

# Fermionic zero modes on a toroidal cosmic string

Abhijit B. Gadde and Urjit A. Yajnik

*Indian Institute of Technology Bombay, Mumbai 400076*

## Abstract

We consider a toroidal configuration of cosmic string in 3+1 dimensions in an abelian Higgs model, a compactification of the Nielsen-Olesen string. This object is classically unstable. We explicitly compute the number of permitted zero modes for majorana fermions coupled to such a string. As in the case of indefinitely long strings, there are  $|n|$  zero modes for winding number sector  $n$ , and correspondingly, induced fermionic charge  $n/2$  which can be fractional. According to a previously proved result, this implies quantum mechanical stability for objects with odd winding number. The result is of significance to cosmology in classes of unified theories permitting such cosmic strings.

## 1 Introduction

Solitons are classically stable solutions of field equations, made possible when the spontaneous breaking of gauge symmetry permits topologically non-trivial boundary conditions. Examples of such solitons exist in one dimension [1] called kinks, where the gauge field is absent, in two dimensions [2] called Nielsen-Olesen strings which occur in abelian gauge theories, in three dimensions [3, 4], being 't Hooft-Polyakov monopoles occurring in Yang-Mills theories. More interesting systems are those in which the interaction of the fermi fields with the soliton is also considered. Under certain conditions, existence of fermionic zero modes results in fractional fermion number being induced on the classical solution [1, 5]. Such systems can not relax to trivial vacuum in isolation [6] due to Quantum Mechanics. This possibility was first emphasised in [7] and its consequences to possible particle like states in  $SO(10)$  Grand Unified Theory were studied in [8]. Fractional fermion number phenomenon is also of importance in condensed matter systems like conducting polymers [9].

In this paper, we study toroidal configurations of the Nielsen-Olesen string. From classical arguments, this object can be shown to be unstable with respect to shrinking under its own tension. However, it exists as an extremum of the action in  $3 + 1$  dimensions and carries finite total energy, and can be of fundamental significance to cosmology. Here we have studied the interaction of this object with a majorana fermion field and have shown the existence of fermionic zero modes. Significance of such solutions to cosmology was studied in [10] and more recently in [11, 12, 13]. We use the same fermionic coupling as was first studied by Jackiw and Rossi [14], wherein the mass of the fermion is derived entirely from spontaneous symmetry breaking. Our main result about relation between winding number and number of zero modes is the same as [14], although some of the details are different. Explicit arguments for the stability of objects with fractional fermion number were spelt out in [6], which will essentially apply in the present case as well.

Several standard caveats apply to the present work. We treat the fermions as a quantum perturbation to a classical background and ignore the back reaction of the fermions to the string. While we study the highly symmetric object, the torus, the result about zero modes should apply in the general situation subject to some modifications, which however should not modify the main results regarding induced stability. In particular the “zero-energy” solutions will no longer be so on a generic closed loop geometry, however the modes, if singleton, more generally in odd number should remain so as long as the essential topological aspects of the boundary conditions

far from the string are not modified. Finally we also assume the metastability of the loop to ensure its existence over time scales long enough to treat zero-modes as occurring on essentially static background.

We solve the Majorana-Dirac equation in toroidal coordinates, and for this purpose begin with a review of the same in sec. 2. In sec. 3 the equations are formulated. The asymptotic form of the fermionic wave function is studied in sec. 4. In sec. 5, the behaviour near the core of the torus, which determines the number of zero-energy solutions is studied, and is found to reveal that the number is the same as for the uncompactified Nielsen-Olesen string. Sec. 6 is devoted to summary and conclusion.

## 2 Toroidal Coordinates

The coordinate transformations from the cartesian to the toroidal co-ordinates [16] are given by the following relations

$$x = \frac{a \sinh v \cos \varphi}{\cosh v - \cos u} \quad (1)$$

$$y = \frac{a \sinh v \sin \varphi}{\cosh v - \cos u} \quad (2)$$

$$z = \frac{a \sin u}{\cosh v - \cos u} \quad (3)$$

where  $v$  ranges from 0 to  $\infty$ ,  $u$  ranges from 0 to  $2\pi$  and  $\varphi$  ranges from 0 to  $2\pi$ . The parameter  $a$  sets the size of the family of torii given by  $v = \text{constant}$ . The variable  $\varphi$  parameterises the length of the loop while  $u$  winds around any segment of the loop given by  $\varphi = \text{constant}$ . The coordinates have the property that as  $v$  tends to infinity, we approach the core of the loop. Spatial infinity is approached when  $u$  and  $v$  simultaneously approach zero.

In the following it is convenient to introduce  $\xi = (u + iv)/2$  and calculate the metric elements  $h_v = |\partial \vec{r} / \partial v|$  both in terms of  $u, v$  and  $\xi, \bar{\xi}$

$$h_u = \frac{a}{\cosh v - \cos u} = \frac{a}{2 \sin \xi \sin \bar{\xi}} \quad (4)$$

$$h_v = \frac{a}{\cosh v - \cos u} = \frac{a}{2 \sin \xi \sin \bar{\xi}} \quad (5)$$

$$h_\varphi = \frac{a \sinh v}{\cosh v - \cos u} = \frac{a \sinh v}{2 \sin \xi \sin \bar{\xi}} \quad (6)$$

The expression for the gradient takes the form,

$$\nabla = \hat{v} \frac{1}{h_v} \frac{\partial}{\partial v} + \hat{u} \frac{1}{h_u} \frac{\partial}{\partial u} + \hat{\varphi} \frac{1}{h_\varphi} \frac{\partial}{\partial \varphi} \quad (7)$$

## 3 Formulation of the Equation

The Abelian Higgs model with gauge field  $A_\mu$  and a charged scalar field  $\phi$  has the Lagrangian

$$\mathcal{L} = (D_\mu \phi)^* (D^\mu \phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{4} (|\phi|^2 - \eta^2)^2 \quad (8)$$

with  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  and the covariant derivative

$$D_\mu = \left( \frac{\partial}{\partial x^\mu} - iq h_\mu A_\mu \right) \quad (9)$$

and the scalar field  $\phi$  is taken to have charge  $q = e$ . This Lagrangian can be extended for the purpose of studying to zero modes [14][15], to include majorana fermions of charge  $q = \frac{1}{2}e$ ,

$$\mathcal{L}_{fermion} = \bar{\psi}i\sigma^\mu D_\mu\psi - \frac{1}{2}[ig_Y\bar{\psi}\phi\psi^c + (h.c.)] \quad (10)$$

where  $\sigma^\mu = (-I, \sigma^i)$ ,  $I$  being the  $2 \times 2$  identity matrix;  $\psi^c = i\sigma^2\psi^*$  is the charge conjugate of  $\psi$ , and  $g_Y$  denotes the Yukawa coupling.

Since  $\psi$  will be reserved for later use, we begin by writing the field equation in terms of the variable  $\tilde{\psi}$

$$\sigma^\mu D_\mu\tilde{\psi} - g_Y\phi\tilde{\psi}^c = 0 \quad (11)$$

We begin by writing the equations of motion of the fermion in usual cylindrical polar coordinates,  $(r, \varphi, z)$  with the string loop of radius  $a$  and with cross-section  $\ll a^2$  laid out symmetrically around the origin in the  $z = 0$  plane. The  $\varphi$  coordinate remains the same upon transforming to the toroidal coordinates. In 2-component notation,

$$\begin{bmatrix} -e^{i\varphi}[D_r + \frac{i}{r}D_\varphi] & D_z + D_t \\ D_z - D_t & e^{-i\varphi}[D_r - \frac{i}{r}D_\varphi] \end{bmatrix} \begin{bmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{bmatrix} = g_Y\phi \begin{bmatrix} \tilde{\psi}_1^* \\ \tilde{\psi}_2^* \end{bmatrix} \quad (12)$$

Since we are looking for zero modes i.e. time independent solutions we take the background fields to possess the ansatz  $A_0 = 0$ . Further, the lowest energy and therefore the most symmetric background solution can be assumed  $\varphi$  independent, and we choose  $A_\varphi = 0$ . However  $\varphi$  explicitly appears in the equations for  $\tilde{\psi}$  and factoring out this dependence requires us to introduce the ansatz  $\tilde{\psi}_1 = e^{-i\varphi/2}\psi_1$  and  $\tilde{\psi}_2 = e^{i\varphi/2}\psi_2$ . This amounts to anti-periodic boundary condition appropriate to a fermion as we traverse the length of the loop. Then the equations obeyed by  $\psi_1$  and  $\psi_2$  are

$$\begin{bmatrix} -[D_r + \frac{1}{2r}] & D_z \\ D_z & [D_r + \frac{1}{2r}] \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = g_Y\phi \begin{bmatrix} \psi_1^* \\ \psi_2^* \end{bmatrix} \quad (13)$$

Thus the problem of solving fermionic equations is restricted essentially to the half plane of the cylindrical polar coordinates,  $r \in [0, \infty)$  and  $z \in (-\infty, \infty)$ . Substituting  $\psi_1 = i\psi_2 = i\psi$  reduces the two equations to the complex equation

$$[D_r + iD_z]\psi + \frac{1}{2r}\psi = g_Y\phi\psi^* \quad (14)$$

We now transform to the toroidal coordinates. Since  $\partial_r$  and  $A_r$  transform identically. Thus, transforming to toroidal coordinates the equation looks as follows,

$$[\sin^2\bar{\xi}(D_u + iD_v) + i\frac{\sin\xi\sin\bar{\xi}}{2\sinh v}]\psi = \frac{\phi}{2i}ag_Y\psi^* \quad (15)$$

A useful substitution now is  $\psi = f(u, v)\sin\xi$ , which leads to the equation for  $f$ ,

$$[\sin\xi\sin\bar{\xi}(D_u + iD_v) + i\frac{\sin^2\xi}{2\sinh v}]f = \frac{\phi}{2i}ag_Yf^* \quad (16)$$

We work in the vacuum sector of winding number  $n$ , i.e., given any segment of the loop, the scalar field  $\phi$  changes phase by  $2\pi n$  around it. In toroidal coordinates this amounts a dependence  $e^{inu}$ . While topologically this is not distinct from trivial vacuum, it has restricted topological stability against breaking of any segment of the loop. Only the shrinking of the loop as a whole can continuously connect it to the trivial vacuum. Thus if latter deformation is forbidden the configuration becomes stable. As a direct generalisation of the Nielsen-Olesen string, the background field configuration is taken to have the ansatz

$$\phi = ik(u, v)\eta e^{inu} \quad (17)$$

$$A_\mu = -n \frac{g(u, v)}{ae} \delta_u^\mu \sin \xi \sin \bar{\xi} \quad (18)$$

where  $g(u, v)$  and  $k(u, v)$  are real functions whose behaviour is  $g, k \rightarrow 0$  near the loop, i.e., as  $v \rightarrow \infty$  so that the solution is regular in core of the loop, and  $g(u, v), k(u, v) \rightarrow 1$  at spatial infinity given by the simultaneous limit  $u, v \rightarrow 0$ . Note that  $\sin \xi \sin \bar{\xi}$  reproduces the well known behaviour  $1/r$  for usual infinitely long string in cylindrical coordinates and becomes pure gauge far from the core of the string. We denote  $g_Y \eta = m$  where  $m$  is the mass of the free fermions far from the string. After substituting above scalar and gauge ansatz, eq. (16) takes the form

$$[\sin \xi \sin \bar{\xi} (\frac{\partial}{\partial \bar{\xi}} - i \frac{n}{2} g) + i \frac{\sin^2 \xi}{2 \sinh v}] f = \frac{1}{2} a k m e^{inu} f^* \quad (19)$$

Using the technique of Jackiw and Rossi, we try the following ansatz for  $f$ .

$$f = X e^{il(\xi + \bar{\xi})} + Y^* e^{i(n-l)(\xi + \bar{\xi})} \quad (20)$$

and equating coefficients of  $e^{ilu}$  and  $e^{i(n-l)u}$  we get two separate equations.

$$[\sin \xi \sin \bar{\xi} (\frac{\partial}{\partial \bar{\xi}} - i(\frac{n}{2} g - l)) + i \frac{\sin^2 \xi}{2 \sinh v}] X = \frac{1}{2} a k m Y \quad (21)$$

$$[\sin \xi \sin \bar{\xi} (\frac{\partial}{\partial \bar{\xi}} - i(\frac{n}{2} g - (n-l))) + i \frac{\sin^2 \xi}{2 \sinh v}] Y^* = \frac{1}{2} a k m X^* \quad (22)$$

taking complex conjugate of the Eq. (22),

$$[\sin \xi \sin \bar{\xi} (\frac{\partial}{\partial \xi} + i(\frac{n}{2} g - (n-l))) - i \frac{\sin^2 \bar{\xi}}{2 \sinh v}] Y = \frac{1}{2} a k m X \quad (23)$$

## 4 Asymptotic analysis

In the asymptotic limit, as mentioned above,  $g$  and  $k$  can be approximated by 1. So in the asymptotic limit the equations (21) and (23) respectively become,

$$[\xi \bar{\xi} (\frac{\partial}{\partial \bar{\xi}} - ip) - \frac{1}{2} \frac{\xi^2}{(\xi - \bar{\xi})}] X = \frac{1}{2} a m Y \quad (24)$$

$$[\xi \bar{\xi} (\frac{\partial}{\partial \xi} - ip) + \frac{1}{2} \frac{\bar{\xi}^2}{(\xi - \bar{\xi})}] Y = \frac{1}{2} a m X \quad (25)$$

where  $p = (\frac{n}{2} - l)$ . We substitute,  $X = A \sqrt{\frac{4i\xi\bar{\xi}}{\xi - \bar{\xi}}}$  similarly  $Y = B \sqrt{\frac{4i\xi\bar{\xi}}{\xi - \bar{\xi}}}$ . The equations are simplified to,

$$\xi \bar{\xi} (\frac{\partial}{\partial \bar{\xi}} - ip) A = \frac{1}{2} a m B \quad (26)$$

$$\xi \bar{\xi} (\frac{\partial}{\partial \xi} - ip) B = \frac{1}{2} a m A \quad (27)$$

combining Eq. (26) and (27) we get,

$$[\xi \bar{\xi} (\frac{\partial}{\partial \xi} - ip) \xi \bar{\xi} (\frac{\partial}{\partial \bar{\xi}} - ip)] A = (ma/2)^2 A \quad (28)$$

Substituting  $\xi = t^{-1} e^{i\theta}$ , Eq. (28) is simplified to

$$[\frac{\partial^2}{\partial t^2} + (\frac{2ip}{t^2} - \frac{1}{t}) \frac{\partial}{\partial t} - (\frac{p^2}{t^4} + \frac{2ip}{t^3} + (\frac{ma}{2})^2) + \frac{1}{t^2} (\frac{\partial^2}{\partial \theta^2} + 2i \frac{\partial}{\partial \theta})] A = 0 \quad (29)$$

This, in asymptotic limit, i.e. as  $|\xi| \rightarrow 0$ , i.e. as  $t \rightarrow \infty$ , becomes

$$\left[\frac{\partial^2}{\partial t^2} - \frac{1}{t} \frac{\partial}{\partial t} - \left(\frac{ma}{2}\right)^2\right]A = 0 \quad (30)$$

this second order differential equation can be solved to yield an exponentially converging solution, whose asymptotic behaviour is  $\sim e^{-mat/2}$ . Incorporating  $\sin \xi$  factor, the asymptotic behaviour of the fermionic wave-function  $\psi \sim (e^{-mat/2})/t$  making it normalisable.

## 5 Counting the number of solutions

To count the total number of fermion zero modes present on soliton in  $n$  vortex sector, we observe the  $v$  dependance of the solution near the loop i.e. as  $v \rightarrow \infty$ . As mentioned above  $g$  and  $k$  both tend to zero as we approach the loop. So substituting  $g = 0 = k$  the equations (21) and (23), near the loop, respectively become,

$$\left(\frac{d}{dv} + l\right)X = 0 \quad (31)$$

$$\left(\frac{d}{dv} + (n-l)\right)Y = 0 \quad (32)$$

So,  $X \sim e^{-lv}$  and  $Y \sim e^{-(n-l)v}$ . And so the behaviour of  $\psi$  near the loop is,

$$\psi = \sin \xi (X e^{ilu} + Y^* e^{i(n-l)u}) = \sin \xi (C_1 e^{2il\xi} + C_2 e^{2i(n-l)\xi}) \quad (33)$$

In the limit  $v \rightarrow \infty$  the right hand side of eq. (33) is dominated by the terms

$$\psi \longrightarrow \frac{1}{2i} (C_1 e^{i2(l-\frac{1}{2})\xi} + C_2 e^{i2[(n-1)-(l-\frac{1}{2})]\xi}) \quad (34)$$

Recalling  $\xi = (u + iv)/2$  and denoting  $l - \frac{1}{2}$  by  $l'$ , and requiring  $\psi$  to remain finite near the loop i.e. as  $v \rightarrow \infty$ , we need,

$$0 \leq l' \leq (n-1) \quad (35)$$

This gives us total of  $n$  complex normalisable solutions, the same result as [14] for the infinitely long string. It should be noted that we have the  $\varphi$  dependence  $e^{\pm i\varphi}$ . If the length parameter along the loop is denoted  $\tilde{z}$ , this can be written as  $e^{\pm i\tilde{z}/2\pi a}$ . This explicit dependence on  $\tilde{z}$  disappears in the limit  $a \rightarrow \infty$  and we recover the translation invariant ansatz for the zero modes utilised in [14]. Taking  $l'$  to be integer (rather than half-integer) gives larger number of solutions and makes the latter single valued as functions of  $u$ , which also accords with the treatment for infinitely long string. So the compactification of Nielsen-Olesen string has not altered the number of zero modes it carries.

## 6 Conclusion

We have proved the existence of  $|n|$  fermionic zero modes on a static toroidal string with topological winding  $n$ . Unlike the non-compact Nielsen-Olesen strings which are infinitely long and often treated as essentially  $2 + 1$  dimensional solitons, toroidal strings are genuinely  $3 + 1$  dimensional configurations of finite energy. So the existence of the latter and the existence of related zero modes are very important from the point of view of cosmology. Our result shows that the toroidal geometry supports the same number of zero modes as the infinitely long string and reassures us that the unbounded string can be recovered as a limiting case of the toroidal configurations considered here.

The boundary condition implied by the behaviour  $e^{\pm i\varphi/2}$  with azimuthal angle  $\varphi$  shows that for small loops, when the loop is indistinguishable from a particle, its wave function obeys the same boundary conditions as an elementary fermion of spin  $1/2$ . Physically such states should be

discovered as heavy fermions of spin  $1/2$ . Further, the occurrence of zero modes would imply, just as in the case of unbounded string, that the loop acquires fermionic charge  $|n|/2$ . If this charge is half-integral, it would be impossible for the loop to disintegrate in isolation without conflicting with Quantum Mechanics. The arguments detailed in [6] apply without significant modification.

When these considerations are further applied to the collective dynamics of the string, new situations need to be addressed. Consider a loop of large radius which folds and begins to cross itself. In the absence of experimental evidence and absence of conclusive theoretical calculation two possibilities are usually considered, one where the two colliding segments pass through and the other where they inter-commute, producing two smaller loops. Since the winding number of the two child strings would be the same as the parent string the number zero modes on each of the child strings would be the same as the parent string. If therefore the parent string had half-integer fermion number, the final state would have integer fermion number. To avoid conflict with quantum mechanical principles we must insist that the inter-commuting process cannot occur for the strings with odd number of zero-modes.

In [6] it was explicitly shown that a single non-compact string cannot decay in isolation even if metastable. However no conclusion could be reached about formation of loops formed by self-intersection of a non-compact string. With the results of the present paper we can conclude that formation of loop by such a process is also forbidden for non-compact strings with odd number of zero-modes, for the same reason as in preceding paragraph.

Loops stabilised by quantum mechanical considerations would be extremely important to Cosmology, where such loops can constitute Cold Dark Matter [17][18]. We may assume that the process of shrinking of the loop under its own tension can continue till some small radius is reached, presumably of the order of the Compton wavelength of the fermions. Provided that fermions are much lighter, such a length would be large compared to the cross-section of the string characterised by gauge boson and scalar masses. Such a state would then be indistinguishable for classical purposes from a fundamental particle. While all the mutually interacting particles would decay into the lightest available particle state subject to conserved quantum numbers, heavy states such as stabilised string loops would persist and serve as Dark Matter. Conversely, unified theories implying unacceptable abundance of such stabilised loops would be ruled out by such considerations.

## Acknowledgement

This work is supported by a grant from Department of Science and Technology.

## References

- [1] R. Jackiw and C. Rebbi, Phys. Rev. **D13**, 3398(1976)
- [2] H. Nielsen and P. Olesen, Nucl. Phys. **B61**, 45(1973)
- [3] G. 't Hooft, Nucl. Phys. **B79**, 276(1974)
- [4] A. Polyakov, JETP Lett. **20**, 194(1974)
- [5] R. Jackiw, Rev. Mod. Phys. **49**, 681(1977)
- [6] N. Sahu and U. A. Yajnik, Phys. Lett. **B596**, 1(2004)
- [7] H. de Vega, Phys. Rev. **D18**, 2932(1978)
- [8] A. Stern, Phys. Rev. Lett. **52**, 2118(1983)
- [9] R. Jackiw and J. R. Schrieffer, Nucl. Phys. **B190**, 253(1981)
- [10] A. Stern and U. A. Yajnik, Nucl. Phys. **B267**, 158 (1986)

- [11] S. C. Davis, A. C. Davis and W. B. Perkins, *Phys. Lett. B* **408**, 81 (1997)
- [12] A. C. Davis, S. C. Davis and W. B. Perkins, *Proceedings of COSMO99*, arXiv:hep-ph/0005091
- [13] A. C. Davis, T. W. B. Kibble, M. Pickles and D. A. Steer, *Phys. Rev.* **D62** 083516 (2000)
- [14] R. Jackiw and P. Rossi, *Nucl. Phys.* **B190**, 681(1981)
- [15] S. C. Davis, *Int. J. Theor. Phys.* **38**, 2889-2900 (1999)
- [16] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, Vol. II, McGraw-Hill, 1953
- [17] E. W. Kolb and M. S. Turner, *The early Universe*, Addison-Wesley Pub. Co. 1990
- [18] D. N. Spergel et al, *Astrophys. J. Suppl.* **148**, 175(2003)