arXiv:hep-th/0701077v1 9 Jan 2007

PARABOSONIC AND PARAFERMIONIC ALGEBRAS. GRADED STRUCTURE AND HOPF STRUCTURES

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Abstract. Parabosonic $P_B^{(n)}$ and parafermionic $P_F^{(n)}$ algebras are described as quotients of the tensor algebras of suitably choosen vector spaces. Their (super-) Lie algebraic structure and consequently their (super-) Hopf structure is shortly discussed. A bosonisation-like construction is presented, which produces an ordinary Hopf algebra $P_{B(K^{\pm})}^{(n)}$ starting from the super Hopf algebra $P_B^{(n)}$.

1. Introduction and Definitions

Throughout this paper we are going to use the following notation conventions:
If x and y are any monomials of the tensor algebra of some k-vector space, we are going to call commutator the following expression:

$$[x, y] = x \otimes y - y \otimes x \equiv xy - yx$$

and anticommutator the following expression:

$$\{x, y\} = x \otimes y + y \otimes x \equiv xy + yx$$

By the field k we shall always mean $\mathbb{R}$ or $\mathbb{C}$, and all tensor products will be considered over k unless stated so. Finally we freely use Sweedler's notation for the comultiplication throughout the paper.

Let us consider the k-vector space V_B freely generated by the elements: b_i^+, b_j^-, $i, j = 1, ..., n$. Let $T(V_B)$ denote the tensor algebra of V_B (i.e.: the free algebra generated by the elements of the basis). In $T(V_B)$ we consider the (two-sided) ideal I_B generated by the following elements:

$$\left[b_m^-, \{b_k^+, b_l^-\}\right] - 2\delta_{km} b_l^-$$

$$\left[b_m^-, \{b_k^-, b_l^-\}\right]$$

$$\left[b_m^+, \{b_k^-, b_l^-\}\right] + 2\delta_{lm} b_k^- + 2\delta_{km} b_l^-$$

and all adjoints. (Since we do not yet consider any particular representation the term "adjoint" has no other meaning but to denote different

(linearly independent) elements of the algebra. We say for example that b_i^- and b_i^+ are adjoint elements or that $b_i^+ b_j^-$ and $b_j^+ b_i^-$ are adjoint elements. But when we consider representations of such algebras then we see that the "physically interesting" modules are those in which the "adjoint" elements become adjoint operators of a Hilbert space.)
We now have the following:

Definition 1.1. *We define the parabosonic algebra in* $2n$ *generators* $P_B^{(n)}$ *(*n *parabosons) to be the quotient algebra of the tensor algebra of* V_B *with the ideal* I_B*. In other words:*

$$P_B^{(n)} = T(V_B)/I_B$$

In a similar way we may describe the parafermionic algebra in n generators: Let us consider the k-vector space V_F freely generated by the elements: $f_i^+, f_j^-,\ i,j = 1,...,n$. Let $T(V_F)$ denote the tensor algebra of V_F (i.e.: the free algebra generated by the elements of the basis). In $T(V_F)$ we consider the (two-sided) ideal I_F generated by the following elements:

$$\left[f_m^-, [f_k^+, f_l^-]\right] - 2\delta_{km} f_l^-$$

$$\left[f_m^-, [f_k^-, f_l^-]\right]$$

$$\left[f_m^+, [f_k^-, f_l^-]\right] + 2\delta_{lm} f_k^- - 2\delta_{km} f_l^-$$

and all adjoints.
We get the following definition:

Definition 1.2. *We define the parafermionic algebra in* $2n$ *generators* $P_F^{(n)}$ *(*n *parafermions) to be the quotient algebra of the tensor algebra of* V_F *with the ideal* I_F*. In other words:*

$$P_F^{(n)} = T(V_F)/I_F$$

Parafermionic and parabosonic algebras first appeared in the physics literature by means of generators and relations, in the pionnering works of Green [6] and Greenberg and Messiah [5]. Their purpose was to introduce generalizations of the usual bosonic and fermionic algebras of quantum mechanics, capable of leading to generalized versions of the Bose-Einstein and Fermi-Dirac statistics (see: [16]).

2. (super-)Lie and (super-)Hopf algebraic structure of $P_B^{(n)}$ and $P_F^{(n)}$

Due to it's simpler nature, parafermionic algebras were the first to be identified as the universal enveloping algebras (UEA) of simple Lie algebras. This was done almost at the same time by S.Kamefuchi, Y.Takahashi in [10] and by C. Ryan, E.C.G. Sudarshan in [23]. In fact the following proposition was shown in the above mentioned references:

Proposition 2.1. *The parafermionic algebra in $2n$ generators is isomorphic to the universal enveloping algebra of the simple complex Lie algebra B_n (according to the well known classification of the simple complex Lie algebras), i.e:*

$$P_F^{(n)} \cong U(B_n)$$

An immediate consequence of the above identification is that parafermionic algebras are (ordinary) Hopf algebras, with the generators $f_i^{\pm}$, $i = 1, ..., n$ being the primitive elements. The Hopf algebraic structure of $P_F^{(n)}$ is completely determined by the well known Hopf algebraic structure of the Lie algebras, due to the above isomorphism. For convenience we quote the relations explicitly:

$$\Delta(f_i^{\pm}) = f_i^{\pm} \otimes 1 + 1 \otimes f_i^{\pm}$$

$$\varepsilon(f_i^{\pm}) = 0$$

$$S(f_i^{\pm}) = -f_i^{\pm}$$

The algebraic structure of parabosons seemed to be somewhat more complicated. The presence of anticommutators among the (trilinear) relations defining $P_B^{(n)}$ "breaks" the usual (Lie) antisymmetry and makes impossible the identification of the parabosons with the UEA of any Lie algebra. It was in the early '80 's that was conjectured [16], that due to the mixing of commutators and anticommutators in $P_B^{(n)}$ the proper mathematical "playground" should be some kind of Lie superalgebra (or: $\mathbb{Z}_2$-graded Lie algebra). Starting in the early '80 's, and using the recent (by that time) results in the classification of the finite dimensional simple complex Lie superalgebras which was obtained by Kac (see: [7, 8]), T.D.Palev managed to identify the parabosonic algebra with the UEA of a certain simple complex Lie superalgebra. In [19], [21], T.D.Palev shows the following proposition:

Proposition 2.2. *The parabosonic algebra in $2n$ generators is isomorphic to the universal enveloping algebra of the classical simple complex Lie superalgebra $B(0, n)$ (according to the classification of the simple complex Lie superalgebras given by Kac), i.e:*

$$P_B^{(n)} \cong U(B(0, n))$$

Note that $B(0, n)$ in Kac's notation, is the classical simple complex orthosymplectic Lie superalgebra denoted $osp(1, 2n)$ in the notation traditionally used by physicists until then.

The universal enveloping algebra $U(L)$ of a Lie superalgebra L is not a Hopf algebra, at least in the ordinary sense. $U(L)$ is a $\mathbb{Z}_2$-graded associative algebra (or: superalgebra) and it is a super-Hopf algebra in a sense that we briefly describe: First we consider the braided tensor product algebra $U(L)\underline{\otimes}U(L)$, which means the vector space $U(L) \otimes U(L)$ equipped with the associative multiplication:

$$(a \otimes b) \cdot (c \otimes d) = (-1)^{|b||c|} ac \otimes bd$$

for b, c homogeneous elements of $U(L)$, and $|.|$ denotes the degree of an homogeneous element (i.e.: $|b| = 0$ if b is an even element and $|b| = 1$ if b is an odd element). Note that $U(L)\underline{\otimes}U(L)$ is also a superalgebra or: $\mathbb{Z}_2$-graded associative algebra. Then $U(L)$ is equipped with a coproduct

$$\Delta : U(L) \to U(L)\underline{\otimes}U(L)$$

which is an superalgebra homomorphism from $U(L)$ to the braided tensor product algebra $U(L)\underline{\otimes}U(L)$:

$$\Delta(ab) = \sum (-1)^{|a_2||b_1|} a_1 b_1 \otimes a_2 b_2 = \Delta(a) \cdot \Delta(b)$$

for any a, b in $U(L)$, with $\Delta(a) = \sum a_1 \otimes a_2$, $\Delta(b) = \sum b_1 \otimes b_2$, and a_2, b_1 homogeneous. Δ is uniquely determined by it's value on the generators of $U(L)$ (i.e.: the basis elements of L):

$$\Delta(x) = 1 \otimes x + x \otimes 1$$

Similarly, $U(L)$ is equipped with an antipode $S : U(L) \to U(L)$ which is not an algebra anti-homomorphism (as in ordinary Hopf algebras) but a braided algebra anti-homomorphism (or: "twisted" anti-homomorphism) in the following sense:

$$S(ab) = (-1)^{|a||b|} S(b)S(a)$$

for any homogeneous $a, b \in U(L)$.
All the above description is equivalent to saying that $U(L)$ is a a Hopf algebra in the braided category of $\mathbb{C}\mathbb{Z}_2$-modules, or: a braided group, where the braiding is induced by the non-trivial quasitriangular structure of the $\mathbb{C}\mathbb{Z}_2$ Hopf algebra i.e. by the non-trivial R-matrix:

$$R_g = \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g)$$

where $1, g$ are the elements of the $\mathbb{Z}_2$ group which is now written multiplicatively.

In view of the above description, an immediate consequence of proposition 2.2, is that the parabosonic algebras $P_B^{(n)}$ are super-Hopf algebras, with the generators $b_i^{\pm}$, $i = 1, ..., n$ being the primitive elements. It's super-Hopf algebraic structure is completely determined by the super-Hopf algebraic structure of Lie superalgebras, due to the above mentioned isomorphism. Namely the following relations determine completely the super-Hopf algebraic structure of $P_B^{(n)}$:

$$\Delta(b_i^{\pm}) = 1 \otimes b_i^{\pm} + b_i^{\pm} \otimes 1$$

$$\varepsilon(b_i^{\pm}) = 0$$

$$S(b_i^{\pm}) = -b_i^{\pm}$$

3. Bosonisation: ordinary Hopf algebraic structures for parabosons

A general scheme for "transforming" a Hopf algebra B in the braided category ${}_H\mathcal{M}$ (H: some quasitriangular Hopf algebra) into an ordinary one, namely the smash product Hopf algebra: $B \rtimes H$, such that the two algebras have equivalent module categories, has been developed during '90 's. The original reference is [11] (see also [12, 13]). The technique is called bosonisation, the term coming from physics. This technique uses ideas developed in [22], [14]. It is also presented and applied in [3], [4], [1].
In the special case that B is some super-Hopf algebra, then: $H = \mathbb{C}\mathbb{Z}_2$, equipped with it's non-trivial quasitriangular structure, formerly mentioned. In this case, the technique simplifies and the ordinary Hopf algebra produced is the smash product Hopf algebra $B \rtimes \mathbb{C}\mathbb{Z}_2$. The grading in B is induced by the $\mathbb{C}\mathbb{Z}_2$-action on B:

$$g \triangleright x = (-1)^{|x|} x$$

for x homogeneous in B. (Note that for self-dual Hopf algebras -such as $\mathbb{C}\mathbb{Z}_2$- the notions of action and coaction coincide). This action is "absorbed" in $B \rtimes \mathbb{C}\mathbb{Z}_2$, and becomes an inner automorphism:

$$gxg = (-1)^{|x|} x$$

where we have identified: $b \rtimes 1 \equiv b$ and $1 \rtimes g \equiv g$ in $B \rtimes \mathbb{C}\mathbb{Z}_2$. This inner automorphism is exactly the adjoint action of g on $B \rtimes \mathbb{C}\mathbb{Z}_2$ (as an ordinary Hopf algebra). The following proposition is proved -as an example of the bosonisation technique- in [12]:

Proposition 3.1. *Corresponding to every super-Hopf algebra B there is an ordinary Hopf algebra $B \rtimes \mathbb{C}\mathbb{Z}_2$, it's bosonisation, consisting of*

B extended by adjoining an element g with relations, coproduct, counit and antipode:

$$g^2 = 1 \qquad gb = (-1)^{|b|} bg \qquad \Delta(g) = g \otimes g \quad \Delta(b) = \sum b_1 g^{|b_2|} \otimes b_2$$

$$S(g) = g \quad S(b) = g^{-|b|} \underline{S}(b) \qquad \varepsilon(g) = 1 \qquad\qquad \varepsilon(b) = \underline{\varepsilon}(b)$$

where $\underline{S}$ and $\underline{\varepsilon}$ denote the original maps of the super-Hopf algebra B. Moreover, the representations of the bosonised Hopf algebra $B \rtimes \mathbb{C}\mathbb{Z}_2$ are precisely the super-representations of the original superalgebra B.

The application of the above proposition in the case of the parabosonic algebra $P_B^{(n)} \cong U(B(0,n))$ is straightforward: we immediately get it's bosonised form $P_{B(g)}^{(n)}$ which by definition is:

$$P_{B(g)}^{(n)} \equiv P_B^{(n)} \rtimes \mathbb{C}\mathbb{Z}_2 \cong U(B(0,n)) \rtimes \mathbb{C}\mathbb{Z}_2$$

Let us describe now a slightly different construction (see: [2]), which achieves the same object: the determination of an ordinary Hopf structure for the parabosonic algebra $P_B^{(n)}$.
Defining:

$$N_{lm} = \frac{1}{2}\{b_l^+, b_m^-\}$$

we notice that these are the generators of the Lie algebra $u(n)$:

$$[N_{kl}, N_{mn}] = \delta_{lm} N_{kn} - \delta_{kn} N_{ml}$$

We introduce now the elements:

$$\mathcal{N} = \sum_{i=1}^{n} N_{ii} = \frac{1}{2} \sum_{i=1}^{n} \{b_i^+, b_i^-\}$$

which are exactly the linear Casimirs of $u(n)$.
We can easily find that they satisfy:

$$[\mathcal{N}, b_i^{\pm}] = \pm b_i^{\pm}$$

We now introduce the following elements:

$$K^+ = \exp(i\pi\mathcal{N}) \equiv \sum_{m=0}^{\infty} \frac{(i\pi\mathcal{N})^m}{m!}$$

and:

$$K^- = \exp(-i\pi\mathcal{N}) \equiv \sum_{m=0}^{\infty} \frac{(-i\pi\mathcal{N})^m}{m!}$$

Utilizing the above power series expressions and the commutation relations formerly calculated for $\mathcal{N}$ we get after lengthy but straightforward calculations:

$$K^+K^- = K^-K^+ = 1$$

and also:

$$\{K^+, b_i^{\pm}\} = 0 \quad \{K^-, b_i^{\pm}\} = 0$$

We finally have the following proposition:

Proposition 3.2. *Corresponding to the super-Hopf algebra $P_B^{(n)}$ there is an ordinary Hopf algebra $P_{B(K^\pm)}^{(n)}$, consisting of $P_B^{(n)}$ extended by adjoining two elements K^+, K^- with relations, coproduct, counit and antipode:*

$$\Delta(b_i^{\pm}) = b_i^{\pm} \otimes 1 + K^{\pm} \otimes b_i^{\pm} \qquad \Delta(K^{\pm}) = K^{\pm} \otimes K^{\pm}$$

$$\varepsilon(b_i^{\pm}) = 0 \qquad \varepsilon(K^{\pm}) = 1$$

$$S(b_i^{\pm}) = b_i^{\pm} K^{\mp} \qquad S(K^{\pm}) = K^{\mp}$$

$$K^+K^- = K^-K^+ = 1 \qquad \{K^+, b_i^{\pm}\} = 0 = \{K^-, b_i^{\pm}\}$$

Proof. The proposition may be proved by lengthy but straightforward calculations. See [2]. □

The above constructed algebra $P_{B(K^\pm)}^{(n)}$, is an ordinary Hopf algebra in the sense that the comultiplication is extended to the whole of $P_{B(K^\pm)}^{(n)}$ as an algebra homomorphism :

$$\Delta : P_{B(K^\pm)}^{(n)} \to P_{B(K^\pm)}^{(n)} \otimes P_{B(K^\pm)}^{(n)}$$

where $P_{B(K^\pm)}^{(n)} \otimes P_{B(K^\pm)}^{(n)}$ is considered as the tensor product algebra with the usual product:

$$(a \otimes b)(c \otimes d) = ac \otimes bd$$

for any $a, b, c, d \in P_{B(K^\pm)}^{(n)}$ and the antipode extends as usual as an algebra anti-homomorphism.

The relation between the above constructed ordinary Hopf algebras, $P_{B(g)}^{(n)}$ and $P_{B(K^\pm)}^{(n)}$ remains to be seen.

This research was co-funded by the European Union in the framework of the program "Pythagoras II", contract number 80897.

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