Refining the classification of the irreps of the 1D*N*-Extended Supersymmetry

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Abstract

The linear finite irreducible representations of the algebra of the 1D N-Extended Supersymmetric Quantum Mechanics are discussed in terms of their "connectivity" (a symbol encoding information on the graphs associated to the irreps). The classification of the irreducible representations with the same fields content and different connectivity is presented up to $N \leq 8$.

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1 Introduction

The structure of the irreducible representations of the N-extended supersymmetric quantum mechanics has been elucidated only recently (see [1, 2, 3, 4, 5]). One is concerned with the problem of classifying the finite linear irreducible representations of the supersymmetry algebra

$$\{Q_i, Q_j\} = 2\delta_{ij}H,$$

$$[Q_i, H] = 0,$$
(1)

where Q_i are N odd supercharges (i = 1, ..., N), while the bosonic central extension H can be regarded as a hamiltonian (therefore $H \equiv i\frac{d}{dt}$) of a supersymmetric quantum mechanical system. The finite linear irreps of (1) consist of an equal finite number n of bosonic and fermionic fields (depending on a single coordinate t, the time) upon which the supersymmetry operators act linearly.

In [1] it was proven that all (1) irreps fall into classes of equivalence determined by the irreps of an associated Clifford algebra. As one of the corollaries, a relation between n (the total number of bosonic, or fermionic, fields entering the irrep) and the value N of the extended supersymmetry was established.

A dimensionality $d_i = d_1 + \frac{i-1}{2}$ (d_1 is an arbitrary constant) can be assigned to the fields entering an irrep. The difference in dimensionality between a given bosonic and a given fermionic field is a half-integer number. The fields content of an irrep is the set of integers (n_1, n_2, \ldots, n_l) specifying the number n_i of fields of dimension d_i entering the irrep. Physically, the n_l fields of highest dimension are the auxiliary fields which transform as a time-derivative under any supersymmetry generator. The maximal value l (corresponding to the maximal dimensionality d_l) is known as the length of the irrep. Either n_1, n_3, \ldots correspond to the bosonic fields (therefore n_2, n_4, \ldots specify the fermionic fields) or viceversa. In both cases the equality $n_1+n_3+\ldots=n_2+n_4+\ldots=n$ is guaranteed. A multiplet is bosonic (fermionic) if its n_1 component fields of lower dimensions are bosonic (fermionic). The representation theory does not discriminate the overall bosonic or fermionic nature of the multiplet.

In [2] the allowed (n_1, n_2, \ldots, n_l) fields contents of the N-extended (1) superalgebra were classified (the results were explicitly furnished for $N \leq 10$). In [5] it was further pointed out that an equivalence relation could be introduced in such a way that the fields content uniquely specifies the irreps in the given class. On physical grounds, irreps with different fields content produce quite different supersymmetric physical systems. For instance, the fields content determines the dimensionality of the target space of the one-dimensional N-extended supersymmetric sigma models, see e.g. [6]. Similarly, dimensional reductions of supersymmetric field theories produce extended supersymmetric one-dimensional quantum mechanical systems with specific field contents, see e.g. [7]. The classification of the (1) irreps fields contents has very obvious physical meaning. This part of the program of classifying irreps, due to [2], can now be considered completed.

The (1) irreps were investigated in [3] in terms of filtered Clifford modules. In [3] and [4] it was pointed out that certain irreps admitting the same fields content can be regarded as inequivalent. These results were obtained by analyzing the "connectivity properties" (more on that later) of certain graphs associated to the irreps. A notion of equivalence

class among irreps (spotting their difference in "connectivity") was introduced. In [4], two examples were explicitly presented. They involved a pair of N = 6 irreps with (6, 8, 2)fields content and a pair of N = 5 irreps with (6, 8, 2) fields content. In [4] the classification of the irreps which differ by connectivity was left as an open problem.

In this letter we point out that, using the approach of [2], we can easily classify the connectivity properties of the irreps of given fields contents. The explicit results will be presented for $N \leq 8$. Since the $N \leq 4$ cases are trivial, the connectivity being uniquely determined by the fields content, we explicitly present the results for N = 5, 6, 7, 8.

The connectivity of the irreps (inspired by the graphical presentation of the irreps known as "Adinkras" [8]) can be understood as follows. For the class of irreducible representations under consideration, any given field of dimension d is mapped, under a supersymmetry transformation, either

a) to a field of dimension $d + \frac{1}{2}$ belonging to the multiplet^{*} or,

b) to the time-derivative of a field of dimension $d - \frac{1}{2}$.

If the given field belongs to an irrep of the N-extended (1) supersymmetry algebra, therefore $k \leq N$ of its transformations are of type a), while the N - k remaining ones are of type b). Let us now specialize our discussion to a length-3 irrep (the interesting case for us). Its fields content is given by $(n_1, n, n - n_1)$, while the set of its fields is expressed by $(x_i; \psi_j; g_k)$, with $i = 1, \ldots, n_1, j = 1, \ldots, n, k = 1, \ldots, n - n_1$. The x_i 's are 0-dimensional fields (the ψ_j are $\frac{1}{2}$ -dimensional and the g_k 1-dimensional fields, respectively). The connectivity associated to the given multiplet is defined in terms of the ψ_g symbol. It encodes the following information. The $n \frac{1}{2}$ -dimensional fields ψ_j are partitioned in the subsets of m_r fields admitting k_r supersymmetry transformations of type a) (k_r can take the 0 value). We have $\sum_r m_r = n$. The ψ_g symbol is expressed as

$$\psi_g \equiv m_{1k_1} + m_{2k_2} + \dots \tag{2}$$

As an example, the N = 7 (6, 8, 2) multiplet admits connectivity $\psi_g = 6_2 + 2_1$ (see (8)). It means that there are two types of fields ψ_j . 6 of them are mapped, under supersymmetry transformations, in the two auxiliary fields g_k . The two remaining fields ψ_j are only mapped into a single auxiliary field.

An analogous symbol, x_{ψ} , can be introduced. It describes the supersymmetry transformations of the x_i fields into the ψ_j fields. This symbol is, however, always trivial. An *N*-irrep with $(n_1, n, n - n_1)$ fields content always produce $x_{\psi} \equiv n_{1N}$.

Using the methods of [2], we are able to classify here the admissible ψ_g connectivities of the irreps. The pair of N = 6 (6, 8, 2) irreps and the pair of N = 5 (6, 8, 2) irreps of [4] fall into the two admissible classes of ψ_g connectivity for the corresponding values of Nand fields content.

In [4] the two sets of three ordered numbers (for length-3 multiplets), $S = [s_1, s_2, s_3]$ and $T = [t_1, t_2, t_3]$, the "sources" and "targets" respectively, have been introduced. The integer s_i gives the number of fields of dimension $d_i = \frac{i-1}{2}$ which do not result as an

a)-supersymmetry transformation of at least one field of dimension $d_i - \frac{1}{2}$. The integer t_i gives the number of fields of dimension $d_i = \frac{i-1}{2}$ which only admit supersymmetry transformations of type *b*). For a multiplet of $(n_1, n, n - n_1)$ fields content, necessarily

^{*}or to its opposite, the sign of the transformation being irrelevant for our purposes.

 $s_1 = n_1, s_3 = 0$, together with $t_1 = 0$ and $t_3 = n - n_1$. S, and T are fully determined once s_2 and t_2 , respectively, are known. The complete list of ψ_g connectivities for length-3 multiplets contains more information than S and T. As for the targets, it is obvious that t_2 can be recovered from ψ_g . As for the sources, using the $(n_1, n, n - n_1) \leftrightarrow (n - n_1, n, n_1)$ irreps duality discussed in [2], s_2 is recovered from the ψ_g connectivity of the associated dual multiplet. In Section **3** we produce the list of the allowed connectivities. We prove that the connectivity symbol ψ_g allows to discriminate inequivalent irreps which are not discriminated by the sources and targets S, T (of given heighth) introduced in [4]. In Section **4** we summarize the previous results, presenting the full list of $N \leq 8$ irreps differing by sources and targets, as well as the full list of $N \leq 8$ irreps with the same sources and targets and different ψ_g connectivity. We explicitly present the N = 5 supersymmetry transformations for two such irreps. We also present them graphically (the associated "adinkras"). We postpone to the Conclusions a discussion of the possible interpretations of our finding.

This paper is structured as follows. In the next Section, the needed ingredients and [2] conventions are reviewed. The main results are presented in Section **3**. The irreps connectivities are furnished for all cases which can potentially produce inequivalent results (therefore, for the N = 5, 6, 7 length-3 and length-4 irreps). In Section **4** it is pointed out that the ψ_g connectivities computed in Section **3** can discriminate irreps which are not discriminated by the sets of "sources and targets" numbers employed in [4]. Further comments and open problems are discussed in the Conclusions. To make the paper self-consistent, an Appendix with our conventions of the Cl(0,7) Clifford generators (used to construct the N = 5, 6, 7, 8 supersymmetry operators) is added.

2 Basic notions and conventions

In this Section we summarize the basic notions, results and conventions of [2] that will be needed in the following. Up to $N \leq 8$, inequivalent connectivities are excluded for N = 1, 2, 3, 4 and can only appear, in principle, for N = 5, 6, 7, 8. The irreps of the N = 5, 6, 7, 8 supersymmetric extensions can be obtained through a dressing of the N = 8length-2 root multiplet (see [2] and the comment in [5]). For simplicity, we can therefore limit the discussion of the [2] construction starting from the N = 8 length-2 root multiplet. It involves 8 bosonic and 8 fermionic fields entering a column vector (the bosonic fields are accommodated in the upper part). The 8 supersymmetry operators \hat{Q}_i $(i = 1, \ldots, 8)$ in the (8, 8) N = 8 irrep are given by the matrices

$$\widehat{Q}_j = \begin{pmatrix} 0 & \gamma_j \\ -\gamma_j \cdot H & 0 \end{pmatrix}, \qquad \widehat{Q}_8 = \begin{pmatrix} 0 & \mathbf{1}_8 \\ \mathbf{1}_8 \cdot H & 0 \end{pmatrix}$$
(3)

where the γ_j matrices (j = 1, ..., 7) are the 8 × 8 generators of the Cl(0, 7) Clifford algebra and $H = i\frac{d}{dt}$ is the hamiltonian. The Cl(0, 7) Clifford irrep is uniquely defined up to similarity transformations and an overall sign flipping [9]. Without loss of generality we can unambiguously fix the γ_j matrices to be given as in the Appendix. Each γ_j matrix (and the $\mathbf{1}_8$ identity) possesses 8 non-vanishing entries, one in each column and one in each row. The whole set of non-vanishing entries of the eight (A.1) matrices fills the entire $8 \times 8 = 64$ squares of a "chessboard". The chessboard appears in the upper right block of (3).

The length-3 and length-4 N = 5, 6, 7, 8 irreps (no irrep with length l > 4 exists for $N \leq 9$, see [2]) are acted upon by the Q_i 's supersymmetry transformations, obtained from the original \hat{Q}_i operators through a dressing,

$$\widehat{Q}_i \quad \to \quad Q_i = D\widehat{Q}_i D^{-1},\tag{4}$$

realized by a diagonal dressing matrix D. It should be noticed that only the subset of "regular" dressed operators Q_i (i.e., having no $\frac{1}{H}$ or higher poles in its entries) act on the new irreducible multiplet. Apart from the self-dual (4, 8, 4) N = 5, 6 irreps, without loss of generality, for our purpose of computing the irreps connectivities, the diagonal dressing matrix D which produces an irrep with $(n_1, n, n - n_1)$ fields content can be chosen to have its non-vanishing diagonal entries given by $\delta_{pq}d_q$, with $d_q = 1$ for $q = 1, \ldots, n_1$ and $q = n + 1, \ldots, 2n$, while $d_q = H$ for $q = n_1 + 1, \ldots, n$. Any permutation of the first n entries produces a dressing which is equivalent, for computing both the fields content and the ψ_g connectivity, to D. The only exceptions correspond to the N = 5 (4, 8, 4) and N = 6 (4, 8, 4) irreps. Besides the diagonal matrix D as above, inequivalent irreps can be obtained by a diagonal dressing D' with diagonal entries $\delta_{pq}d'_q$, with $d'_q = H$ for q = 4, 6, 7, 8 and $d'_q = 1$ for the remaining values of q.

Similarly, the $(n_1, n_2, n - n_1, n - n_2)$ length-4 multiplets are acted upon by the Q_i operators dressed by D, whose non-vanishing diagonal entries are now given by $\delta_{pq}d_q$, with $d_q = 1$ for $q = 1, \ldots, n_1$ and $q = 2n - n_2 + 1, \ldots, 2n$, while $d_q = H$ for $q = n_1 + 1, \ldots, 2n - n_2$.

The N = 5, 6, 7, 8 length-2 (8, 8) irreps are unique (for the given value of N), see [5]. It is also easily recognized that all N = 8 length-3 irreps of given fields content produce the same value of ψ_g connectivity (2). For what concerns the length-3 N = 5, 6, 7 irreps the situation is as follows. Let us consider the irreps with (k, 8, 8 - k) fields content. Its supersymmetry transformations are defined by picking an N < 8 subset from the complete set of 8 dressed Q_i operators. It is easily recognized that for N = 7, no matter which supersymmetry operator is discarded, any choice of the seven operators produces the same value for the ψ_g connectivity. Irreps with different connectivity can therefore only be found for N = 5, 6. The $\binom{8}{6} = 28$ choices of N = 6 operators fall into two classes, denoted as A and B, which can, potentially, produce (k, 8, 8 - k) irreps with different connectivity. Similarly, the $\binom{8}{5} = 56$ choices of N = 5 operators fall into two A and B classes which can, potentially, produce irreps of different connectivity. For some given (k, 8, 8 - k) irrep, the value of ψ_g connectivity computed in both N = 5 (as well as N = 6) classes can actually coincide. In the next Section we will show when this feature indeed happens.

To be specific, we present a list of representatives of the supersymmetry operators for

each N and in each N = 5, 6 A, B class. We have, with diagonal dressing D,

$$N = 8 \equiv Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8$$

$$N = 7 \equiv Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7$$

$$N = 6 (case A) \equiv Q_1, Q_3, Q_4, Q_5, Q_6, Q_7$$

$$N = 6 (case B) \equiv Q_1, Q_2, Q_3, Q_4, Q_5, Q_6$$

$$N = 5 (case A) \equiv Q_3, Q_4, Q_5, Q_6, Q_7$$

$$N = 5 (case B) \equiv Q_2, Q_3, Q_4, Q_5, Q_6$$
(5)

and, with diagonal dressing D' for the (4, 8, 4) irreps,

$$N = 6 (case A') \equiv Q_1, Q_3, Q_4, Q_5, Q_6, Q_7$$

$$N = 5 (case A') \equiv Q_3, Q_4, Q_5, Q_6, Q_7$$
(6)

We are now in the position to compute the connectivities of the irreps (the results are furnished in the next Section). Quite literally, the computations can be performed by filling a chessboard with pawns representing the allowed configurations.

3 Classification of the irreps connectivities

In this Section we report the results of the computation of the allowed connectivities for the N = 5, 6, 7 length-3 and length-4 irreps. As discussed in the previous Section, the only values of $N \leq 8$ which allow the existence of multiplets with the same fields content but inequivalent connectivities are N = 5 and N = 6. We also produce the S and T allowed sources and targets numbers for the irreps. As recalled in the Introduction, the S sources can be recovered from a symbol, denoted as " $_x\psi$ ", expressing the partitions of the $n\frac{1}{2}$ dimensional fields ψ_j in terms of the $h_r \leq N$ number of supersymmetry transformations of a) type which map the x_i fields on a given $\frac{1}{2}$ -dimensional field. Due to the irrep $(n_1, n, n - n_1) \leftrightarrow (n - n_1, n, n_1)$ duality discussed in [2], $_x\psi$ is recovered from the ψ_g connectivity of its dual irrep. Indeed

$$_{x}\psi[(k,n,n-k)_{*}] = \psi_{g}[(n-k,n,k)_{*}]$$
(7)

(the suffix $* \equiv A, B$ has been introduced in order to discriminate, when needed, the A and B subcases of N = 5, 6).

Our results concerning the allowed ψ_q connectivities of the length-3 irreps are reported

length - 3	N = 7	N = 6	N = 5
(7, 8, 1)	$7_1 + 1_0$	$6_1 + 2_0$	$5_1 + 3_0$
(6, 8, 2)	$6_2 + 2_1$	$6_2 + 2_0 (A)$	$4_2 + 2_1 + 2_0 (A)$
		$4_2 + 4_1 (B)$	$2_2 + 6_1 (B)$
(5, 8, 3)	$5_3 + 3_2$	$4_3 + 2_2 + 2_1 (A)$	$4_3 + 3_1 + 1_0 (A)$
		$2_3 + 6_2 (B)$	$1_3 + 5_2 + 2_1 (B)$
(4, 8, 4)	$4_4 + 4_3$	$4_4 + 4_2 (A)$	$4_4 + 4_1 (A)$
		$2_4 + 4_3 + 2_2 (A')$	$1_4 + 3_3 + 3_2 + 1_1 (A')$
		$8_3 (B)$	$4_3 + 4_2 (B)$
(3, 8, 5)	$3_5 + 5_4$	$2_5 + 2_4 + 4_3 (A)$	$1_5 + 3_4 + 4_2 (A)$
		$6_4 + 2_3 (B)$	$2_4 + 5_3 + 1_2 (B)$
(2, 8, 6)	$2_6 + 6_5$	$2_6 + 6_4 (A)$	$2_5 + 2_4 + 4_3 (A)$
		$4_5 + 4_4 (B)$	$6_4 + 2_3 (B)$
(1, 8, 7)	$1_7 + 7_6$	$2_6 + 6_5$	$3_5 + 5_4$

in the following table (the A, A', B cases of N = 5, 6 are specified)

(8)

The ψ_g connectivities of the N = 5 (and N = 6) A and B subcases collapse to the same value for the (1, 8, 7) and (7, 8, 1) irreps, proving that these multiplets do not admit inequivalent connectivities.

It is helpful to produce tables with the values of the ψ_g connectivity, the S sources and the T targets for the irreps admitting inequivalent connectivities. For N = 6 we get

N = 6:	connectivities	sources	targets
$(6, 8, 2)_A$	$6_2 + 2_0$	S = [6, 0, 0]	T = [0, 2, 2]
$(6, 8, 2)_B$	$4_2 + 4_1$	S = [6, 0, 0]	T = [0, 0, 2]
$(5, 8, 3)_A$	$4_3 + 2_2 + 2_1$	S = [5, 0, 0]	T = [0, 0, 3]
$(5, 8, 3)_B$	$2_3 + 6_2$	S = [5, 0, 0]	T = [0, 0, 3]
$(4, 8, 4)_A$	$4_4 + 4_2$	S = [4, 0, 0]	T = [0, 0, 4]
$(4, 8, 4)_{A'}$	$2_4 + 4_3 + 2_2$	S = [4, 0, 0]	T = [0, 0, 4]
$(4, 8, 4)_B$	8_{3}	S = [4, 0, 0]	T = [0, 0, 4]
$(3, 8, 5)_A$	$2_5 + 2_4 + 4_3$	S = [3, 0, 0]	T = [0, 0, 5]
$(3, 8, 5)_B$	$6_4 + 2_3$	S = [3, 0, 0]	T = [0, 0, 5]
$(2, 8, 6)_A$	$2_6 + 6_4$	S = [2, 2, 0]	T = [0, 0, 6]
$(2, 8, 6)_B$	$4_5 + 4_4$	S = [2, 0, 0]	T = [0, 0, 6]

(9)

For N = 5 we obtain

N = 5:	connectivities	sources	targets
$(6, 8, 2)_A$	$4_2 + 2_1 + 2_0$	S = [6, 0, 0]	T = [0, 2, 2]
$(6, 8, 2)_B$	$2_2 + 6_1$	S = [6, 0, 0]	T = [0, 0, 2]
$(5, 8, 3)_A$	$4_3 + 3_1 + 1_0$	S = [5, 0, 0]	T = [0, 1, 3]
$(5, 8, 3)_B$	$1_3 + 5_2 + 2_1$	S = [5, 0, 0]	T = [0, 0, 3]
$(4, 8, 4)_A$	$4_4 + 4_1$	S = [4, 0, 0]	T = [0, 0, 4]
$(4, 8, 4)_{A'}$	$1_4 + 3_3 + 3_2 + 1_1$	S = [4, 0, 0]	T = [0, 0, 4]
$(4, 8, 4)_B$	$4_3 + 4_2$	S = [4, 0, 0]	T = [0, 0, 4]
$(3, 8, 5)_A$	$1_5 + 3_4 + 4_2$	S = [3, 1, 0]	T = [0, 0, 5]
$(3, 8, 5)_B$	$2_4 + 5_3 + 1_2$	S = [3, 0, 0]	T = [0, 0, 5]
$(2, 8, 6)_A$	$2_5 + 2_4 + 4_3$	S = [2, 2, 0]	T = [0, 0, 6]
$(2, 8, 6)_B$	$6_4 + 2_3$	S = [2, 0, 0]	T = [0, 0, 6]

We postpone to Section 4 the discussion of our results.

3.1 Connectivities of the length-4 multiplets

Up to $N \leq 8$, the only admissible $(n_1, n_2, n - n_1, n - n_2)$ length-4 fields contents for the $(x_i; \psi_j; g_k; \omega_l)$ irreps are given below (see [2]). Here x_i $(i = 1, ..., n_1)$ denote the 0dimensional fields, ψ_j $(j = 1, ..., n_2)$ denote the $\frac{1}{2}$ -dimensional fields, g_k $(k = 1, ..., n - n_1)$ denote the 1-dimensional fields and, finally, ω_l $(l = 1, ..., n - n_2)$ denote the $\frac{3}{2}$ -dimensional auxiliary fields.

The analysis of the connectivities of the length-4 irreps is done as in the case of the length-3 irreducible multiplets. Since we have an extra set of fields w.r.t. the length-3 multiplets, the results can be expressed in terms of one more non-trivial symbol. Besides ψ_g , we introduce the g_{ω} symbol as well. The definition of g_{ω} follows the definition of ψ_g in (2). The difference of g_{ω} w.r.t. ψ_g is that the g_k fields enter now in the place of the ψ_j fields, while the ω_l fields enter in the place of the g_k fields.

Contrary to the case of the length-3 irreps, the connectivity of the length-4 irreps is uniquely specified in terms of N and the length-4 fields content. The complete list of results is presented in the following table

length - 4	su.sies	ψ_g	g_{ω}
(1,7,7,1):	N = 7	7_6	7_1
	N = 6	$1_6 + 6_5$	$6_1 + 1_0$
	N = 5	$2_5 + 5_4$	$5_1 + 2_0$
(2,7,6,1):	N = 6	$1_6 + 6_4$	6_1
	N = 5	$1_5 + 2_4 + 4_3$	$5_1 + 1_0$
(2, 6, 6, 2):	N = 6	6_4	6_2
	N = 5	$2_4 + 4_3$	$4_2 + 2_1$
(1, 6, 7, 2):	N = 6	6_{5}	$6_2 + 1_0$
	N = 5	$1_5 + 5_4$	$4_2 + 2_1 + 1_0$
(1,5,7,3):	N = 5	5_{4}	$4_3 + 3_1$
(3,7,5,1):	N = 5	$3_4 + 4_2$	5_1
(1,3,3,1):	N = 3	3_2	3_1

(11)

(10)

4 On "irreps connectivities" versus "sources and targets"

From the results presented in (9) and (10) we obtain two corollaries. At first we notice that, besides the N = 6 (6, 8, 2) and N = 5 (6, 8, 2) pairs of cases presented in [4], there exists four extra pairs, for $N \leq 8$, of inequivalent irreps with the same fields content which differ by the values of the sources and targets. The whole list of such pairs is given by

$$N = 6: (6, 8, 2)_A \leftrightarrow (6, 8, 2)_B$$

$$N = 6: (2, 8, 6)_A \leftrightarrow (2, 8, 6)_B$$

$$N = 5: (6, 8, 2)_A \leftrightarrow (6, 8, 2)_B$$

$$N = 5: (5, 8, 3)_A \leftrightarrow (5, 8, 3)_B$$

$$N = 5: (3, 8, 5)_A \leftrightarrow (3, 8, 5)_B$$

$$N = 5: (2, 8, 6)_A \leftrightarrow (2, 8, 6)_B$$
(12)

The above list produces the complete classification of inequivalent $N \leq 8$ irreps that are discriminated by different values of S and T alone.

On the other hand, a second corollary of the (9) and (10) results shows the existence of extra irreps sharing the same fields content $(n_1, n, n-n_1)$, the same sources $S = [s_1, s_2, s_3]$ and the same targets $T = [t_1, t_2, t_3]$ which, nevertheless, admit different ψ_g connectivity. They are given by

$$N = 6: \quad (3, 8, 5)_A \leftrightarrow (3, 8, 5)_B$$

$$N = 6: \quad (4, 8, 4)_A \leftrightarrow (4, 8, 4)_{A'} \leftrightarrow (4, 8, 4)_B$$

$$N = 6: \quad (5, 8, 3)_A \leftrightarrow (5, 8, 3)_B$$

$$N = 5: \quad (4, 8, 4)_A \leftrightarrow (4, 8, 4)_{A'} \leftrightarrow (4, 8, 4)_B$$
(13)

In order to convince the reader of the existence of such irreps with same sources and targets but different connectivity it is useful to explicitly present the supersymmetry transformations (depending on the ε_i global fermionic parameters) in at least one case. We write below a pair of N = 5 irreps (the $(4, 8, 4)_A$ and the $(4, 8, 4)_B$ multiplets) differing by connectivity, while admitting the same number of sources and the same number of targets. It is also convenient to visualize them graphically as adinkras (see [8]). The graphical presentation at the end of this Section is given as follows. Three rows of (from bottom to up) 4, 8 and 4 dots are associated with the x_i , ψ_j and g_k fields, respectively. Supersymmetry transformations are represented by lines of 5 different colors (since N =5). Solid lines are associated to transformations with a positive sign, dashed lines with a negative sign. It is easily recognized that in the type A graph there are $4 \psi_j$ points with four colored lines connecting them to the g_k points. In the type B graph we have $4 \psi_j$ points with three colored lines and the 4 remaining ψ_j points with two colored lines connecting them to the g_k points.

The supersymmetry transformations are explicitly given by

i) The $N = 5 (4, 8, 4)_A$ transformations:

$$\begin{split} \delta x_1 &= \varepsilon_2 \psi_3 + \varepsilon_4 \psi_5 + \varepsilon_3 \psi_6 + \varepsilon_1 \psi_7 + \varepsilon_5 \psi_8 \\ \delta x_2 &= \varepsilon_2 \psi_4 + \varepsilon_3 \psi_5 - \varepsilon_4 \psi_6 - \varepsilon_5 \psi_7 + \varepsilon_1 \psi_8 \\ \delta x_3 &= -\varepsilon_2 \psi_1 - \varepsilon_1 \psi_5 - \varepsilon_5 \psi_6 + \varepsilon_4 \psi_7 + \varepsilon_3 \psi_8 \\ \delta x_4 &= -\varepsilon_2 \psi_2 + \varepsilon_5 \psi_5 - \varepsilon_1 \psi_6 + \varepsilon_3 \psi_7 - \varepsilon_4 \psi_8 \\ \delta \psi_1 &= -i\varepsilon_2 \dot{x}_3 - \varepsilon_4 g_1 - \varepsilon_3 g_2 - \varepsilon_1 g_3 - \varepsilon_5 g_4 \\ \delta \psi_2 &= -i\varepsilon_2 \dot{x}_4 - \varepsilon_3 g_1 + \varepsilon_4 g_2 + \varepsilon_5 g_3 - \varepsilon_1 g_4 \\ \delta \psi_3 &= i\varepsilon_2 \dot{x}_1 + \varepsilon_1 g_1 + \varepsilon_5 g_2 - \varepsilon_4 g_3 - \varepsilon_3 g_4 \\ \delta \psi_4 &= i\varepsilon_2 \dot{x}_2 - \varepsilon_5 g_1 + \varepsilon_1 g_2 - \varepsilon_3 g_3 + \varepsilon_4 g_4 \\ \delta \psi_5 &= i\varepsilon_4 \dot{x}_1 + i\varepsilon_3 \dot{x}_2 - i\varepsilon_1 \dot{x}_3 + i\varepsilon_5 \dot{x}_4 + \varepsilon_2 g_3 \\ \delta \psi_6 &= i\varepsilon_3 \dot{x}_1 - i\varepsilon_4 \dot{x}_2 - i\varepsilon_5 \dot{x}_3 - i\varepsilon_4 \dot{x}_4 - \varepsilon_2 g_1 \\ \delta \psi_8 &= i\varepsilon_5 \dot{x}_1 + i\varepsilon_1 \dot{x}_2 + i\varepsilon_3 \dot{x}_3 - i\varepsilon_4 \dot{x}_4 - \varepsilon_2 g_2 \\ \delta g_1 &= -i\varepsilon_4 \dot{\psi}_1 - i\varepsilon_3 \dot{\psi}_2 + i\varepsilon_1 \dot{\psi}_3 - i\varepsilon_5 \dot{\psi}_4 - i\varepsilon_2 \dot{\psi}_7 \\ \delta g_2 &= -i\varepsilon_3 \dot{\psi}_1 + i\varepsilon_4 \dot{\psi}_2 + i\varepsilon_5 \dot{\psi}_3 + i\varepsilon_1 \dot{\psi}_4 - i\varepsilon_2 \dot{\psi}_8 \\ \delta g_3 &= -i\varepsilon_1 \dot{\psi}_1 + i\varepsilon_5 \dot{\psi}_2 - i\varepsilon_4 \dot{\psi}_3 - i\varepsilon_3 \dot{\psi}_4 + i\varepsilon_2 \dot{\psi}_5 \\ \delta g_4 &= -i\varepsilon_5 \dot{\psi}_1 - i\varepsilon_1 \dot{\psi}_2 - i\varepsilon_3 \dot{\psi}_3 + i\varepsilon_4 \dot{\psi}_4 + i\varepsilon_2 \dot{\psi}_6 \end{split}$$

$$(14)$$

ii) The $N = 5 \ (4, 8, 4)_B$ transformations:

$$\begin{split} \delta x_{1} &= \varepsilon_{5}\psi_{2} + \varepsilon_{2}\psi_{3} + \varepsilon_{4}\psi_{5} + \varepsilon_{3}\psi_{6} + \varepsilon_{1}\psi_{7} \\ \delta x_{2} &= -\varepsilon_{5}\psi_{1} + \varepsilon_{2}\psi_{4} + \varepsilon_{3}\psi_{5} - \varepsilon_{4}\psi_{6} + \varepsilon_{1}\psi_{8} \\ \delta x_{3} &= -\varepsilon_{2}\psi_{1} - \varepsilon_{5}\psi_{4} - \varepsilon_{1}\psi_{5} + \varepsilon_{4}\psi_{7} + \varepsilon_{3}\psi_{8} \\ \delta x_{4} &= -\varepsilon_{2}\psi_{2} + \varepsilon_{5}\psi_{3} - \varepsilon_{1}\psi_{6} + \varepsilon_{3}\psi_{7} - \varepsilon_{4}\psi_{8} \\ \delta \psi_{1} &= -i\varepsilon_{5}\dot{x}_{2} - i\varepsilon_{2}\dot{x}_{3} - \varepsilon_{4}g_{1} - \varepsilon_{3}g_{2} - \varepsilon_{1}g_{3} \\ \delta \psi_{2} &= i\varepsilon_{5}\dot{x}_{1} - i\varepsilon_{2}\dot{x}_{4} - \varepsilon_{3}g_{1} + \varepsilon_{4}g_{2} - \varepsilon_{1}g_{4} \\ \delta \psi_{3} &= i\varepsilon_{2}\dot{x}_{1} + i\varepsilon_{5}\dot{x}_{4} + \varepsilon_{1}g_{1} - \varepsilon_{4}g_{3} - \varepsilon_{3}g_{4} \\ \delta \psi_{4} &= i\varepsilon_{2}\dot{x}_{2} - i\varepsilon_{5}\dot{x}_{3} + \varepsilon_{1}g_{2} - \varepsilon_{3}g_{3} + \varepsilon_{4}g_{4} \\ \delta \psi_{5} &= i\varepsilon_{4}\dot{x}_{1} + i\varepsilon_{3}\dot{x}_{2} - i\varepsilon_{1}\dot{x}_{4} - \varepsilon_{5}g_{2} + \varepsilon_{2}g_{3} \\ \delta \psi_{6} &= i\varepsilon_{3}\dot{x}_{1} - i\varepsilon_{4}\dot{x}_{2} - i\varepsilon_{1}\dot{x}_{4} - \varepsilon_{2}g_{1} + \varepsilon_{5}g_{4} \\ \delta \psi_{7} &= i\varepsilon_{1}\dot{x}_{1} + i\varepsilon_{4}\dot{x}_{3} + i\varepsilon_{3}\dot{x}_{4} - \varepsilon_{2}g_{2} - \varepsilon_{5}g_{3} \\ \delta g_{1} &= -i\varepsilon_{4}\dot{\psi}_{1} - i\varepsilon_{3}\dot{\psi}_{2} + i\varepsilon_{1}\dot{\psi}_{4} - i\varepsilon_{5}\dot{\psi}_{5} - i\varepsilon_{2}\dot{\psi}_{8} \\ \delta g_{3} &= -i\varepsilon_{1}\dot{\psi}_{1} - i\varepsilon_{4}\dot{\psi}_{3} - i\varepsilon_{3}\dot{\psi}_{4} + i\varepsilon_{2}\dot{\psi}_{5} - i\varepsilon_{5}\dot{\psi}_{8} \\ \delta g_{4} &= -i\varepsilon_{1}\dot{\psi}_{2} - i\varepsilon_{3}\dot{\psi}_{3} + i\varepsilon_{4}\dot{\psi}_{4} + i\varepsilon_{2}\dot{\psi}_{6} + i\varepsilon_{5}\dot{\psi}_{7} \end{split}$$
(15)

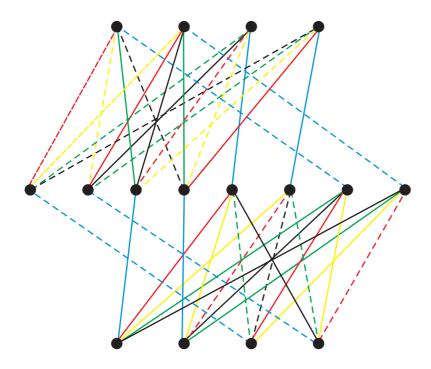


Figure 1: Adinkra of the N = 5 (4, 8, 4) multiplet of $4_4 + 4_1$ connectivity (type A).

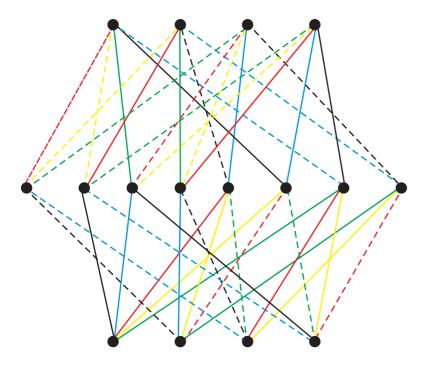


Figure 2: Adinkra of the N = 5 (4, 8, 4) multiplet of $4_3 + 4_2$ connectivity (type B).

5 Conclusions

In this paper we computed the allowed connectivities of the finite linear irreducible representations of the (1) supersymmetry algebra. For length-3 irreps the connectivity is encoded in the ψ_g symbol (2) which specifies how the fields in an irrep are linked together by supersymmetry transformations. For $N \leq 8$ we classified which irreps with the same fields content admit different connectivities (they only exist for N = 5, 6). As a corollary, we classified the irreps with inequivalent "sources and targets".

After the first version of this paper appeared, a revision of [4] was produced. It was pointed out the existence of an extra irrep, missed in our first version, corresponding to the N = 5 $(4, 8, 4)_{A'}$ irrep. The extra cases w.r.t. our previous version of this paper can only appear for self-dual (4, 8, 4) multiplets. Besides the N = 5 $(4, 8, 4)_{A'}$ irrep, we found a second extra case given by the N = 6 $(4, 8, 4)_{A'}$ irrep.

Concerning the [4] reply to our previous comments, we limit ourselves to point out that we defined and introduced the ψ_g symbol as a *quantitative way* of discriminating irreps of inequivalent connectivities. The quantitative discrimination explicitly discussed in [4], based on the number of sources and targets of given height, only allows to spot the difference between the inequivalent irreps appearing in the first version of [4] (and few extra cases). It fails to spot a difference for the large class of inequivalent irreps here discussed.

The approach here discussed can be straighforwardly generalized to compute the connectivities of the $N \geq 9$ irreps of [2]. Concerning physical applications, irreps were classified according to their fields content in [2]. The differences in fields content have obvious physical meanings (as already recalled, irreps with different fields content produce, e.g., one-dimensional supersymmetric sigma models which are embedded in target manifolds of different dimensionality, see [6]). In order to understand the physical implications of the irreps with same fields content but different connectivity, it would be quite important to construct off-shell invariant actions for such irreps. As far as we know, the construction of such off-shell invariant actions has not been accomplished yet. For N = 8a large class of off-shell invariant actions, for each given irrep, has been constructed in [6]. The list in [6] is not exhaustive (see, e.g., [2], where an extra off-shell invariant action was produced). It is possible, but unlikely, that the problem of constructing off-shell invariant actions for multiplets with different connectivities could be solved with the [6] formalism of constrained superfields (since we are dealing with N > 4 systems). It is unclear in fact how to constrain the superfields in the cases under consideration. On the other hand, the linear supersymmetry transformations of the irreps are already given. It therefore looks promising to use the "linear" approach developed in [2]. We are planning to address this problem in the future. Another issue deserving investigation concerns the puzzling similarities shared by both linear and non-linear representations of the (1) supersymmetry algebra, see e.g. [10] for a recent discussion. One of the main motivations of the present work concerns the understanding of the features of the large-N supersymmetric quantum mechanical systems, due to their implications in the formulation of the *M*-theory, see the considerations in [11] and [7]. The dimensional reduction of the 11-dimensional maximal supergravity (thought as the low-energy limit of the *M*-theory) produces an N = 32supersymmetric one-dimensional quantum mechanical system.

Comments on hep-th/0611060v2

The wise Reader is warmly invited to skip this part, which adds nothing to the paper and is only intended to reply to some statements contained in hep-th/0611060v2.

The authors of hep-th/0611060v2 stated:

1) that in the first version of this paper we misquoted Theorem 4.1 in Ref. [5] of their paper.

We point out that

a) We cannot have misquoted Ref. [5] because we did not cite Ref. [5] (math-ph/0512016).

b) We correctly quoted the statements made in hep-th/0611060 about the application of Theorem 4.1.

Indeed, in hep-th/0611060, page 4, it is written (let's call it *Statement A*):

"Theorem 4.1 and Corollary 4.2 from Ref. [5] ensure that every Adinkra is **uniquely specified**, respectively, either by its *set of targets* or by its *set of sources* and the height assignment of these." (boldface ours).

In page 2 it is explained what sources and targets are:

"a node is a *source* if no lower node connects to it, and a *target* if no higher node connects to it."

Nodes are defined a little before, in the same page 2:

"Adinkras represent component bosons in a supermultiplet as white nodes, and fermions as black nodes. A white and a black node are connected by an edge, ...".

The claim in version 2, footnote (5), that the set of sources

"Being a *subgraph* of the Adinkra, they are specified by their connection to the rest of the Adinkra"

is incorrect. Sources or targets are vertices (a set of sources is a set of points). A graph or a subgraph contains points *and edges*.

The missed information in hep-th/0611060 is that the topology of the Adinkra is supposed to be given. In math-ph/0512016 it is written at page 12: "Theorem 4.1 Suppose we are given (1) a topology of an Adinkra (that is, a graph that could be the underlying graph of an Adinkra), ...".

In hep-th/0611060v1, two pairs of examples are discussed s.t. Statement A correctly applies: the inequivalence in these pairs of examples is spotted by the set of sources and targets of given heigth (expressed as (s_1, s_2, s_3) , (t_1, t_2, t_3)). No further information is required. In our paper we produced a class of examples s.t. Statement A no longer applies and further information is required. For these cases we introduced and defined the ψ_g (or, equivalently, the $_x\psi$) symbol to spot the difference between inequivalent irreps. In hep-th/0611060v1 no other examples, besides the two pairs, were given and no discussion was made on how to spot the differences for the more general class of cases we produced. When discussing our examples in hep-th/0611060v2, the authors are forced to rephrase, in words, the information contained in our defined ψ_g and $_x\psi$ symbols. At page 8 we read "the distinction is clearly displayed not by their number, but by the different connectivity to the rest: In the left Adinkra, the four source-bosons connect to four of the fermions by a single edge, and by four edges to the other four fermions; not so in the right-hand side one. No field redefinition can erase this topological distinction." To spot the difference between the two Adinkras here the authors employ, without mentioning it,

the $_x\psi$ symbol defined in this paper (in formula (6) of the previous version, formula (7) of the present version).

2) On page 3, in footnote (2) ("We find it hard to pinpoint what Ref. [12] in fact does claim: ...") and footnote (3) ("Ref [12] in fact lists two N = 5 irreps ...") the hep-th/0611060 authors act pretending to ignore the second letter sent to them by us and M. Rojas (the authors of Ref. [12], namely JHEP **0603** (2006) 098) on Dec. 13, 2006, which answered the questions raised. For completeness, the letter, e-mailed to all six authors of hep-th/0611060, is reported below.

"Dear colleagues,

with respect to your reply, we summarize our considerations in the following list of statements.

1) In our KRT paper we classified the irreps of the 1D N-susy algebra according to the number n_i of fields of dimension $d_i = d_0 + i/2$ entering an irrep. The complete list of $(n_1, n_2, ..., n_l)$ symbols (from now on the "field content of the irreps"), explicitly presented up to $N \leq 10$ (formulas 4.10 for $N \leq 8$, B.3 for N = 9, B.7 and B.11 for N = 10 and l > 4, while the l = 2,3 cases were known from PT) was given. $(n_1, ..., n_l)$ belongs to the list if and only if there exists at least one N-irrep with the given field content.

This is a complete and, as far as we know, correct result (no counterexample so far has been found).

One valid counterexample would amount to

a) find a $(n_1, ..., n_l)$ field content incorrectly inserted (for some given N) in the list or

b) find a $(n_1, ..., n_l)$ field content incorrectly not inserted (for some given N) in the list.

Your two N = 6 (6,8,2) examples do not qualify to counterexample of the above statements ((6,8,2) is a valid field content of N = 6, already present in PT).

2) We acknowledge a somewhat confused presentation in KRT of the $N = 3, 5 \mod 8$ irreps, due to the "double oxidation" of N = 3, 5. Unlike the other values of N, the $N = 3 \mod (\text{length-2})$ irrep can be oxidized to either the N = (4, 0) susy or the N = (3, 3) pseudosusy. On the other hand, the reduction to N = 3 of both N = (3, 3) and N = (4, 0) produces one and the same irrep (Similarly for N = 5). This last statement was not clear to us when writing KRT. It is clear now (it is a consequence of the length-2 \leftrightarrow Clifford algebra correspondence).

3) Irreps and their equivalence classes: KRT was written for a physical, not a mathematical journal. It contains the classification at point 1) as well as other results (invariant actions, etc.) relevant for physicists. Less important (in a physicist's perspective) issues were skipped. This includes the straightforward and somehow formal definition of the class of transformations acting on irreps defining an equivalence relation s.t. any two irreps with the same field content fall into the same class of equivalence (it will be soon presented elsewhere). As stated in our previous mail, we have always been aware of the possibility that a refinement of our classification could be introduced by choosing another group of transformations defining another equivalence relation.

Your results in no way contradict our findings. They complement it. You do not produce any irrep of, let's say, N = 6 with, let's say, (3, 7, 5, 1) field content.

In KRT we produced a valid and complete classification of the field contents of the irreps. Using a valid analogy (simple Lie algebras over C are classified by Dynkin's dia-

grams, while simple Lie algebras over R are obtained by the real forms) we can say that our KRT classification corresponds to the "Dynkin diagrams" of the irreps. Your two (according to your definitions) inequivalent N = 6 (6,8,2) irreps can be regarded as two "real forms" of our unique "Dynkin diagram".

Concerning your reply, you do not formulate any real objection to our classification at point 1). You limit to object to our presentation of it as a "complete classification of the irreps". The fact that the equivalence relation was not explicitly presented in KRT (for the reasons mentioned at point 3) does not however authorize you to assume that this equivalence relation does not exist and/or that your definition of the equivalence relation is the only acceptable.

Considering the other points raised in your reply (the status of the N = 5 reduced from N = 8) they can be easily answered. A detailed clarification of these issues will be soon produced in a forthcoming paper.

Sincerely Yours,

Zhanna Kuznetsova, Moises Rojas and Francesco Toppan E-mails: zhanna@cbpf.br mrojas@cbpf.br toppan@cbpf.br Notes: KRT: ref. Kuznetsova-Rojas-Toppan, JHEP 0603 (2006) 098. PT: ref. Pashnev-Toppan, JMP 42 (2001) 5257. " As promised in the letter, these issues were further clarified by one of us (F.T.) in hepth/0612276.

Appendix

We present here for completeness the set (unique up to similarity transformations and an overall sign flipping) of the seven 8×8 gamma matrices γ_i which generate the Cl(0,7)Clifford algebra. The seven gamma matrices, together with the 8-dimensional identity $\mathbf{1}_8$, are used in the construction of the N = 5, 6, 7, 8 supersymmetry irreps, as explained in the main text.

$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0$	$\gamma_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0$
$\gamma_{3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\gamma_4 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0$
$\gamma_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0$	$\gamma_{6} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$
$\gamma_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$1_{8} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} $ (A.1)

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