are the same otherwise flavour symmetry would be explicitly broken). Consequently, the weight factor (24) of the generic closed connected fat graph Γ with L matter loops becomes

$$W_{\Gamma}^{V}(E, V, n_{p}, L, n_{v}) = (\mathbf{N}_{f})^{L} \kappa^{2\overline{g}-2+L} \gamma_{e}^{h} \prod_{p} g_{p}^{n_{p}} \prod_{i=3,4} v_{i}^{n_{i}}, \qquad (26)$$
$$\rho = \mathbf{N}_{f}/\mathbf{N}.$$

In this limit, matter loops are not further suppressed: the technical advantage is that one has at own disposal two natural coupling constants γ_e and ρ which measure respectively the strength of the gravitational and of the matter loops.

Thus, one can write the following formal expression for the free energy F:

$$F = \sum_{\substack{\Gamma closed \\ connected}} W_{\Gamma}^{V}(E, V, n_{p}, L, n_{v}) F_{\Gamma}^{V} =$$
(27)

$$=\sum_{\overline{g},h,L,n_p,n_i} \left(\mathbf{N}^{2-2\overline{g}} \gamma_e^h \rho^L \prod_p g_p^{n_p} \prod_{i=3,4} v_i^{n_i} \right) F_{\Gamma}^V$$
(28)

where, as in the previous section, F_{Γ}^{V} represents the spacetime-momentum dependent part of fat graph Γ in which also matter loops and vertices have been included in the large **N** limit with ρ fixed.

5 Comparing General Relativity and Gauge Theory expansions, Holography and Higher Spins

Here some differences between the General Relativity and Gauge Theory large **N** expansions will be discussed and the relation with the Holographic Principle will be analyzed.

The most evident difference between the two theories manifests itself when there is no matter: in the purely gluonic sector of large N SU(N) the corrections are suppressed by powers of $1/N^2$ while in the purely gravitational sector of large \mathbf{N} (BF formulation of) General Relativity the corrections are of order of powers of 1/N (which are of the same order of matter loops corrections). As it has been already mentioned, this is due to the contribution of non orientable fat graphs. Thus, gravity seems to be able to "imitate" matter: this should not appear really as a surprise. Since the works of Kaluza and Klein, many purely gravitational higher dimensional models have been constructed in which gravity in higher dimensions appears in lower dimensions as gravity plus matter. One could except that for pure gravity in four dimensions the Kaluza Klein idea does not provide with matter-like gravitational solutions. In fact, exact solutions of vacuum four-dimensional Einstein equations which can be interpreted as spin 1 particles (see [6]) and (more surprisingly) as spin 1/2particles (see |14|, for recent results and an updated list of references see |16|) have been constructed. The present results tell that this property of gravity to be able to "look like" matter should survive at a quantum level.

There is another difference which is less evident but, perhaps, more intriguing in a Holographic perspective. As it has been stressed in the previous sections, the Lagrange field ϕ (which, in the double line notation, carries four internal lines) gives rise to an effective quadruple vertex for the field *B*. In fact, this effective quadruple vertex is not completely analogous to a standard quadruple vertex: there is an interesting point missing in this picture. Let us imagine to give a very large but not infinite mass to ϕ (in other words, we are using a very powerful "magnifying glass" to disclose the internal structure of the effective quadruple vertex). It is clear that, to the eves of a gauge theorists something strange is happening: many connected fat graphs with ϕ vertices appear as disconnected fat graphs of some more usual Gauge Theory in which there are not fields represented by four (or more) color lines. In other words, it is not difficult to imagine, for example, some *Matrix Model* which, in its large N expansion, admits these fat graphs: however, this *Matrix Model* (in which only fields carrying two internal lines appear) would consider these fat graphs as disconnected and, therefore, not relevant for computing the free energy. Of course, in the General Relativity case, these graphs are not disconnected and do contribute to the free energy since ϕ is a basic field of the theory. Thus, in the General Relativity case, there are many more fat graphs contributing to the free energy which in a Gauge Theory with fields described by single and double lines would be neglected (see fig. (10)). The physical interpretation of this fact could be related to the Holographic Principle (see, for a detailed review, [5]). The reason is that, quite generically, since there are "many more"⁸ terms contributing to the free energy, the free energy itself is likely to be "higher"⁹.

To provide this last sentence with an analytical proof would require the analysis of the space-time-momentum dependent part of a generic fat graph and is a completely hopeless task. However, there are two quite sound arguments supporting it. Firstly, in order for the free energy in the General Relativity case not to be "higher" (in the sense specified above), there should be many fortuitous cancellations in the sum giving rise to the free energy among terms with different topological weights W_{Γ}^{V} . In other words, quite unlikely, the contribution to the free energy of a given "*GT-disconnected*" fat graph ¹⁰ should be cancelled by the contribution(s) of graph(s) with different genus, different number of "color" and matter loops and a different distribution of vertices. The meaning of this fact is "Holographic" in nature: the free energy can be written as

F = H - TS

where H is the internal energy, T the temperature and S the entropy. A "higher" free energy can be seen as a "lower" entropy and this is precisely what one would expect in a holographic theory: the Holographic Principle

⁸ Here "many more" means "many more with respect to a gauge theory having the same fat graphs in the topological expansion but having, in the 't Hooft notation, only fields represented by single and double lines."

⁹ Here "higher" means "higher than in a gauge theory which has the same fat graphs in the topological expansion but has only field represented by single and double lines".

¹⁰ Which means "Disconnected if interpreted as fat graphs of a Gauge Theory with only single and double line fields, but connected when fields represented by more than two internal lines (such as ϕ) are taken into account."

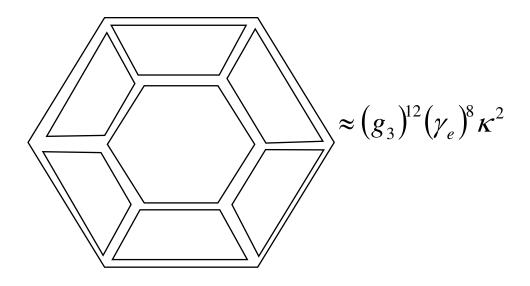


Fig. 10. This planar graph would appear disconnected into two pieces without a basic field represented by four internal lines (ϕ in this case) which, in fact, makes it connected. This implies that in theories in which there are fields represented by more than two internal lines the free energy receives many more contributions.

implies a striking reduction of the degrees of freedom (see, for example, [5]) and, therefore, of the entropy with respect to a local Quantum Field Theory ¹¹. The main role to achieve this decreasing of the entropy has been played by the field ϕ which, in the 't Hooft notation, is represented by four internal lines: obviously, the more internal lines are needed to represent a given field, the more such a field is able to decrease the entropy because of the many "*GT-disconnected*" fat graphs (see fig. (11) in which there is a fat graph with a higher spin fields Φ_{HS} represented by eight internal lines interacting with ordinary fields through the coupling constant g_{HS}).

The second argument supporting this scheme is related to *string theory*: *string theory* is expected to be a holographic theory but, unfortunately, is very far from being solved. However, in *string theory* are predicted an infinite number of higher spin fields which have very interesting geometrical properties (see, for example, [4]). Such fields, in the present notation, would be represented by many internal lines (according to their spin: the higher the spin, the more the

¹¹ In Quantum Field Theory the entropy, when suitably regolarized, is proportional to the volume of the space where the fields live.

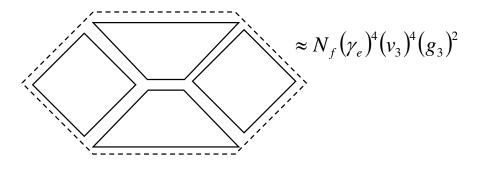


Fig. 11. In the presence of fields represented by more internal lines (eight in this case) there are fat graphs which would appear disconnected (into three pieces in this case) and, in fact, are connected due to the higher spin field. Here there is an example of a theory with a field represented by eight internal lines: similar planar fat graphs give contributions to the free energy which are absent in ordinary gauge theories.

internal lines). Thus, if there are no fortuitous cancellations, the entropy of the theory with these higher spin fields included would be strongly reduced. Hence, higher spin fields could play the main role in making *string theory* Holographic. Up to now, a microscopic mechanism able to explain, at least qualitatively, what kind of interactions could reduce the entropy as required by the Holographic Principle has not been found yet. The present results suggest that such a microscopic mechanism could be related to the interactions of higher spin fields which, being represented in the present notation by multiple internal lines, could give rise to the desired reduction of the entropy. It is interesting to note that this is the first precise microscopical mechanism which could be able to explain the Holographic Principle and it is based on higher spin fields which are very natural objects in *string theory*.

Eventually, it is worth to note the close parallelism between the BF formulation of General Relativity and the *unfolded* formulation of higher spin dynamics due to Vasiliev [25]: in both cases, the dynamics is formulated as a trivial "topological" dynamics plus a constraint which gives a non trivial content to the theory. In the *unfolded* formulation of higher spin dynamics the basic equations can be reduced to (see, for detailed reviews, [4])

$$d\omega = \omega \wedge \omega, \quad \omega = dx^{\nu} \omega_{\nu}^{a} T_{a} \tag{29}$$

$$\nabla_{\omega}\overline{B} = 0, \quad \overline{B} = \overline{B}^{A}T_{A} \tag{30}$$

$$\chi(\overline{B}) = 0 \tag{31}$$

where ω are one forms taking values in some Lie (super)algebra \pounds with generators T_a , \overline{B} are zero forms taking values is some (in general) different representation of \pounds , ∇_{ω} is the covariant derivative associated to ω and $\chi(\overline{B})$ is an algebraic constraint which is invariant under the gauge transformations of the first two equations (29) and (30). If one would neglect eq. (31), then eqs. (29) and (30) would be solved by pure gauge fields. Indeed, eqs. (29) and (30) bear a strong resemblance with the eq. (3) of the BF model, while the few differences appear to be technical in nature. The main suggestion related to such a close parallelism between the BF formulation of General Relativity and the *unfolded* formulation of higher spin dynamics is that the BF formulation of General Relativity could be very useful to find a local Lagrangian for interacting higher spin fields.

6 Conclusions and Perspectives

In this paper a large **N** expansion for General Relativity has been proposed. It is based on the BF formulation of General Relativity in which the Einstein-Hilbert action is splitted into a topological term plus a constraint. The scheme proposed allows to overcome some technical problems present in other proposals - such as the impossibility to evaluate the exact dependence of a given fat graph on the small expansion parameter(s). This method allowed to show that, unlike ordinary Gauge Theory, in the purely gravitational sector of the theory in the large N expansion the subleading terms are of order of powers of 1/N (and not $1/N^2$ as it happens in ordinary Gauge Theory) and so they are of the same order of matter loops corrections. The technical reason is that, being the gauge group SO(N-1,1) whose fundamental representation is real, in the topological expansion non orientable fat graphs cannot be excluded. This can be related to the fact that General Relativity is, in a sense, able to "imitate" matter: besides the well known Kaluza-Klein mechanism, classical exact solutions of vacuum four dimensional Einstein equations describing spin 1/2 and spin 1 particles are available too. The present results tell that such a property should be kept by the theory also at a quantum level. It is also possible to include matter in this scheme: it has been stressed that it is not clear how to include scalars in this picture. At a first glance, it seems that scalars, which have not "SO(N-1,1)-color", could be analogous to baryons in $SU(\mathbf{N})$: this could explain why they are so heavy (so heavy that they have

not been observed yet) and weakly interacting. Another interesting outcome of the analysis is the role of fields represented by more than two internal lines (higher spin fields). The presence of higher spin fields implies that, quite generically (this means "unless fortuitous cancellations occur") the free energy is higher or, equivalently, the entropy is lower than in ordinary Gauge Theory. This could be the microscopical mechanism responsible for the Holographic Principle which implies a striking reduction of the degrees of freedom. Moreover, higher spin fields are very natural objects in string theory. There are many directions worth to be further analyzed. First of all, it would be very important and rich of phenomenological consequences (from particles physics to cosmology) to clarify the nature of scalars in this scheme and, in particular, if they could be considered as a sort of baryons. A deeper understanding of the higher spin fields in a Holographic perspective is also welcome: the dynamics of higher spins, as this method clarifies, is likely to have a very strong influence on the microscopical entropy.

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