

**TOWARD LOGARITHMIC EXTENSIONS OF $\widehat{sl}(2)_k$
CONFORMAL FIELD MODELS**

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ABSTRACT. For positive integer $p = k + 2$, we construct a logarithmic extension of the $\widehat{sl}(2)_k$ conformal field theory of integrable representations by taking the kernel of two fermionic screening operators in a three-boson realization of $\widehat{sl}(2)_k$. The currents $W^-(z)$ and $W^+(z)$ of a W -algebra acting in the kernel are determined by a highest-weight state of dimension $4p - 2$ and charge $2p - 1$, and a $(\theta = 1)$ -twisted highest-weight state of the same dimension $4p - 2$ and charge $-2p + 1$. We construct $2p$ W -algebra representations, evaluate their characters, and show that together with the $p - 1$ integrable representation characters they generate a modular group representation whose structure is described as a deformation of the $(9p - 3)$ -dimensional representation $\mathcal{R}_{p+1} \oplus \mathbb{C}^2 \otimes \mathcal{R}_{p+1} \oplus \mathcal{R}_{p-1} \oplus \mathbb{C}^2 \otimes \mathcal{R}_{p-1} \oplus \mathbb{C}^3 \otimes \mathcal{R}_{p-1}$, where \mathcal{R}_{p-1} is the $SL(2, \mathbb{Z})$ representation on integrable representation characters and \mathcal{R}_{p+1} is a $(p + 1)$ -dimensional $SL(2, \mathbb{Z})$ representation known from the logarithmic $(p, 1)$ model. The dimension $9p - 3$ is conjecturally the dimension of the space of torus amplitudes, and the \mathbb{C}^n with $n = 2$ and 3 suggest the Jordan cell sizes in indecomposable W -algebra modules. Under Hamiltonian reduction, the W -algebra currents map into the currents of the triplet W -algebra of the logarithmic $(p, 1)$ model.

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1. INTRODUCTION

Logarithmic conformal field theories in two dimensions [1, 2] are attracting some attention from different standpoints: in the context closest to the subject of this paper, in [3, 4, 5, 6] and, as regards the extended (W -)algebras, in [7, 8, 9, 10, 11, 12]; in the context of boundary conformal field theories, in [13, 14, 15]; mathematically, various aspects of logarithmic conformal models and related structures in vertex-operator algebras were considered in [16, 17, 18, 19]; relations to statistical-mechanics models have been studied in [20, 21, 22]; various aspects of logarithmic models were elaborated in [8, 23, 24, 10, 25, 26, 27, 28, 29, 30]; relations to quantum groups and a “non-semisimple” extension of the Kazhdan–Lusztig correspondence [31] were investigated in [11, 32, 33, 15]. Logarithmic conformal field theories can be viewed as an extension of rational conformal field theories [34, 35, 36, 37] to the case involving indecomposable representations of the chiral algebra. Identification of the chiral algebra itself requires some care in logarithmic models: in the known examples, starting with the pioneering works [7, 2, 8], the chiral algebra is not the “naive,” manifest symmetry algebra (e.g., Virasoro) but its nonlinear extension, i.e., some W -algebra (cf. [4, 10, 11, 12]).

A systematic way to define a logarithmic conformal field theory model is to take the kernel of the differential in a complex associated with screening operators acting in appropriate free-field spaces. Constructed this way, logarithmic models are a natural generalization of rational ones (which are just the cohomology of the same differential, cf. [38, 39]), but can also be defined in the case where the cohomology is trivial and therefore the rational model is empty [10]. Furthermore, defining logarithmic models in terms of a kernel of screenings suggests chiral W -algebras of these models; in the known $(p, 1)$ and (p, q) cases, the W -algebra that is the symmetry of the model is the maximum local algebra acting in the kernel.

In this paper, the “screening-based” approach is used to logarithmically extend the well-known $\widehat{sl}(2)_k$ minimal models of integrable representations. Part of the motivation is in the general popularity of WZW-related models and the possibility of constructing coset models in particular. But success is not guaranteed a priori.

Two *related* difficulties can be perceived in carrying the previously developed methods over to theories where the “naive” symmetry algebra (the one that is manifest before

identifying the W -algebra) is an affine Lie algebra. First, the characters acquire a dependence on $\nu \in \mathbb{C}$, in addition to the modular parameter $\tau \in \mathfrak{h}$; then, whenever the W -algebra characters involve derivatives of theta-functions (which is a typical feature of logarithmic conformal field theories), predictable complications with modular transformation properties occur. Second, representations multiply under the spectral flow action, to infinity in general; this seems to take us even farther away from the rational class than the generous setting of logarithmic conformal field theory may allow. But the situation with infinitely many inequivalent representations produced by the spectral flow is in fact already encountered in the more familiar setting of admissible representations [40] of affine Lie algebras, $\widehat{sl}(2)$ in particular. If the characters are understood in appropriate analytic-continuation terms, the number of the resulting *character functions* is finite (cf. [41, 42, 43]) and, moreover, a finite-dimensional modular group representation is realized on them. An extra complication occurring in the logarithmic/nonsemisimple case is that the space of torus amplitudes is not exhausted by the characters, and therefore some other functions, which are not characters, come into play. In the “ ν -free” cases studied previously, these *generalized characters* typically had the form of characters times polynomials in τ , with the degree of the polynomials determined by the Jordan cell size [12]; we have to see how this behavior is affected by the appearance of a ν variable. Continuing with challenges encountered in the study of logarithmic conformal field theories, we mention that their W -symmetries are rather complicated algebras whose representation theory is poorly understood in general.

Our aim is to report that despite these complications, it is nevertheless possible to achieve certain consistency in constructing logarithmic extensions of the minimal $\widehat{sl}(2)$ models following the strategy “*screenings* \rightarrow *kernel* \rightarrow *W-algebra* \rightarrow *characters* \rightarrow *generalized characters and modular transformations.*” *Consistency* here refers to modular transformations, whose closure is a very strong consistency check for various structures in conformal field theory. It has been observed in somewhat different situations in [42, 41] (and maybe elsewhere) that the closure of a set of character functions under the spectral flow tends to imply their closure under modular transformations. To a certain extent, this is also the case with the proposed logarithmic $\widehat{sl}(2)$ theory, where, as in other logarithmic models, generalized characters occur in addition, but where also “absorbing” the explicit ν dependence requires introducing a matrix automorphy factor, i.e., changing a (right) $SL(2, \mathbb{Z})$ -action $\gamma : f(\tau, \nu) \mapsto f(\gamma\tau, \gamma\nu)$, $\gamma \in SL(2, \mathbb{Z})$, into $\gamma : f(\tau, \nu) \mapsto j(\gamma; \tau, \nu)f(\gamma\tau, \gamma\nu)$, where j is a function on $SL(2, \mathbb{Z}) \times \mathfrak{h} \times \mathbb{C}$ (matrix-valued if f is a vector) satisfying the cocycle condition

$$j(\gamma\gamma'; \tau, \nu) = j(\gamma'; \tau, \nu) j(\gamma; \gamma'\tau, \gamma'\nu), \quad j(1; \tau, \nu) = 1.$$

We start with screenings that single out the $\widehat{sl}(2)$ algebra as their centralizer in a three-boson realization. There are two a priori inequivalent possibilities for this, involving

either one bosonic and one fermionic or two fermionic screenings. The option chosen in this paper is the one with *two fermionic screenings*. Two fermionic screenings Q_- and Q_+ give rise to complexes of a somewhat unusual \mathfrak{K} -shape

(1.1)

which is an $\widehat{\mathfrak{sl}}(2)$ version of the butterfly resolution in [44] (also see [45]). The sites denote twisted (and typically highly reducible) $\widehat{\mathfrak{sl}}(2)_k$ -modules.¹ The two types of arrows, north-east and south-east, correspond to the two fermionic screenings, whose composition gives the middle link. *For positive integer $k+2$, the two screenings commute with each other.* Whenever the cohomology is nontrivial, it sits at the “right eye” (*) and gives an integrable representation. We also construct and use several acyclic butterfly complexes.

That the resolution involves $\widehat{\mathfrak{sl}}(2)$ -representations of arbitrarily large twists leads to a number of problems (e.g., reasonably weighted sums of their characters diverge). In following the “screenings \rightarrow kernel \rightarrow W -algebra $\rightarrow \dots$ ” strategy, we therefore take the kernels not in all the modules constituting the butterfly resolution but only in the untwisted ones (those at the horizontal symmetry line). Accordingly, the W -algebra that we identify maps only horizontally between the $\widehat{\mathfrak{sl}}(2)_k$ -modules associated with the sites in (1.1). The currents $W^-(z)$ and $W^+(z)$ generating the W -algebra are determined by singular vectors representing a highest-weight state of dimension $4p-2$ and charge $2p-1$, and a $(\theta=1)$ -twisted highest-weight state of the same dimension $4p-2$ and charge $-2p+1$, where $p=k+2$. More precisely, we let E_n and F_n be the $\widehat{\mathfrak{sl}}(2)$ generators (with $[E_m, F_n] = km\delta_{m+n,0} + 2H_{m+n}$) and write $|\lambda; \theta\rangle$ for a highest-weight vector with spin λ and twist θ (see Sec. 2.1.2 for the details). Let $\lambda^+(r,s) = \frac{r-1}{2} - p\frac{s-1}{2}$ and $\lambda^-(r,s) = -\frac{r+1}{2} + p\frac{s}{2}$. The W -algebra currents $W^-(z)$ and $W^+(z)$ are the operators corresponding to the states

$$(1.2) \quad |W^-\rangle = (F_{-1})^{3p-1}(E_0)^{2p-1}(F_{-1})^{p-1}(E_0)^{-1}(F_{-1})^{-p-1}|\lambda^+(p-1, 3); 1\rangle,$$

$$(1.3) \quad |W^+\rangle = (E_{-1})^{3p-1}|\lambda^-(3p-1, 1); 0\rangle,$$

¹The butterfly resolution differs from Felder-type resolutions [38, 39] not only in its shape but also in that the modules farther away from the horizontal symmetry axis of the butterfly have progressively higher twists (parameters of spectral flow transformations). The spectral flow can be visualized to map vertically in (1.1), with the result that the butterfly starts “flying.”

where negative powers are to be understood as explained in [46].

For this W -algebra, we construct $2p$ of its representations, denoted by \mathcal{Y}_r^\pm , $1 \leq r \leq p$, evaluate their characters $\chi_r^\pm(\tau, \nu)$, and study their spectral-flow and modular transformation properties. Because the representation theory of this W -algebra is largely unexplored,² we actually use the spectral flow to generate a set of character functions on which the modular group action may be expected to close if an appropriate number of generalized characters (involving polynomials in τ) are added. In a sense, we compensate for the poorly known W -algebra representation theory by seeking a modular group representation generated from a set of those W -algebra characters that we can explicitly evaluate by honest representation theory. The precise result is as follows.

Main result. *The modular group representation generated from W -algebra characters in the logarithmic $\widehat{sl}(2)_k$ model with positive integer $p = k + 2$ is a deformation, via a matrix automorphy factor, of the direct sum*

$$(1.4) \quad \mathcal{R}_{p+1} \oplus \mathbb{C}^2 \otimes \mathcal{R}_{p+1} \oplus \mathcal{R}_{\text{int}}(p) \oplus \mathbb{C}^2 \otimes \mathcal{R}_{\text{int}}(p) \oplus \mathbb{C}^3 \otimes \mathcal{R}_{\text{int}}(p),$$

where $\mathcal{R}_{\text{int}}(p)$ is the $(p-1)$ -dimensional $SL(2, \mathbb{Z})$ representation on the integrable $\widehat{sl}(2)_k$ characters, \mathcal{R}_{p+1} is a $(p+1)$ -dimensional representation, \mathbb{C}^2 is the defining two-dimensional representation, and \mathbb{C}^3 is its symmetrized square; the matrix automorphy factor becomes equal to the identity matrix at $\nu = 0$.

Fulfilling the general expectation [9, 6], we next show (Theorem 5.1) that the Hamiltonian reduction indeed relates the logarithmic $\widehat{sl}(2)_k$ to the logarithmic $(p = k + 2, 1)$ models: the W -algebra generators in (1.2) and (1.3) map under the Hamiltonian reduction to generators of the triplet W -algebra [7, 2, 8] of the $(p, 1)$ logarithmic model, which were defined in [10] in terms of a Virasoro screening. We note that the $SL(2, \mathbb{Z})$ representation on generalized characters in the logarithmic $(p, 1)$ model was evaluated as $\mathcal{R}_{p+1} \oplus \mathbb{C}^2 \otimes \mathcal{R}_{\text{int}}(p)$ in [11]; the Hamiltonian reduction argument, in particular, “explains” the occurrence of $\mathcal{R}_{\text{int}}(p)$ there.³

The total dimension $9p - 3$ of the $SL(2, \mathbb{Z})$ -representation in (1.4) is a likely candidate for the dimension of the space of torus amplitudes, if such a finite-dimensional space can be constructed at all following one of the more direct approaches. The $n = 2$ and 3 in the \mathbb{C}^n tensor factors entering (1.4) suggest the Jordan cell sizes to be encountered in indecomposable W -algebra modules.

² W -algebras in logarithmic models can be viewed as extensions of “minimal” (e.g., Virasoro or, in our case, $\widehat{sl}(2)$) algebras by (descendants of) certain vertex operators. Vertex-operator extensions have attracted some general interest, e.g., in [47, 48, 49, 50, 51].

³Hamiltonian reduction at the level of conformal blocks (solutions of the Knizhnik–Zamolodchikov equations, see [52] and the references therein) was studied in [9] to analyze logarithmic extensions of (p, q) minimal models and, in particular, gave evidence in favor of the existence of a W -algebra in these models.

The ν variable in the argument of the characters can in hindsight be seen to result in producing deformations of direct sums of $SL(2, \mathbb{Z})$ representations. The $(2p + (p - 1))$ -dimensional space spanned by the W -algebra characters $(\chi_r^\pm(\tau, \nu))_{1 \leq r \leq p}$ and the integrable $\widehat{\mathfrak{sl}}(2)_k$ -representation characters $(\chi_r(\tau, \nu))_{1 \leq r \leq p-1}$ is extended to the $(9p - 3)$ -dimensional space in (1.4) due to two mechanisms. First, the spectral flow closes if $2p$ functions $\omega_r^\pm(\tau, \nu)$, $1 \leq r \leq p$, are added. We treat them on equal footing with characters, although their representation-theory meaning is not discussed here. Second, as with the “ ν -free” characters of logarithmic conformal field theories known previously [11, 12], certain combinations of the $\chi_r^\pm(\tau, \nu)$, $\chi_r(\tau, \nu)$, $\omega_r^\pm(\tau, \nu)$ become parts of multiplets, i.e., transform in representations of the form $\mathbb{C}^n \otimes \pi$, where π is some $SL(2, \mathbb{Z})$ representation and \mathbb{C}^n is \mathbb{C}^2 or \mathbb{C}^3 realized on polynomials in τ of the respective degrees 1 and 2. But it then turns out that the explicit occurrences of the ν variable in modular transformation formulas results not only in “mixing” different modular-group representations with each other but also in proliferating the number of functions involved, as is already clear from the $\nu \mapsto \nu/(c\tau + d)$ transformations introducing a fractional-linear factor. This behavior can be “absorbed” into a matrix automorphy factor, isolating which leaves us just with the representation in (1.4).⁴

Notation. We fix the level k as any complex number not equal to -2 in Sec. 2 and as $k \in \{0, 1, 2, \dots\}$ (and occasionally $k = -1$) starting with 3.2. We also use the notation $p = k + 2$, with the apologies for a certain lack of consistency, in that a formula or two neighboring formulas may contain both k and p . Similar negligence is shown regarding another global notation, $j = \frac{r-1}{2}$: both j and r are used interchangeably.

Remark. The paper contains quite a few pictures of the subquotient structure of the relevant modules and maps between them. An alternative way of delivering the same information would be a comparable abundance of formal notation, making sense out of which would anyway require some visualization. The reader inclined to giving each object a special name and a defining formula that makes all the parameters explicit must be able to reconstruct the details from the numerous labels in the pictures (e.g., as in Fig. 3.1 (p. 16) below).

This paper is organized as follows. In Sec. 2, we fix the notation and conventions and recall standard facts about the $\widehat{\mathfrak{sl}}(2)$ algebra, the spectral flow, and singular vectors in $\widehat{\mathfrak{sl}}(2)$ Verma modules, and then introduce the bosonization and the corresponding screenings to be used in what follows. In Sec. 3, we construct the butterfly resolution

⁴The need to introduce a matrix automorphy factor in order to extract a finite-dimensional $SL(2, \mathbb{Z})$ -representation of course reflects certain “pathologies” inherent in the adopted setting, where the number of free fields (3) is larger than the number of screenings (2). As can be seen in [53], a more natural object from the “screening/quantum-group” standpoint may be given by the coset $\widehat{\mathfrak{sl}}(2)/u(1)$, but we leave its “logarithmization” for the future.

of integrable representations and several acyclic butterfly complexes. This gives enough information for constructing the W -algebra generators $W^\pm(z)$ and representations \mathcal{Y}_r^\pm , $1 \leq r \leq p$, and evaluating the characters of the latter in Sec. 4. We also study the spectral-flow and modular transformation properties of the characters (extended by other functions) in Sec. 4. In Sec. 5, we evaluate the Hamiltonian reduction of the $W^\pm(z)$ generators, showing that they map into the generators of the triplet W -algebra of the logarithmic $(p, 1)$ -models. Section 6 is a list of things that have not been done in this paper but are potentially interesting, even if some of them prove impracticable.

Appendix A pertains entirely to Sec. 3 and serves to recall the embedding structure of $\widehat{sl}(2)_k$ Verma modules; most readers may ignore it altogether. Appendix B sets the notation and summarizes some facts about theta-functions, extensively used in Sec. 4. Appendix C contains a rather explicit description of extensions among $SL(2, \mathbb{Z})$ representations in their “functional” realization, which occur in Sec. 4.

2. THE $\widehat{sl}(2)$ ALGEBRA

In this section, we set the notation for the $\widehat{sl}(2)$ algebra, its twisted modules, and singular vectors in Verma modules. We then introduce the three-boson realization of $\widehat{sl}(2)$ and the corresponding screenings.

2.1. The algebra, spectral flow, and twisted Verma modules. The level- k affine algebra $\widehat{sl}(2)_k$ is defined by the commutation relations

$$(2.1) \quad \begin{aligned} [H_m, E_n] &= E_{m+n}, & [H_m, F_n] &= -F_{m+n}, & [H_m, H_n] &= \frac{k}{2} m \delta_{m+n,0}, \\ [E_m, F_n] &= k m \delta_{m+n,0} + 2H_{m+n}, \end{aligned}$$

with $m, n \in \mathbb{Z}$. In terms of the currents $X(z) = \sum_{n \in \mathbb{Z}} X_n z^{-n-1}$, $X = E, H, F$, the above commutation relations are reformulated as the OPEs

$$(2.2) \quad \begin{aligned} H(z)E(w) &= \frac{E(w)}{z-w}, & H(z)F(w) &= \frac{-F(w)}{z-w}, \\ E(z)F(w) &= \frac{k}{(z-w)^2} + \frac{2H(w)}{z-w}, & H(z)H(w) &= \frac{k/2}{(z-w)^2}. \end{aligned}$$

and the Sugawara energy-momentum tensor is given by the standard expression

$$T_{\text{Sug}}(z) = \frac{1}{k+2} \left(\frac{1}{2} E(z)F(z) + \frac{1}{2} F(z)E(z) + H(z)H(z) \right)$$

(here and below, normal ordering is understood; for brevity, we sometimes write $AB(z)$ instead of the normal-ordered product $A(z)B(z)$). Generators of the Virasoro algebra with central charge $c = \frac{3k}{k+2}$ are then introduced via $T_{\text{Sug}}(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$.

For each $\theta \in \mathbb{Z}$, there is an $\widehat{sl}(2)_k$ automorphism given by the so-called spectral flow transformation (see [54])

$$(2.3) \quad \mathcal{U}_\theta : \quad E_n \mapsto E_{n+\theta}, \quad F_n \mapsto F_{n-\theta}, \quad H_n \mapsto H_n + \frac{k}{2} \theta \delta_{n,0}$$

Spectral-flow transforming any $\widehat{sl}(2)$ -module \mathcal{C} gives *twisted* modules $\mathcal{U}_\theta \mathcal{C} = \mathcal{C};_\theta$.⁵

For any $\widehat{sl}(2)_k$ -module \mathcal{C} , its character is

$$(2.4) \quad \chi^{\mathcal{C}}(q, z) = \text{Tr}_{\mathcal{C}}(q^{L_0 - \frac{c}{24}} z^{H_0}).$$

We let $\chi^{\mathcal{C};_\theta}(q, z)$ denote the character $\chi^{\mathcal{C};_\theta}(q, z)$ of twisted modules. In what follows, we frequently use the following elementary result.

2.1.1. Lemma ([41]). *Let \mathcal{C} be an $\widehat{sl}(2)_k$ -module. Then*

$$(2.5) \quad \chi^{\mathcal{C};_\theta}(q, z) = q^{\frac{k}{4}\theta^2} z^{-\frac{k}{2}\theta} \chi^{\mathcal{C}}(q, zq^{-\theta}).$$

2.1.2. Twisted Verma modules. We next fix our conventions regarding twisted Verma modules. For $\lambda \in \mathbb{C}$ and $\theta \in \mathbb{Z}$, the twisted Verma module $\mathcal{M}_{\lambda;\theta}$ is freely generated by $E_{\leq \theta-1}$, $F_{\leq -\theta}$, and $H_{\leq -1}$ from a twisted highest-weight vector $|\lambda; \theta\rangle$ defined by the conditions

$$(2.6) \quad \begin{aligned} E_{\geq \theta} |\lambda; \theta\rangle &= H_{\geq 1} |\lambda; \theta\rangle = F_{\geq -\theta+1} |\lambda; \theta\rangle = 0, \\ (H_0 + \frac{k}{2}\theta) |\lambda; \theta\rangle &= \lambda |\lambda; \theta\rangle. \end{aligned}$$

It follows that

$$L_0 |\lambda; \theta\rangle = \Delta_{\lambda;\theta} |\lambda; \theta\rangle, \quad \Delta_{\lambda;\theta} = \frac{\lambda^2 + \lambda}{k+2} - \theta\lambda + \frac{k}{4}\theta^2.$$

For $k\theta \neq 0$, we must therefore distinguish between the eigenvalue of H_0 on a twisted highest-weight state and the spectral-flow-independent parameter λ (which, e.g., determines the existence of singular vectors in $\mathcal{M}_{\lambda;\theta}$). We say that the eigenvalue of H_0 is the charge and λ is the spin of $|\lambda; \theta\rangle$. Setting $\theta = 0$ gives the usual (“untwisted”) Verma modules. We write $|\lambda\rangle = |\lambda; 0\rangle$ and, similarly, $\mathcal{M}_\lambda = \mathcal{M}_{\lambda;0}$.

We write $|\alpha\rangle \doteq |\lambda; \theta\rangle$ whenever conditions (2.6) are satisfied for a state $|\alpha\rangle$.

The character of a twisted Verma module $\mathcal{M}_{\lambda;\theta}$ can be conveniently written in terms of $h = \lambda - \frac{k}{2}\theta$, the eigenvalue of H_0 in (2.6), as

$$(2.7) \quad \chi_{\lambda;\theta}^{\mathcal{M}}(q, z) = (-1)^\theta \frac{q^{\frac{(h-\theta+\frac{1}{2})^2}{k+2} - \frac{1}{8}} z^{h-\theta}}{\vartheta_{1,1}(q, z)}$$

(see (B.3) for $\vartheta_{1,1}$).

Twists, although producing nonequivalent modules, do not alter the submodule grid structure, and we can therefore reformulate a classic result as follows.

2.2. Theorem ([55, 46]). *A singular vector exists in a twisted $\widehat{sl}(2)$ Verma module $\mathcal{M}_{\lambda;\theta}$ if and only if λ can be written as $\lambda = \lambda^+(r, s)$ or $\lambda = \lambda^-(r, s)$ with $r, s \in \mathbb{N}$, where*

⁵An automorphism α of an algebra \mathfrak{a} maps an \mathfrak{a} -module M into a module αM on which the algebra acts as $a \cdot (\alpha m) = \alpha(\alpha^{-1}(a) \cdot m)$, $a \in \mathfrak{a}$, $m \in M$. The \mathfrak{a} -representations on M and αM are *not* equivalent in general.

$$\begin{aligned}\lambda^+(r, s) &= \frac{r-1}{2} - (k+2)\frac{s-1}{2}, \\ \lambda^-(r, s) &= -\frac{r+1}{2} + (k+2)\frac{s}{2}.\end{aligned}$$

Whenever $\lambda = \lambda^+(r, s)$, the singular vector is given by

$$(2.8) \quad |\text{MFF}^+(r, s; \theta | \lambda)\rangle = (F_{-\theta})^{r+(s-1)(k+2)} (E_{\theta-1})^{r+(s-2)(k+2)} (F_{-\theta})^{r+(s-3)(k+2)} \dots \\ \cdot (E_{\theta-1})^{r-(s-2)(k+2)} (F_{-\theta})^{r-(s-1)(k+2)} |\lambda; \theta\rangle.$$

Whenever $\lambda = \lambda^-(r, s)$, the singular vector is given by

$$(2.9) \quad |\text{MFF}^-(r, s; \theta | \lambda)\rangle = (E_{\theta-1})^{r+(s-1)(k+2)} (F_{-\theta})^{r+(s-2)(k+2)} (E_{\theta-1})^{r+(s-3)(k+2)} \dots \\ \cdot (F_{-\theta})^{r-(s-2)(k+2)} (E_{\theta-1})^{r-(s-1)(k+2)} |\lambda; \theta\rangle.$$

We recall that these formulas yield polynomial expressions in the currents via repeated application of (the spectral-flow transform of) the formulas

$$(2.10) \quad \begin{aligned}(F_0)^\alpha E_m &= -\alpha(\alpha-1)F_m(F_0)^{\alpha-2} - 2\alpha H_m(F_0)^{\alpha-1} + E_m(F_0)^\alpha, \\ (F_0)^\alpha H_m &= \alpha F_m(F_0)^{\alpha-1} + H_m(F_0)^\alpha, \\ (E_{-1})^\alpha F_m &= -\alpha(\alpha-1)E_{m-2}(E_{-1})^{\alpha-2} - k\alpha \delta_{m-1,0}(E_{-1})^{\alpha-1} \\ &\quad + 2\alpha H_{m-1}(E_{-1})^{\alpha-1} + F_m(E_{-1})^\alpha, \\ (E_{-1})^\alpha H_m &= -\alpha E_{m-1}(E_{-1})^{\alpha-1} + H_m(E_{-1})^\alpha,\end{aligned}$$

which can be derived for positive integer α and then continued to arbitrary complex α .

2.2.1. For $s = 1$, singular vectors (2.8) and (2.9) do not require any algebraic rearrangements and take the simple form

$$(2.11) \quad |\text{MFF}^+(r, 1; \theta | \lambda)\rangle = (F_{-\theta})^r |\lambda; \theta\rangle, \quad |\text{MFF}^-(r, 1; \theta | \lambda)\rangle = (E_{\theta-1})^r |\lambda; \theta\rangle.$$

2.2.2. Another special case to be used in what follows occurs for positive integer $p = k + 2$ and $\lambda = \lambda^+(p, s)$. From (2.8), we then have

$$(2.12) \quad |\text{MFF}^+(p, s)\rangle = (F_0)^{sp} (E_{-1})^{(s-1)p} |\lambda\rangle.$$

If $s = 1$, we return to (2.11), but if $s \geq 2$, then the corresponding state $|\lambda\rangle$ with $\lambda = p - \frac{ps+1}{2}$ also admits the singular vector $|\text{MFF}^-(p(s-1), 1)\rangle$, through which $|\text{MFF}^+(p, s)\rangle$ is actually seen to factor in (2.12).

Similarly, if $\lambda = \lambda^-(p, s)$, the corresponding singular vector becomes

$$(2.13) \quad |\text{MFF}^-(p, s)\rangle = (E_{-1})^{sp} (F_0)^{(s-1)p} |\lambda\rangle,$$

which factors through $|\text{MFF}^+(p(s-1), 1)\rangle$ whenever $s \geq 2$.

2.3. Integrable representation characters. For positive integer $k + 2$, the characters of the integrable representations \mathcal{J}_r , $r = 1, \dots, k + 1$, are given by

$$(2.14) \quad \chi_r(q, z) \equiv \chi_r^{\mathcal{J}}(q, z) = \frac{\theta_{r,p}(q, z) - \theta_{-r,p}(q, z)}{\Omega(q, z)}, \quad r = 1, \dots, p - 1$$

where

$$(2.15) \quad \Omega(q, z) = q^{\frac{1}{8}} z^{\frac{1}{2}} \vartheta_{1,1}(q, z).$$

The integrable representation characters are holomorphic in $z \in \mathbb{C}$ and transform under the spectral flow as $\chi_{r;1}(q, z) = \chi_{p-r}(q, z)$.

2.4. Bosonization and fermionic screenings. We keep the level k fixed, temporarily at any complex value not equal to -2 . We introduce the well-known bosonization of the $\widehat{\mathfrak{sl}}(2)_k$ algebra associated with two fermionic screenings, following the conventions in [53] (where two $\widehat{\mathfrak{sl}}(2)$ bosonizations are discussed from a unified standpoint and a more general case is also considered; the bosonization chosen here is termed *symmetric* in [53], for the reasons that become quite obvious when it is compared with the other, nonsymmetric bosonization also discussed there).

2.4.1. “Symmetric” three-boson realization. Let ξ , ψ_- , and ψ_+ be three vectors in \mathbb{C}^3 with the scalar products

$$(2.16) \quad \begin{aligned} \xi \cdot \xi &= 0, & \xi \cdot \psi_- &= 1, & \xi \cdot \psi_+ &= -1, \\ \psi_- \cdot \psi_- &= 1, & \psi_- \cdot \psi_+ &= k + 1, \\ \psi_+ \cdot \psi_+ &= 1 \end{aligned}$$

(the determinant of the Gram matrix is equal to $-2(k + 2)$, and hence the vectors are defined uniquely modulo an overall rotation). We introduce a triple of scalar fields $\varphi = (\varphi_1, \varphi_2, \varphi_3)$, in the standard basis, with the OPEs

$$\partial \varphi_i(z) \partial \varphi_j(w) = \frac{\delta_{i,j}}{(z-w)^2},$$

where $\partial f(z) = \frac{\partial f(z)}{\partial z}$. For any $a \in \mathbb{C}^3$, let $a \cdot \partial \varphi$ (as well as $a \cdot \varphi$) denote the Euclidean scalar product.

It is easy to verify that the currents

$$(2.17) \quad \begin{aligned} E(z) &= \psi_+ \cdot \partial \varphi(z) e^{\xi \cdot \varphi(z)}, \\ H(z) &= \frac{1}{2}(k\xi + \psi_- - \psi_+) \cdot \partial \varphi(z), \\ F(z) &= \psi_- \cdot \partial \varphi(z) e^{-\xi \cdot \varphi(z)} \end{aligned}$$

satisfy the $\widehat{\mathfrak{sl}}(2)_k$ OPEs. We refer to these formulas as the three-boson realization (bosonization) of $\widehat{\mathfrak{sl}}(2)_k$ (its relation to the Wakimoto representation [56, 57] is established by bosonizing the first-order $\beta\gamma$ system involved in the Wakimoto representation). The

bosonized form of the Sugawara energy-momentum tensor is given by a standard formula involving the inverse of the Gram matrix of ξ , ψ_- , and ψ_+ [53]. We keep the notation L_n for the corresponding Virasoro generators.

2.4.2. Screenings. The bosonization in (2.17) is associated with two fermionic screenings

$$(2.18) \quad Q_- = \oint e^{\psi_- \cdot \varphi}, \quad Q_+ = \oint e^{\psi_+ \cdot \varphi}.$$

Simple calculation shows that they indeed commute with the $\widehat{sl}(2)_k$ currents in (2.17).

In what follows, we no longer use the components $(\partial\varphi_1, \partial\varphi_2, \partial\varphi_3)$ of $\partial\varphi$, and instead sometimes use the notation

$$\partial\varphi_-(z) = \psi_- \cdot \partial\varphi(z), \quad \partial\varphi_+(z) = \psi_+ \cdot \partial\varphi(z), \quad \partial\varphi_0(z) = \xi \cdot \partial\varphi(z).$$

3. TWISTED WAKIMOTO-TYPE MODULES AND BUTTERFLY COMPLEXES

The bosonization of $\widehat{sl}(2)_k$ introduced in **2.4** gives rise to Wakimoto-type free-field modules [56, 57]. The aim of this section is to construct complexes of twisted free-field modules using the two fermionic screenings. We begin in **3.1** with defining the relevant vertex operators and give simple formulas for the action of $\widehat{sl}(2)$ generators on them; these formulas are then used in evaluating singular vectors in terms of the above bosonization and screenings. A foregone conclusion is that “half” the singular vectors in Wakimoto-type modules vanish. The emerging pattern can then be rephrased as the existence of the desired complexes. In **3.2**, we construct the butterfly resolutions of integrable representations, with the cohomology concentrated in the “right eye.” The complexes in **3.3** and **3.4** are acyclic.

The reader may wish to skip this long and rather technical section and come back to the results in it when they are actually needed, and proceed directly to Sec. **4**.

3.1. Vertex operators and states. We introduce the family of vertex operators

$$(3.1) \quad U_{\lambda, h, \theta_-, \theta_+}(z) = e^{(h\xi + (\frac{\lambda}{k+2} - \theta_-)\psi_- + (\frac{\lambda}{k+2} - \theta_+)\psi_+) \cdot \varphi(z)}$$

and the corresponding states $|\lambda, h, \theta_-, \theta_+\rangle\rangle$. The parameterization is redundant, the vertex being unchanged under

$$(3.2) \quad \lambda \mapsto \lambda + p\alpha, \quad \theta_{\pm} \mapsto \theta_{\pm} + \alpha$$

for arbitrary α . When we restrict to positive integer $k+2$ in what follows, we take $2\lambda + 1 \in \{1, \dots, 2(k+2)\}$, $h \mp \lambda \in \mathbb{Z}$, and $\theta_{\pm} \in \mathbb{Z}$, with θ_- and θ_+ of the same sign (hence the two, not four, wings of the butterfly).

The $|\lambda, h, \theta_-, \theta_+\rangle\rangle$ states are Virasoro primaries and, moreover, $\widehat{sl}(2)_k$ *relaxed highest-weight states*: the Verma-module highest-weight conditions in (2.6) are relaxed to

$$\begin{aligned}
(3.3) \quad & E_{\geq \theta_+ + 1} |\lambda, h, \theta_-, \theta_+\rangle = 0, \\
& F_{\geq -\theta_+ + 1} |\lambda, h, \theta_-, \theta_+\rangle = 0, \\
& H_{\geq 1} |\lambda, h, \theta_-, \theta_+\rangle = 0, \quad \theta = \theta_- - \theta_+
\end{aligned}$$

(which, strictly speaking, requires $\theta_- - \theta_+ \in \mathbb{Z}$); moreover, as is easy to verify,

$$\begin{aligned}
(3.4) \quad & E_{\theta_- - \theta_+} |\lambda, h, \theta_-, \theta_+\rangle = (\lambda - h - \theta_-(k+1) - \theta_+) |\lambda, h+1, \theta_-, \theta_+\rangle, \\
& F_{\theta_+ - \theta_-} |\lambda, h, \theta_-, \theta_+\rangle = (\lambda + h - \theta_- - (k+1)\theta_+) |\lambda, h-1, \theta_-, \theta_+\rangle.
\end{aligned}$$

It also follows that

$$H_0 |\lambda, h, \theta_-, \theta_+\rangle = h |\lambda, h, \theta_-, \theta_+\rangle$$

(and therefore h is the charge of the state, in accordance with with the terminology introduced in **2.1.2**).

The reader is invited to appreciate the significance of the factors $(\lambda - h - \theta_-(k+1) - \theta_+)$ and $(\lambda + h - \theta_- - (k+1)\theta_+)$ in (3.4) in “strengthening” the relaxed highest-weight conditions in (3.3) to the twisted highest-weight conditions (2.6) by appropriately choosing the parameters (e.g., h). We now elaborate on this and several other simple technical details (the next subsection may be skipped until its results are actually used).

3.1.1. First, it is obvious that

$$E_{\theta_- - \theta_+} |\lambda, h, \theta_-, \theta_+\rangle = 0 \iff Q_+ U_{\lambda, h, \theta_-, \theta_+}(z) = 0.$$

Moreover, when this condition is satisfied, the state $|\lambda, h+1, \theta_-, \theta_+\rangle$ is mapped by Q_+ (see **2.4.2**) into a twisted highest-weight state with the twist $\theta_- - \theta_+ + 1$. This is an immediate consequence of the OPE

$$e^{\psi_+ \cdot \varphi(z)} U_{\lambda, h, \theta_-, \theta_+}(w) = (z-w)^{-h+\lambda-\theta_-(k+1)-\theta_+} e^{\psi_+ \cdot \varphi(z)} U_{\lambda, h, \theta_-, \theta_+}(w)$$

(with the normal ordered product in the right-hand side). Properly developing this observation shows that

$$\begin{aligned}
(3.5) \quad & (E_{\theta_- - \theta_+ - 1})^N U_{\lambda, \lambda - \theta_-(k+1) - \theta_+, \theta_-, \theta_+}(z) = \\
& = (-1) \dots (-N) Q_+ U_{\lambda, \lambda - \theta_-(k+1) - \theta_+ + N, \theta_-, \theta_+ + 1}(z).
\end{aligned}$$

For the U operator in the left-hand side, we have

$$(3.6) \quad U_{\lambda, \lambda - \theta_-(k+1) - \theta_+, \theta_-, \theta_+} \doteq |\lambda - (k+2) \frac{\theta_- + \theta_+}{2}; \theta_- - \theta_+\rangle.$$

It also follows that the “top” F -mode acts on it as

$$\begin{aligned}
(3.7) \quad & (F_{\theta_+ - \theta_-})^N U_{\lambda, \lambda - \theta_-(k+1) - \theta_+, \theta_-, \theta_+}(z) = \\
& = \frac{\Gamma(2\lambda + 1 - (\theta_- + \theta_+)(k+2))}{\Gamma(2\lambda + 1 - N - (\theta_- + \theta_+)(k+2))} U_{\lambda, \lambda - N - \theta_-(k+1) - \theta_+, \theta_-, \theta_+}(z).
\end{aligned}$$

The ratio of Γ -functions here is of course a simple product of N factors as they follow from (3.4). But (3.7) can be analytically continued to complex values of N , con-

sistently with the continuation underlying the construction of the MFF singular vectors (see (2.10)). Formula (3.5) can be continued similarly, up to an (inessential) sign, if $N!$ is replaced with the Γ -function.

Similarly, $F_{\theta_+ - \theta_-} |\lambda, h, \theta_-, \theta_+\rangle = 0 \iff Q_- U_{\lambda, h, \theta_-, \theta_+}(z) = 0$ and

$$(3.8) \quad (F_{\theta_+ - \theta_-})^M U_{\lambda, -\lambda + \theta_- - 1 + (k+1)\theta_+, \theta_- - 1, \theta_+}(z) = \\ = (-1) \dots (-M) Q_- U_{\lambda, -\lambda + \theta_- - 1 - M + (k+1)\theta_+, \theta_-, \theta_+}(z).$$

Up to a sign, this can be continued to complex M by replacing $M!$ with the Γ -function. For the operator in the left-hand side of (3.8), we have

$$(3.9) \quad U_{\lambda, -\lambda + \theta_- - 1 + (k+1)\theta_+, \theta_- - 1, \theta_+} \doteq |-\lambda - 1 + (k+2) \frac{\theta_- + \theta_+}{2}; \theta_- - \theta_+\rangle.$$

It also follows that the ‘‘top’’ E -mode acts on this state as

$$(3.10) \quad (E_{\theta_- - \theta_+ - 1})^N U_{\lambda, -\lambda + \theta_- - 1 + (k+1)\theta_+, \theta_- - 1, \theta_+} = \\ = \frac{\Gamma(2\lambda + 1 - (\theta_- + \theta_+ - 1)(k+2))}{\Gamma(2\lambda + 1 - N - (\theta_- + \theta_+ - 1)(k+2))} U_{\lambda, -\lambda + N + \theta_- - 1 + (k+1)\theta_+, \theta_- - 1, \theta_+}.$$

3.1.2. The above formulas provide a bridge between the MFF singular vectors and the screenings, in accordance with the well-known fact that ‘‘half’’ the singular vectors in Wakimoto modules vanish. For example, if λ is chosen as $\lambda = \frac{r-1}{2}$ with a positive integer r , the $|\text{MFF}^+(r, \theta_- + \theta_+ + 1; \theta_- - \theta_+)\rangle$ singular vector can be constructed on the state in (3.6). Recalling (2.8), we then evaluate the action of the first (rightmost) operator factor $(F_{-\theta_- + \theta_+})^{r - (\theta_- + \theta_+)(k+2)}$ using (3.7). As noted above, Eq. (3.7) is applicable for any complex $k \neq -2$ and hence a complex exponent $r - (\theta_- + \theta_+)(k+2)$. The result vanishes when evaluated at $N = r - (\theta_- + \theta_+)(k+2)$.

Butterfly complexes

From now on, we assume that $p = k+2 \in \{1, 2, \dots\}$ and consider $\widehat{sl}(2)$ modules whose elements are given by the vertices (3.1) and their descendants with

$$(3.11) \quad 2\lambda + 1 \in \{1, \dots, 2p\}, \quad h - \lambda \in \mathbb{Z}$$

and integer θ_{\pm} .

The first consequence of restricting to integer $k \geq -1$ is that the two fermionic screenings Q_- and Q_+ become local with respect to each other and, moreover, (super)commute. Indeed, we have the regular operator product $e^{\psi_- \cdot \varphi(z)} e^{\psi_+ \cdot \varphi(w)} \propto (z-w)^{k+1}$.

The range of $2p$ different values of λ in (3.11) is covered in what follows by considering $\lambda = \frac{r-1}{2}$ with $r = 1, \dots, p-1$ in **3.2** (with the spins of integrable representations), $\lambda = \frac{p}{2} + \frac{r-1}{2}$ in **3.3**, and two remaining values in **3.4**.

3.2. Butterfly resolutions of integrable representations. The aim of this subsection is to construct resolutions of integrable representations of form (1.1), with the map in the center given by the composition $Q_- \circ Q_+$.

In each wing, the modules are labeled by two integers of the same sign. With a ‘‘global’’ numbering of all of them (e.g., with θ_- and θ_+ ranging from minus to plus infinity), those in one of the wings would be labeled with negative integers; but analyzing the structure of modules dependent on expressions like $-\theta_- - \theta_+ - 1$ with negative θ_- and θ_+ is somewhat counter-intuitive, and we therefore choose a ‘‘local’’ numbering in each wing, with positive integers in either case:

$$(3.12) \quad \begin{aligned} m, n \geq 1 & \quad \text{in the left wing,} \\ m, n \geq 0 & \quad \text{in the right wing,} \end{aligned}$$

but with the notation for right-wing objects acquiring a prime.

3.2.1. Left wing. For compactness of the formulas, we use the notation

$$(3.13) \quad j = \frac{r-1}{2}.$$

For positive integer m and n , we define the operator

$$(3.14) \quad U_{m,n}[r](z) = e^{([n-1+m(k+1)-j]\xi + \frac{j}{k+2}(\psi_- + \psi_+) - m\psi_- - m\psi_+)} \cdot \varphi(z)$$

(which is $U_{j, n-1+m(k+1)-j, n, m}(z)$ in the nomenclature of **3.1**) and let $\mathcal{U}_{m,n}[r]$ denote the corresponding Wakimoto-type module, i.e., the $\widehat{s\ell}(2)_k$ -module on the free-field space generated from $U_{m,n}[r]$ (abusing the terminology, we sometimes say for brevity that $\mathcal{U}_{m,n}[r]$ is ‘‘generated’’ from $U_{m,n}[r]$). The extremal diagram of the module (tilted in accordance with the twist $n - m$) can be represented as

$$(3.15)$$

As follows from **3.1**, $U_{m,n}[r]$ defines a relaxed highest-weight state,⁶

$$E_{n-m+1}U_{m,n}[r] = F_{m-n+1}U_{m,n}[r] = 0,$$

and acting on $U_{m,n}[r]$ with $(F_{m-n})^{(m+n)(k+2)-(2j+1)}$ gives the operator/state

$$(3.16) \quad \begin{aligned} L_{m,n}[r](z) &= e^{([j-n(k+1)-m]\xi + \frac{j}{k+2}(\psi_- + \psi_+) - m\psi_- - m\psi_+)} \cdot \varphi(z) \\ &\doteq \left| \frac{r-1}{2} - (k+2)\frac{m+n}{2}; n-m \right\rangle, \end{aligned}$$

⁶For notational simplicity, we no longer distinguish between operators and the corresponding states. We also omit nonzero factors in the normalization of states.

a twisted highest-weight state with the spin $\lambda^+(r, n+m+1) = \lambda^-((m+n+1)(k+2) - r, 1)$. The E_{n-m} and F_{m-n} arrows in (3.15) map into twisted highest-weight states (“charged” singular vectors): for example, as is easy to verify,

$$(3.17) \quad E_{n-m}U_{m,n}[r] \equiv R_{m,n}[r](z) = e^{([n+m(k+1)-j]\xi + \frac{j}{k+2}(\psi_- + \psi_+) - m\psi_- - m\psi_+) \cdot \varphi(z)} \\ \doteq |\lambda^+((m+n+1)p - r, 1); n-m+1\rangle.$$

We also note the OPE $e^{\psi_+ \cdot \varphi(u)}L_{m+1,n}[r](z) = \text{reg}$, whence $Q_+L_{m,n}[r](z) = 0$; it follows similarly that $Q_-R_{m,n}[r](z) = 0$.

Sugawara dimensions of the operators $L_{m,n}[r]$ and $R_{m,n}[r]$ are

$$\Delta_{L_{m,n}[r]} = \frac{(j-n(k+2))(j-n(k+2)+1)}{k+2} - \frac{1}{2}(m-n)(m-n+1), \\ \Delta_{R_{m,n}[r]} = \frac{(j-m(k+2))(j-m(k+2)+1)}{k+2} - \frac{1}{2}(m-n-1)(m-n).$$

The subquotient structure of the module $\mathcal{L}_{m,n}[r]$ “generated” from $L_{m,n}[r]$ is shown in the well-known picture in Fig. 3.1. We use the same convention as in Appendix A to direct arrows toward submodules. The embedding structure of $\mathcal{L}_{m,n}[r]$ can be considered a result of the vanishing of “half” the singular vectors in the free-field realization (2.17). The filled dots in the figure represent operators of the form

$$(3.18) \quad \mathcal{P}[\partial\varphi(z)] e^{(h\xi + \frac{j}{k+2}(\psi_- + \psi_+) - m\psi_- - m\psi_+) \cdot \varphi(z)},$$

where \mathcal{P} is a differential polynomial (in the three currents $\partial\varphi_0(z)$, $\partial\varphi_-(z)$, and $\partial\varphi_+(z)$) and the values of h are shown in square brackets at the corresponding nodes. In particular, for the operators/states labeled K_b^+ , $b = 2, 3, \dots$, in Fig. 3.1, we have

$$K_b^+(z) = \mathcal{P}_b[\partial\varphi(z)] e^{(h_b\xi + \frac{j}{k+2}(\psi_- + \psi_+) - m\psi_- - m\psi_+) \cdot \varphi(z)},$$

where

$$(3.19) \quad h_b = \begin{cases} -\frac{r+1}{2} + n + \frac{b}{2} + (m + \frac{b}{2})(k+1), & b \text{ even,} \\ \frac{r-1}{2} + n + \frac{b-1}{2} + (m + \frac{b-1}{2})(k+1), & b \text{ odd.} \end{cases}$$

The character of the irreducible subquotient \mathcal{K}_b corresponding to K_b^+ follows from (A.4), (A.5), and (2.5): for $b \geq 1$, we have (with the dependence on m and n indicated as a subscript)

$$(3.20) \quad \chi_{m,n}^{\mathcal{K}_{2b}}(q, z) = \frac{(-1)^{n-m}}{q^{\frac{1}{8}} \vartheta_{1,1}(q, z)} \left(\sum_{a \geq 0} + \sum_{a \leq -n-m-2b} \right) \left(q^p \left(\frac{r}{2p} - (m+b+a) \right)^2 z^{-\frac{r+1}{2} + (m+b+a)p} \right. \\ \left. - q^p \left(\frac{r}{2p} - (n+b+a) \right)^2 z^{\frac{r-1}{2} - (n+b+a)p} \right),$$

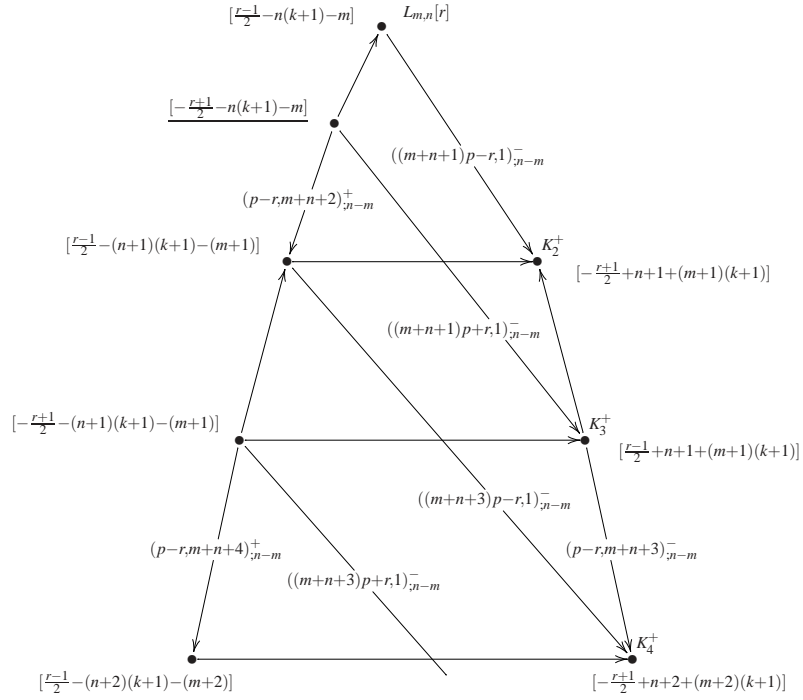


FIGURE 3.1. Subquotient structure of the left-wing twisted Wakimoto module $\mathcal{L}_{m,n}[r]$. For visual clarity, the view is “rotated back” by the spectral flow with $\theta = m - n$ (in the original view, the horizontal arrows are tilted by the angle α such that $\tan \alpha = n - m$). The values of h for operators (3.18) are shown in square brackets (one of these is underlined for later reference). The $(a, 1)_{n-m}^-$ arrows represent nonvanishing singular vectors $|\text{MFF}^-(a, 1; n - m)\rangle$, given by simple formula (2.11).

$$(3.21) \quad \chi_{m,n}^{\mathcal{K}_{2b+1}}(q, z) = \frac{(-1)^{n-m}}{q^{\frac{1}{8}} \vartheta_{1,1}(q, z)} \left(\sum_{a \geq 0} + \sum_{a \leq -n-m-2b-1} \right) \left(q^p \left(\frac{r}{2p} + m + b + a \right)^2 z^{\frac{r-1}{2} + (m+b+a)p} - q^p \left(\frac{r}{2p} + n + b + a \right)^2 z^{-\frac{r+1}{2} - (n+b+a)p} \right).$$

Similarly, and with the same conventions, the subquotient structure of the twisted Wakimoto module $\mathcal{R}_{m,n}[r]$ “generated” from $R_{m,n}[r]$ is shown in Fig. 3.2. In the figure, the overall twist is “undone” by the same amount as for the $\mathcal{L}_{m,n}[r]$ module; therefore, in describing the module $\mathcal{U}_{m,n}[r]$ with the extremal diagram in (3.15), the two diagrams in Figs. 3.1 and 3.2 must be placed next to each other, in accordance with the grades, which means placing the top node of $\mathcal{R}_{m,n}[r]$ $(m+n)p - r + 1$ units of charge to the right of the top node of $\mathcal{L}_{m,n}[r]$. It then follows that starting with the embedding level of K_2^+ , each node of $\mathcal{L}_{m,n}[r]$ has a corresponding node of $\mathcal{R}_{m,n}[r]$ as the nearest right neighbor.

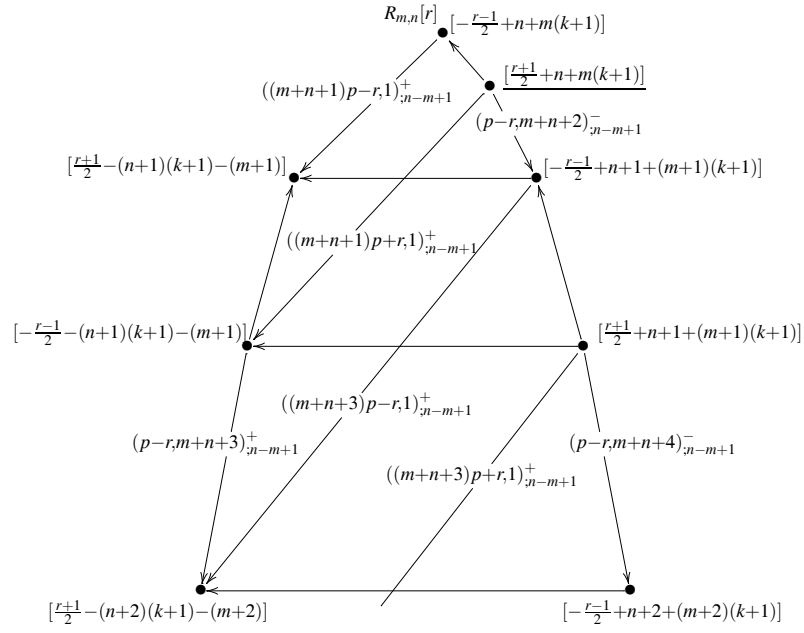


FIGURE 3.2. Subquotient structure of the left-wing twisted Wakimoto module $\mathcal{R}_{m,n}[r]$. For the ease of comparison with Fig. 3.1, the picture is “rotated” by the spectral flow with $\theta = m - n$. As in Fig. 3.1, the values of h are shown in square brackets (one of these is underlined for later reference).

The module $\mathcal{U}_{m,n}[r]$ is an extension of $\mathcal{L}_{m,n}[r]$ and $\mathcal{R}_{m,n}[r]$. We do not describe all of its structure, which we do not need, but describe the occurrence of the kernel of the two screenings below. For this, we first consider the maps provided by the screenings.

There are $\widehat{s\ell}(2)_k$ -homomorphisms

$$(3.22) \quad Q_+ : \mathcal{R}_{m+1,n}[r] \rightarrow \mathcal{L}_{m,n}[r],$$

$$(3.23) \quad Q_- : \mathcal{L}_{m,n+1}[r] \rightarrow \mathcal{R}_{m,n}[r],$$

whose construction can be outlined as follows. At the level of extremal states (see (3.15)), we have seen that $L_{m+1,n}[r](z)$ is annihilated by Q_+ , but the nearest-neighbor state

$$V_{m+1,n}[r] = e^{([j-n(k+1)-m]\xi + \frac{j}{k+2}(\psi_- + \psi_+) - n\psi_- - (m+1)\psi_+)} \cdot \varphi$$

is mapped under Q_+ as

$$Q_+ V_{m+1,n}[r](z) = e^{([j-n(k+1)-m]\xi + \frac{j}{k+2}(\psi_- + \psi_+) - n\psi_- - m\psi_+)} \cdot \varphi(z) = L_{m,n}[r](z).$$

Further acting with E_{n-m-1} gives (up to a nonzero factor)

$$(3.24) \quad Q_+ R_{m+1,n}[r](z) = \\ = (E_{-1+n-m})^{(m+n+1)(k+2)-r} e^{([j-n(k+1)-m]\xi + \frac{j}{k+2}(\psi_- + \psi_+) - n\psi_- - m\psi_+)} \cdot \varphi(z),$$

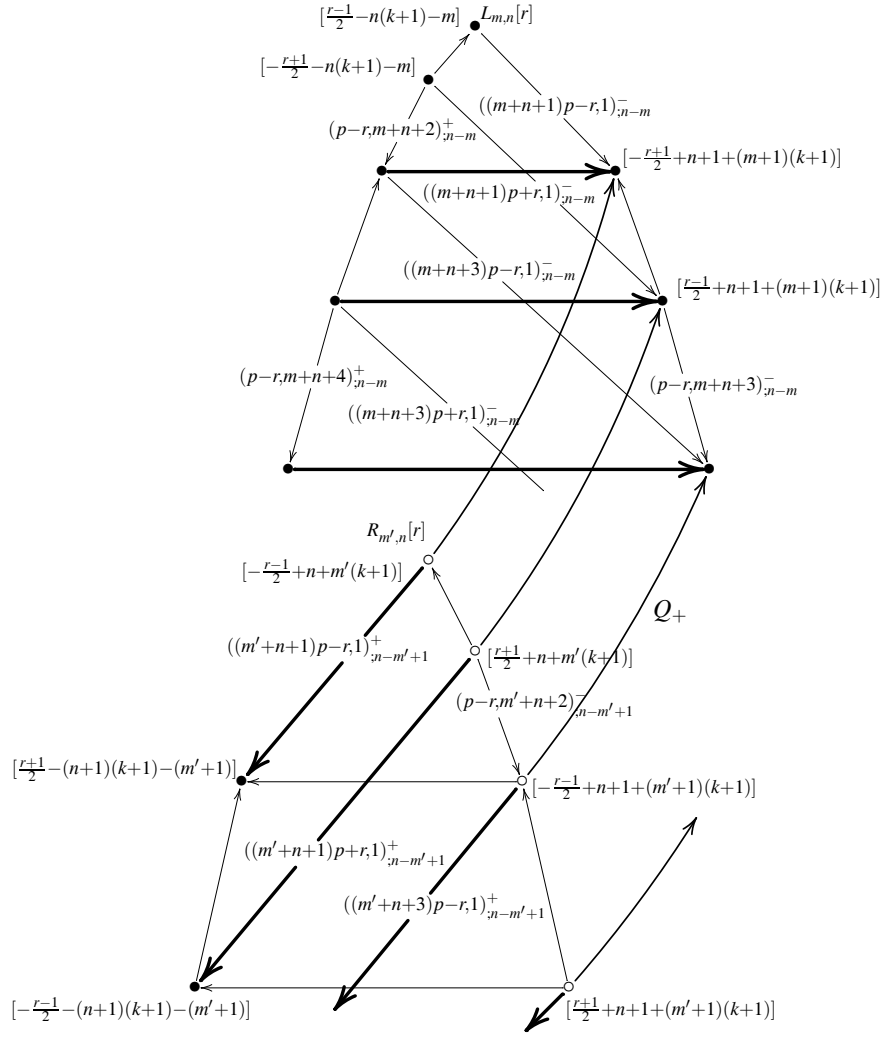


FIGURE 3.3. The left-wing map $Q_+ : \mathcal{R}_{m+1,n}[r] \rightarrow \mathcal{L}_{m,n}[r]$. Filled dots in the $\mathcal{R}_{m+1,n}[r]$ module denote subquotients that are in the kernel of Q_+ .

which is just the $|\text{MFF}^-((m+n+1)(k+2)-r, 1; n-m)\rangle$ singular vector constructed on the $L_{m,n}[r]$ state.

Figure 3.3 shows further details that make up the definition of $Q_+ : \mathcal{R}_{m+1,n}[r] \rightarrow \mathcal{L}_{m,n}[r]$. In the figure, we reproduce the pictures of the $\mathcal{L}_{m,n}[r]$ and $\mathcal{R}_{m',n}[r]$ modules, the latter shown just as in Fig. 3.2 for the ease of comparison, but with m' to be taken equal to $m+1$. Therefore, the twist of $\mathcal{R}_{m',n}[r]$ is $n-m'-1 = n-m-2$, with the result that the tilted $((m'+n+i)p \pm r, 1)^\pm$ -arrows in the lower part of the figure (shown boldfaced) should be drawn horizontally in the conventions applicable to the upper part (we repeat that the module $\mathcal{R}_{m',n}[r]$ is just copied from Fig. 3.2). But the map by Q_+ places these tilted $((m'+n+i)p \pm r, 1)^\pm$ -arrows just over the horizontal arrows in $\mathcal{L}_{m,n}[r]$ (also bold-

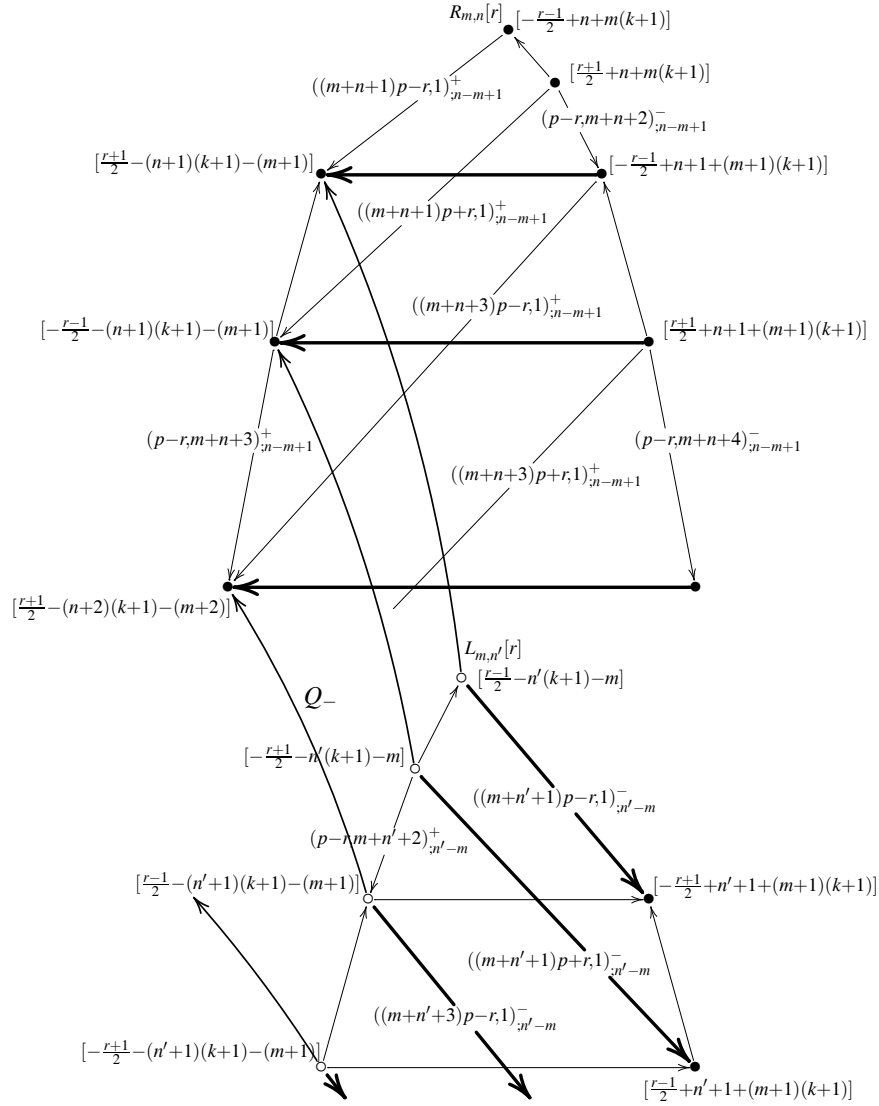


FIGURE 3.4. The left-wing map $Q_- : \mathcal{L}_{m,n+1}[r] \rightarrow \mathcal{R}_{m,n}[r]$. Filled dots in $\mathcal{L}_{m,n+1}[r]$ denote subquotients that are in the kernel of Q_- .

faced for this reason), which are oppositely directed because of the vanishing singular vectors; therefore, the tilted $\mathcal{R}_{m',n}[r]$ -arrows point to the kernel of Q_+ .

The map $Q_- : \mathcal{L}_{m,n+1}[r] \rightarrow \mathcal{R}_{m,n}[r]$ can be described similarly (see Fig. 3.4). We have

$$Q_- U_{m,n+1}[r](z) = e^{([n+m(k+1)-j]\xi + \frac{j}{k+2}(\psi_- + \psi_+) - n\psi_- - m\psi_+)} \cdot \varphi(z) = R_{m,n}[r](z)$$

and

$$(3.25) \quad Q_- L_{m,n+1}[r](z) = (F_{m-n-1})^{(m+n+1)(k+2)-r} R_{m,n}[r](z),$$

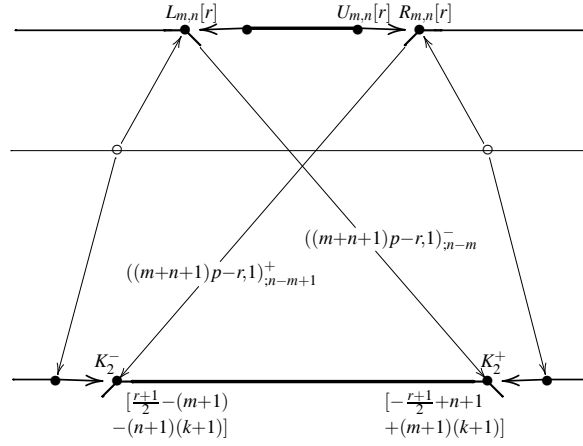


FIGURE 3.5. In the left-wing module $\mathcal{U}_{m,n}[r]$, the first embedding level where $\ker Q_- \cap \ker Q_+$ is nonzero.

which is just the $|\text{MFF}^+(r', 1; \theta)\rangle$ singular vector with $r' = (m+n+1)(k+2) - r$ and $\theta = n - m + 1$, constructed on $R_{m,n}[r]$. The $\mathcal{L}_{m,n'}[r]$ module is shown in Fig. 3.4 just as in Fig. 3.1, with n' to be set equal to $n+1$. The tilted $\mathcal{L}_{m,n'}[r]$ -arrows point to the kernel of Q_- .

In a “linear combination” of the notations used in Eq. (3.15) and Figs. A.1 and A.2, the extremal diagram of $\mathcal{U}_{m,n}[r]$ and the structure within the first several embedding levels are as shown in Fig 3.5. (We do *not* fully describe the structure of the first embedding level. There occurs a submodule in the kernel of Q_- and a submodule in the kernel of Q_+ , but the intersection of the kernels is zero.) As before, expressions in square brackets indicate the h parameter of the corresponding operators (3.18). The arrows pointing at K_2^+ and K_2^- from the respective nearest-neighbor states indicate that there is a *submodule* generated from either of the operators at the target nodes of these arrows,

$$(3.26) \quad K_2^+ = \mathcal{P}_2^+ e^{[-\frac{r+1}{2} + n + 1 + (m+1)(k+1)]\xi + \frac{j}{k+2}(\psi_- + \psi_+) - m\psi_- - m\psi_+} \cdot \varphi$$

$$= (E_{n-m-1})^{(m+n+1)(k+2)-r} e^{([j-n(k+1)-m]\xi + \frac{j}{k+2}(\psi_- + \psi_+) - m\psi_- - m\psi_+)} \cdot \varphi$$

and

$$(3.27) \quad K_2^- = \mathcal{P}_2^- e^{[\frac{r+1}{2} - m - 1 - (n+1)(k+1)]\xi + \frac{j}{k+2}(\psi_- + \psi_+) - m\psi_- - m\psi_+} \cdot \varphi$$

$$= (F_{m-n-1})^{(m+n+1)(k+2)-r} e^{([n+m(k+1)-j]\xi + \frac{j}{k+2}(\psi_- + \psi_+) - m\psi_- - m\psi_+)} \cdot \varphi,$$

where \mathcal{P}_2^\pm are differential polynomials of degree $(k+2)(m+n+1) - r$. This submodule is in the intersection of the kernels $\ker Q_- \cap \ker Q_+$. Moreover, at each next embedding level, $\ker Q_- \cap \ker Q_+$ is generated by the corresponding operators, which for even

embedding levels are given by

$$(3.28) \quad K_{2i}^+ = \mathcal{P}_{2i}^+ e^{([\frac{r+1}{2} + (n+i) + (m+i)(k+1)]\xi + \frac{j}{k+2}(\psi_- + \psi_+) - m\psi_- - m\psi_+) \cdot \varphi},$$

$$(3.29) \quad K_{2i}^- = \mathcal{P}_{2i}^- e^{([\frac{r+1}{2} - (m+i) - (n+i)(k+1)]\xi + \frac{j}{k+2}(\psi_- + \psi_+) - m\psi_- - m\psi_+) \cdot \varphi}.$$

We actually need the socle $\text{soc } \mathcal{U}_{m,n}[r]$ of $\mathcal{U}_{m,n}[r]$ and the kernel

$$(3.30) \quad \mathcal{K}_{m,n}[r] = \ker Q_- \cap \ker Q_+ \Big|_{\text{soc } \mathcal{U}_{m,n}[r]} = \bigoplus_{b \geq 1} \mathcal{K}_{2b}.$$

In what follows, we need the characters of the kernels in the diagonal case $m = n$; it then follows from (3.20) that

$$(3.31) \quad \begin{aligned} \chi^{\mathcal{K}_{m,m}[r]}(q, z) &= \sum_{b \geq 1} \chi^{\mathcal{K}_{2b}}(q, z) = \\ &= \frac{1}{q^{\frac{1}{8}} \vartheta_{1,1}(q, z)} \left[\sum_{a \geq m+1} (a-m) + \sum_{a \leq -m-1} (-a-m) \right] q^{p(\frac{r}{2p}-a)^2} (z^{-\frac{r+1}{2}+ap} - z^{\frac{r-1}{2}-ap}). \end{aligned}$$

3.2.2. Right wing. The structure of the right-wing modules is essentially dual to that of the left-wing modules. We recall the labeling in (3.12). For each pair of nonnegative integers m and n , the key operators in the extremal diagram of the (m, n) th module are

$$(3.32) \quad L'_{m,n}[r](z) = e^{([-j-1-n-m(k+1)]\xi + \frac{j}{k+2}(\psi_- + \psi_+) + n\psi_- + m\psi_+) \cdot \varphi(z)},$$

$$(3.33) \quad \begin{aligned} V'_{m,n}[r](z) &= e^{([-j-n-m(k+1)]\xi + \frac{j}{k+2}(\psi_- + \psi_+) + n\psi_- + m\psi_+) \cdot \varphi(z)} \\ &\doteq |\lambda^+(p-r, m+n+1); m-n+1\rangle, \end{aligned}$$

$$(3.34) \quad \begin{aligned} U'_{m,n}[r](z) &= e^{([j+m+n(k+1)]\xi + \frac{j}{k+2}(\psi_- + \psi_+) + n\psi_- + m\psi_+) \cdot \varphi(z)} \\ &\doteq |\lambda^+(r+p(m+n), 1); m-n\rangle, \end{aligned}$$

$$(3.35) \quad R'_{m,n}[r](z) = e^{([j+n(k+1)+m+1]\xi + \frac{j}{k+2}(\psi_- + \psi_+) + n\psi_- + m\psi_+) \cdot \varphi(z)},$$

where we recall that $j = \frac{r-1}{2}$. The extremal diagram (tilted in accordance with the twist $m-n$) is “turned inside out” compared with (3.15):

$$(3.36) \quad \begin{array}{c} \begin{array}{c} L'_{m,n}[r] \\ \bullet \\ \swarrow \\ E_{m-n} \rightarrow \bullet \\ \searrow \\ V'_{m,n}[r] \end{array} \\ \begin{array}{c} \bullet \\ \swarrow \\ U'_{m,n}[r] \\ \bullet \\ \searrow \\ F_{n-m} \rightarrow \bullet \\ \bullet \\ R'_{m,n}[r] \end{array} \end{array}$$

The spin of the twisted highest-weight state $U'_{m,n}[r]$ is $\lambda^+(r+p(m+n), 1)$. The angles are drawn in accordance with the conventions in Appendix A, to indicate twisted highest-weight states. The submodule “bordered by the two angles” is in $\ker Q_- \cap \ker Q_+$:

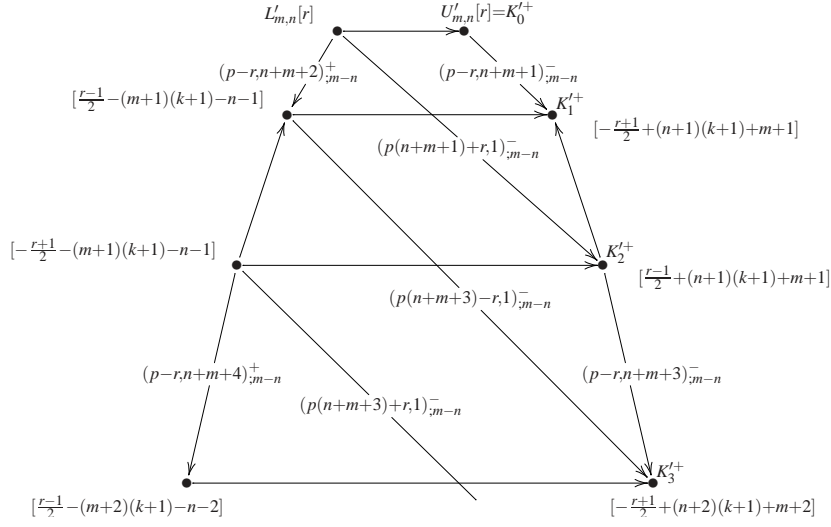


FIGURE 3.6. Subquotient structure of the right-wing twisted Wakimoto module $\mathcal{L}'_{m,n}[r]$.

as is easy to see, there are regular OPEs $e^{\psi_- \cdot \varphi(z)} U'_{m,n}[r](w) \propto (z-w)^{(m+n)p+r-1}$ and $e^{\psi_+ \cdot \varphi(z)} U'_{m,n}[r](w) \propto (z-w)^0$, and hence

$$Q_- U'_{m,n}[r](w) = Q_+ U'_{m,n}[r](w) = 0.$$

Next, we have $Q_+ R'_{m,n}[r](w) = U'_{m+1,n}[r](w)$ and, similarly, $Q_- L'_{m,n}[r](w) = V'_{m,n+1}[r](w)$, which is the right-neighbor state of $L'_{m,n+1}[r](w)$, as shown in (3.36). It is then not difficult to consecutively trace the maps of the lower-lying subquotients under both Q_- and Q_+ . About “half” the subquotient structure is shown in Fig. 3.6.

It follows that $\ker Q_- \cap \ker Q_+$ in $U'_{m,n}[r]$ is spanned by the irreducible subquotients \mathcal{K}'_b generated from the states labeled $K_b'^+$, $b = 0, 1, 2, \dots$, in Fig. 3.6. Their characters are readily found as (with the dependence on m and n indicated as a subscript and the dependence on r suppressed for notational simplicity)

$$(3.37) \quad \chi_{m,n}^{\mathcal{K}'_{2b}}(q, z) = \chi_{n,m}^{\mathcal{K}_{2b+1}}(q, z), \quad b \geq 0,$$

$$(3.38) \quad \chi_{m,n}^{\mathcal{K}'_{2b-1}}(q, z) = \chi_{n,m}^{\mathcal{K}_{2b}}(q, z), \quad b \geq 1$$

(see (3.20)–(3.21)). In particular, setting $b = m = n = 0$ in (3.37) gives the character of the integrable representation \mathcal{J}_r (see 2.3)

$$(3.39) \quad \chi_{0,0}^{\mathcal{K}'_0}(q, z) = \chi_r(q, z).$$

The character of the kernel

$$(3.40) \quad \mathcal{K}'_{m,n}[r] = \ker Q_- \cap \ker Q_+ \Big|_{\text{soc } U'_{m,n}[r]}$$

for each diagonal right-wing site (m, m) is given by

$$(3.41) \quad \begin{aligned} \chi^{\mathcal{K}'_{m,m}[r]}(q, z) &= \sum_{b \geq 1} \chi^{\mathcal{K}'_{2b-1}}(q, z) = \\ &= \frac{1}{q^{\frac{1}{8}} \vartheta_{1,1}(q, z)} \left[\sum_{a \geq m+1} (a-m) + \sum_{a \leq -m-1} (-a-m) \right] q^{\left(\frac{r}{2p}-a\right)^2} \left(z^{-\frac{r+1}{2}+ap} - z^{\frac{r-1}{2}-ap} \right). \end{aligned}$$

3.2.3. The middle. The two-wing creature is obtained by joining the two wings via the map

$$Q_+ \circ Q_- : \mathcal{U}_{1,1}[r] \rightarrow \mathcal{U}'_{0,0}[r].$$

At the ‘‘corner’’ of the right wing, in the module $\mathcal{U}'_{0,0}[r]$, the $U'[r]$ operator (3.34) is given by

$$U'_{0,0}[r](z) = e^{(j\xi + \frac{j}{k+2}(\psi_- + \psi_+)) \cdot \varphi(z)}.$$

At the corner of the left wing, the corresponding operator is the one in (3.14) with $m = n = 1$. Evidently,

$$Q_- U_{1,1}[r](z) = e^{((k+1-j)\xi + \frac{j}{k+2}\psi_- + (\frac{j}{k+2}-1)\psi_+) \cdot \varphi(z)}$$

and it then follows from (3.5) that

$$(-1)^{p-r} (p-r)! Q_+ e^{((k+1-j)\xi + \frac{j}{k+2}(\psi_- + \psi_+) - \psi_+) \cdot \varphi(z)} = (E_{-1})^{p-r} e^{(j\xi + \frac{j}{k+2}(\psi_- + \psi_+)) \cdot \varphi(z)}.$$

Thus,

$$Q_+ Q_- U_{1,1}[r](z) = \frac{1}{(p-r)!} |_{\text{MFF}^-} (p-r, 1; j) \in \mathcal{U}'_{0,0}[r](z).$$

The irreducible quotient generated from $\mathcal{U}'_{0,0}[r]$ is in the cohomology. It is not difficult to trace the action of $Q_+ Q_-$ on the subquotients in $\mathcal{U}_{1,1}[r]$.

The maps constructed above finally give the butterfly resolution of integrable representations.

3.3. Acyclic butterfly complexes. We next consider the free-field modules whose elements are (the states associated with) operators descendant from (3.1) for $2\lambda + 1 \in \{p+1, \dots, 2p-1\}$ (about the second half of the range in (3.11)); we parameterize the λ as

$$\lambda = \frac{p}{2} + \frac{r-1}{2},$$

with $r \in \{1, 2, \dots, k+1\}$. We keep the notation in (3.13).

The required modules can then be constructed by replacing $j \mapsto j + \frac{p}{2}$ in the formulas in **3.2**. In accordance with (3.2), this is equivalent to the shifts $m \mapsto m - \frac{1}{2}$, $n \mapsto n - \frac{1}{2}$ for each module in the left wing. Consequently, we can describe each left-wing module just as in Eqs. (3.14)–(3.15) and Figs. 3.1 and 3.2, but with

$$(3.42) \quad m, n \in \left\{ \frac{1}{2}, \frac{3}{2}, \dots \right\} \quad \text{in the left wing.}$$

Taking *both* m and n half-integer leads to no conflict because only $m + n$ has to be integer (see Figs. 3.1 and 3.2, where all the arrows (singular vectors) depend only on $m + n$; in particular, the module at the “corner” of the left wing is the one with $m = n = \frac{1}{2}$ and hence $m + n = 1$). From (3.20), the character of $\mathcal{K}_{m,m}[r] = \ker Q_- \cap \ker Q_+ \big|_{\text{soc } \mathcal{U}_{m,m}[r]}$ for half-integer m is given by

$$(3.43) \quad \begin{aligned} \chi^{\mathcal{K}_{m,m}[r]}(q, z) &= \sum_{b \geq 1} \chi_{m,m}^{\mathcal{K}_{2b}}(q, z) \\ &= \frac{1}{q^{\frac{1}{8}} \vartheta_{1,1}(q, z)} \left[\sum_{a \geq \mu+1} (a - \mu) + \sum_{a \leq -\mu} (1 - a - \mu) \right] q^{p(\frac{r}{2p} - a + \frac{1}{2})^2} \\ &\quad \times \left(z^{-\frac{r+1}{2} + (a - \frac{1}{2})p} - z^{\frac{r-1}{2} - (a - \frac{1}{2})p} \right), \end{aligned}$$

where $\mu = m + \frac{1}{2}$ takes positive integer values in the left wing.

In the right wing, similarly, the modules corresponding to the spin $\lambda = \frac{p}{2} + \frac{r-1}{2}$ can be described by formulas in **3.2.2** with the shift $m \mapsto m + \frac{1}{2}$, $n \mapsto n + \frac{1}{2}$, and hence

$$(3.44) \quad m, n \in \left\{ \frac{1}{2}, \frac{3}{2}, \dots \right\} \quad \text{in the right wing.}$$

The kernel $\ker Q_- \cap \ker Q_+$ in the socle is given by the sum of \mathcal{K}'_{2b-1} for $b \geq 1$, and its character is easily expressed as in (3.43).

As the result of passing to half-integer m and n , the resolution becomes acyclic.⁷ It turns out that $U'_{\frac{1}{2}, \frac{1}{2}}[r]$ is now in the image of $Q_- \circ Q_+$: in the module $\mathcal{U}_{\frac{1}{2}, \frac{1}{2}}[r]$, we consider the states at the level r relative to the top and at the grades $-\frac{r+1}{2} - \frac{p}{2} + 1, \dots, \frac{r+1}{2} + \frac{p}{2} - 1$. In superimposing the pictures for $\mathcal{L}_{m,m}[r]$ in Fig. 3.1 and $\mathcal{R}_{m,m}[r]$ in Fig. 3.2 as explained in **3.2.1**, these are the states in between the nodes whose grades are underlined in Figs. 3.1 and Fig. 3.2, for $m = n = \frac{1}{2}$. A codimension-1 submodule in this grade is in the kernel of $Q_- \circ Q_+$, but the one-dimensional quotient is mapped onto the states between (and including) $V'_{\frac{1}{2}, \frac{1}{2}}[r]$ and $U'_{\frac{1}{2}, \frac{1}{2}}[r]$ in (3.36).

3.4. “Steinberg” modules.

3.4.1. $\lambda = (k+1)/2$. This case corresponds to setting $r = p$ in the operators considered in **3.2**. As a result, each of the diagrams in Figs. 3.1 and 3.2 collapses into a single embedding chain, in accordance with the degenerations of the MFF singular vectors discussed in **2.2.2**. The details are quite standard, and we omit them. The kernel $\ker Q_- \cap \ker Q_+$ is spanned by irreducible subquotients whose highest-weight vectors have the charges $n + b + m(k+1+b) + \frac{k+1}{2}$, $b \geq 0$. The character of the kernel in the socle of the left wing is given by

⁷It is difficult to resist invoking a superficial analogy and referring to this case as a Neveu–Schwartz one (recall that in the previous case, the cohomology occurred in the “zeroth” module, which is now absent because of half-integer-valued labels).

$$(3.45) \quad \chi^{\mathcal{K}_{m,n}[p]}(q, z) = \frac{(-1)^{n-m}}{q^{\frac{1}{8}} \vartheta_{1,1}(q, z)} \sum_{b \geq 0} \left(q^{p(m+b+\frac{1}{2})^2} z^{\frac{p-1}{2}+(m+b)p} - q^{p(n+b+\frac{1}{2})^2} z^{-\frac{p+1}{2}-(n+b)p} \right).$$

In the socle of the right wing, a somewhat different, but easily reproducible subquotient structure results in the character of $\ker Q_- \cap \ker Q_+$ given by

$$(3.46) \quad \chi^{\mathcal{K}'_{m,n}[p]}(q, z) = \frac{(-1)^{n-m}}{q^{\frac{1}{8}} \vartheta_{1,1}(q, z)} \sum_{b \geq 0} \left(q^{p(n+b+\frac{1}{2})^2} z^{\frac{p-1}{2}+(n+b)p} - q^{p(m+b+\frac{1}{2})^2} z^{-\frac{p+1}{2}-(m+b)p} \right).$$

It is worth seeing how the middle of the resolution restructures compared with the case in **3.2** (making the complex acyclic). In the ‘‘corner’’ of the left wing, we have the operator

$$U_{1,1}[p](w) = e^{\left(\frac{k+1}{2} \xi_+ + \frac{k+1}{2(k+2)} (\psi_- + \psi_+) - \psi_- - \psi_+ \right) \cdot \varphi(w)}.$$

It develops a first-order pole in the OPE with $e^{\psi_- \cdot \varphi(u)}$ and the resulting operator, moreover, has a first-order pole with $e^{\psi_+ \cdot \varphi(z)}$; therefore,

$$Q_+ Q_- U_{1,1}[p](w) = e^{\left(\frac{k+1}{2} \xi_+ + \frac{k+1}{2(k+2)} (\psi_- + \psi_+) \right) \cdot \varphi(w)} = U'_{1,1}[p](w).$$

3.4.2. $\lambda = (2k+3)/2$. For $r = 2p$, the structure of $\mathcal{U}_{m,n}[p]$ also degenerates; in particular, for $m = n = 1$, (the states corresponding to) the operators

$$L_{1,1}[p](z) = e^{\left(-\frac{1}{2} \xi_- - \frac{1}{2p} (\psi_- + \psi_+) \right) \cdot \varphi(z)} \quad \text{and} \quad R_{1,1}[p](z) = e^{\left(\frac{1}{2} \xi_- - \frac{1}{2p} (\psi_- + \psi_+) \right) \cdot \varphi(z)}$$

are ‘‘facing each other,’’ i.e., have no extremal states between them in a picture similar to (3.15). We omit the details to avoid further lengthening this already long section.

4. W -ALGEBRA, ITS REPRESENTATIONS, CHARACTERS, AND MODULAR TRANSFORMATIONS

The aim of this section is to establish the main result stated in the Introduction. In **4.1**, we first identify the W -algebra generators in the centralizer of the screenings. In **4.2**, we construct $2p$ W -algebra representations \mathcal{Y}_r^\pm , $1 \leq r \leq p$, evaluate their characters $\chi_r^\pm(\tau, \nu)$, and establish their spectral-flow transformation properties. The spectral flow closes if $2p$ functions $\omega_r^\pm(\tau, \nu)$, $1 \leq r \leq p$, are added. Modular transformation properties are studied in **4.3**. Certain combinations of the $\chi_r^\pm(\tau, \nu)$, $\omega_r^\pm(\tau, \nu)$ and the integrable characters $\chi_r(\tau, \nu)$, $1 \leq r \leq p-1$, become parts of multiplets, i.e., transform in representations of the form $\mathbb{C}^n \otimes \pi$, where π is some $SL(2, \mathbb{Z})$ -representation and $n = 2$ and 3 , where the \mathbb{C}^2 and \mathbb{C}^3 representations are realized on polynomials in τ of respective degrees 1 and 2 (see **C.2.1** and **C.3.1**). In addition, a certain triangular structure emerges, with terms of

the form ν times “lower” characters occurring in modular transformations of the “higher” characters. The precise result is in Lemmas **4.3.3** and **4.3.5**.

4.1. The $W^\pm(z)$ currents. The W -algebra representing the symmetry of the model is defined as the maximum local algebra acting in the kernel. As discussed in the Introduction, we somewhat restrict this definition by taking only the generators that map between $\widehat{\mathfrak{sl}}(2)_k$ modules of the same twist.

4.1.1. Locality and the vacuum representation. We first select operators that are local with respect to all operators in the kernel. The kernel $\ker Q_- \cap \ker Q_+$ is spanned by operators of the general form

$$(4.1) \quad (\text{modes}) \cdot e^{(a\xi + \frac{r-1}{2p}(\psi_- + \psi_+) + m\psi_- + n\psi_+) \cdot \varphi(z)}$$

with integer a , integer r , and *simultaneously* integer or half-integer m and n , where (modes) are differential polynomials in the $\widehat{\mathfrak{sl}}(2)_k$ currents. In the OPE

$$e^{a\xi \cdot \varphi(z)} e^{(b\xi + \frac{r-1}{2p}(\psi_- + \psi_+) + m\psi_- + n\psi_+) \cdot \varphi(w)} \propto (z-w)^{a(m-n)},$$

the exponent $a(m-n)$ is therefore always integer; noninteger exponents thus occurs only in the OPEs

$$e^{\frac{r-1}{2p}(\psi_- + \psi_+) \cdot \varphi(z)} e^{\frac{r'-1}{2p}(\psi_- + \psi_+) \cdot \varphi(w)} \propto (z-w)^{\frac{(r-1)(r'-1)}{2p}}$$

and in the OPEs involving $e^{(m\psi_- + n\psi_+) \cdot \varphi}$ in the case of half-integer m and n . It follows that the vertices as in (4.1) with $r = 1$ and integer m and n produce integer-valued exponents — no nonlocalities — in the OPEs with all of the vertices encountered in the kernel (in checking the OPE with $e^{(m'\psi_- + n'\psi_+) \cdot \varphi}$, it is essential that m' and n' can only be half-integer simultaneously). We therefore identify the vacuum representation of the W -algebra with the kernel $\ker Q_- \cap \ker Q_+$ in the socle of the butterfly resolution of the $r = 1$ integrable representation. That is, the vacuum representation of the W -algebra is given by

$$\bigoplus_{m \geq 1} \mathcal{K}_{m,m}[1] \oplus \bigoplus_{m \geq 0} \mathcal{K}'_{m,m}[1],$$

where $\mathcal{K}_{m,n}[r]$ are defined in (3.30) and $\mathcal{K}'_{m,n}[r]$ in (3.40).

4.1.2. $W^\pm(z)$ currents. The fields $W^-(z)$ and $W^+(z)$ generating the W -algebra are associated with certain singular vectors as follows.

- $W^-(z)$ corresponds to the singular vector $|\text{MFF}^+(p-1, 3; 1)\rangle$ constructed on the vertex $V'_{1,1}[1] = e^{(-p\xi + \psi_- + \psi_+) \cdot \varphi}$ (see (3.33) and (3.36); from (3.33), this vertex represents the twisted highest-weight state $|\lambda^+(p-1, 3; 1)\rangle$.⁸ Therefore, $W^-(z)$

⁸The $k = 0$ example given below is already sufficiently generic, and may help visualize the positions of the various states.

is a $(\theta = 1)$ -twisted highest-weight operator of dimension $4p - 2$ and charge $-2p + 1$. (Its *spin* is $-2p + 1 + \frac{k}{2} = \lambda^+(1, 4) = \lambda^-(4p - 1, 1)$.) More explicitly,

$$(4.2) \quad W^-(z) = (F_{-1})^{3p-1} (E_0)^{2p-1} (F_{-1})^{p-1} (E_0)^{-1} (F_{-1})^{-p-1} e^{(-p\xi + \psi_- + \psi_+) \cdot \varphi(z)}.$$

- $W^+(z)$ corresponds to the singular vector $|\text{MFF}^-(3p - 1, 1)\rangle$ constructed on the vertex $L_{1,1}[1] = e^{(-p\xi - \psi_- - \psi_+) \cdot \varphi}$ (see (3.16)). This singular vector the $((m + n + 1)p - r, 1)_{;n-m}^-$ arrow in Fig. 3.1, where we now set $m = n = 1$ and $r = 1$. It follows that $W^+(z)$ is a highest-weight operator of the same dimension $4p - 2$ and of the charge $\lambda^+(4p - 1, 1) = \lambda^-(1, 4) = 2p - 1$:

$$(4.3) \quad W^+(z) = (E_{-1})^{3p-1} e^{(-p\xi - \psi_- - \psi_+) \cdot \varphi(z)} = (\partial^{3p-1} e^{\psi_+ \cdot \varphi(z)}) e^{((2p-1)\xi - \psi_- - 2\psi_+) \cdot \varphi(z)}$$

(where we used (3.24) to evaluate a power of E_{-1}).

Once again, the meaning of (4.2) is that $W^-(z)$ is the operator whose corresponding state is given by the appropriate singular vector (expressed as in the MFF formulas) evaluated on the state corresponding to the vertex $e^{(-p\xi + \psi_- + \psi_+) \cdot \varphi(z)}$.

The OPE of the currents starts as

$$W^+(z) W^-(w) = \frac{\mathcal{O}_{p-1}(w)}{(z-w)^{7p-3}} + \dots,$$

where \mathcal{O}_{p-1} is the charge-0 dimension- $(p-1)$ operator at the first embedding level (the level of K_1^+) in Fig. 3.6 with $m = n = 0$ and $r = 1$: up to a nonzero factor, therefore,

$$\mathcal{O}_{p-1} = F_0^{p-1} |\text{MFF}^-(p-1, 1)\rangle = F_0^{p-1} E_{-1}^{p-1} U_1'[0, 0]$$

(where $U_1'[0, 0](w) = 1$ is the unit operator). For $k = 0$ and 1, in particular,

$$\mathcal{O}_1(z) = 2H(z) = \partial\varphi_-(z) - \partial\varphi_+(z),$$

$$\mathcal{O}_2(z) = -4EF(z) + 8HH(z) + 4\partial H(z)$$

$$= \partial\varphi_+ \partial\varphi_+(z) - 4\partial\varphi_+ \partial\varphi_-(z) + \partial\varphi_- \partial\varphi_-(z) + \partial^2\varphi_+(z) + \partial^2\varphi_-(z)$$

up to nonzero factors.

4.1.3. Example. For $k = 0$, some details of the vacuum representation are shown in Fig. 4.1. As regards explicit expressions for the generators, with the negative powers involved in $W^-(z)$ understood in the standard MFF setting in (2.10), we find that the five factors in (4.2) evaluate as

$$\begin{aligned} F_{-1}^{3p-1} E_0^{2p-1} F_{-1}^{p-1} E_0^{-1} F_{-1}^{-p-1} &= F_{-1}^5 E_0^3 F_{-1} E_0^{-1} F_{-1}^{-3} \\ &= 120F_{-3} + 60H_{-2}F_{-1} + 120H_{-1}F_{-2} + 60H_{-1}^2 F_{-1} - 12E_{-1}F_{-1}^2 + E_0^2 F_{-1}^3 \\ &\quad - 30E_0 F_{-2} F_{-1} - 18E_0 H_{-1} F_{-1}^2. \end{aligned}$$

The corresponding free-field expression is

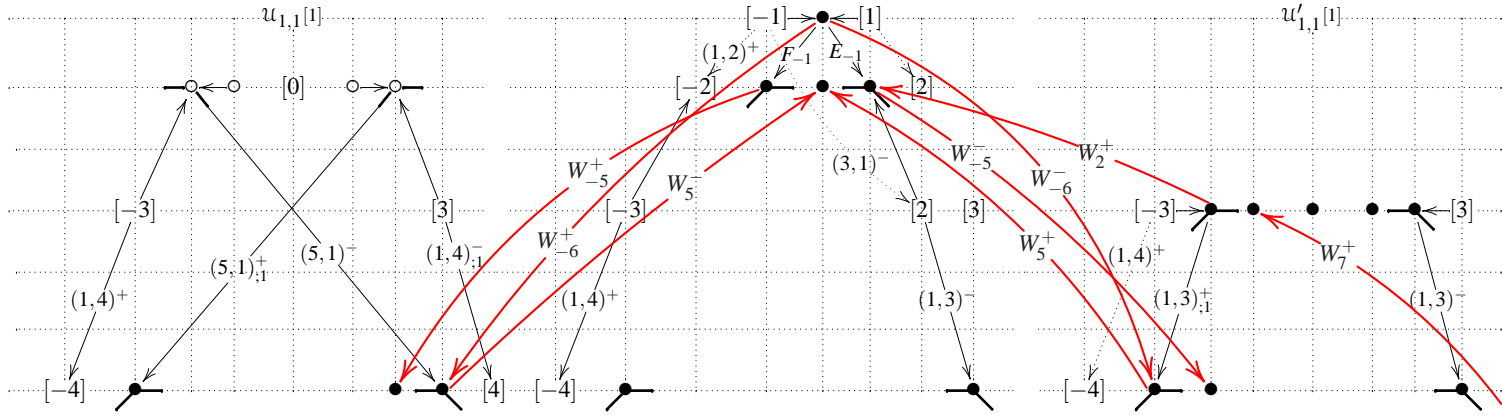


FIGURE 4.1. Some details of the W -algebra vacuum representation at $k = 0$. Three copies of a two-dimensional lattice indicate the (charge, dimension) bigrade. The picture shows three modules: $\mathcal{U}_{1,1}[1]$ (from the left wing of the butterfly) and $\mathcal{U}'_{0,0}[1]$ and $\mathcal{U}'_{1,1}[1]$ (from the right wing). Some sites are shown with their *charges* in square brackets. Boldfaced sites are in $\ker Q_- \cap \ker Q_+$, but not all of them are in the socle: as the maps show, the extremal states of the right-hand module $\mathcal{U}'_{1,1}[1]$ in the grades $-2, \dots, 2$ (the “lid”), although in $\ker Q_- \cap \ker Q_+$, are not in the socle, and hence not in the vacuum representation of the W -algebra (among the descendants of these states, only those in the submodule with extremal states in the grades $(-3, 6), \dots, (3, 6)$ are in the vacuum representation). The states corresponding to the $W^\pm(z)$ generators are at the grades $(\pm(2p-1), 4p-2)$ (in the current case where $k = 0$, at $(3, 6)$ for W^+ and $(-3, 6)$ for W^-); the $W_{-4p+2}^\pm = W_{-6}^\pm$ arrows map into them from the vacuum. “Angles” denote highest-weight state or twisted highest-weight states. Tilted downward $(r', s')^\pm$ -arrows show some of the nonvanishing singular vectors in the corresponding modules $\mathcal{L}_{1,1}[1]$, $\mathcal{R}_{1,1}[1]$, and $\mathcal{L}'_{m,m}[1]$, $\mathcal{R}'_{m,m}[1]$ ($m = 0, 1$), as in Figs. 3.1, 3.2, and 3.6. The twist is additionally indicated with $_{,1}$. Tilted upward arrows are the reverse of arrows that would lead to vanishing singular vectors. In the middle module, the action of E_{-1} and F_{-1} on the vacuum (the top middle state) is also shown.

$$W^-(z) = (-6\partial\varphi_- \partial^2\varphi_+(z) - 18\partial^2\varphi_- \partial\varphi_-(z) - 12\partial^2\varphi_- \partial\varphi_+(z) + 3\partial\varphi_- \partial\varphi_- \partial\varphi_-(z) \\ + 6\partial\varphi_- \partial\varphi_- \partial\varphi_+(z) + 3\partial\varphi_- \partial\varphi_+ \partial\varphi_+(z) + 12\partial^3\varphi_-(z)) e^{(-3\xi_+ + \psi_- + \psi_+) \cdot \varphi(z)}.$$

From (4.3), we also have

$$W^+(z) = (10\partial^3\varphi_+ \partial^2\varphi_+(z) + 5\partial^4\varphi_+ \partial\varphi_+(z) + 15\partial^2\varphi_+ \partial^2\varphi_+ \partial\varphi_+(z) \\ + 10\partial^3\varphi_+ \partial\varphi_+ \partial\varphi_+(z) + 10\partial^2\varphi_+ \partial\varphi_+ \partial\varphi_+ \partial\varphi_+(z) + \partial\varphi_+ \partial\varphi_+ \partial\varphi_+ \partial\varphi_+ \partial\varphi_+(z) \\ + \partial^5\varphi_+(z)) e^{(3\xi_- - \psi_- - \psi_+) \cdot \varphi(z)}.$$

4.2. “Narrow” W -algebra representations \mathcal{Y}_r^\pm and their characters. First, the integrable $\widehat{sl}(2)_k$ representations \mathcal{J}_r , $1 \leq r \leq p-1$, are W -algebra representations. Next, the resolutions in Sec. 3 allow constructing $2p$ W -algebra representations, denoted by \mathcal{Y}_r^\pm , $1 \leq r \leq p$, in what follows. Their underlying vector spaces are the sums of irreducible $\widehat{sl}(2)_k$ subquotients in the kernel $\ker Q_- \cap \ker Q_+$ evaluated in the socle of each module along the horizontal symmetry line of the butterfly in (1.1). That is, \mathcal{Y}_r^+ is the sum of $\mathcal{K}_{m,m}[r]$ in (3.30) and $\mathcal{K}'_{m,m}[r]$ in (3.40):

$$\mathcal{Y}_r^+ = \bigoplus_{m \geq 1} \mathcal{K}_{m,m}[r] \oplus \bigoplus_{m \geq 0}^* \mathcal{K}'_{m,m}[r]$$

where the asterisk at the direct sum is a “lazy notation” to indicate that at $m = 0$, the subquotient given by the integrable representation \mathcal{J}_r (see (3.39)) is to be omitted. Similarly, \mathcal{Y}_r^- is given by an analogous construction in the “Neveu–Schwarz sector,” with summations going over half-integer values:

$$\mathcal{Y}_r^- = \bigoplus_{m \geq \frac{1}{2}} \mathcal{K}_{m,m}[r] \oplus \bigoplus_{m \geq \frac{1}{2}} \mathcal{K}'_{m,m}[r].$$

We next find the characters

$$\chi_r^\pm(q, z) = \chi^{\mathcal{Y}_r^\pm}(q, z).$$

In what follows, we write θ_r for $\theta_{r,p}$ and similarly for theta-function derivatives, and sometimes omit the theta-function argument when it is just (q, z) or, equivalently, (τ, ν) (see Appendix B for the theta-function conventions).

4.2.1. Lemma. *The W -algebra characters $\chi_r^\pm(q, z)$ are given by (see (2.15) for $\Omega(q, z)$)*

$$\chi_r^+(q, z) = \frac{1}{\Omega(q, z)} \left(\frac{r^2}{4p^2} (\theta_{-r} - \theta_r) + \frac{r}{p^2} (\theta'_{-r} + \theta'_r) + \frac{1}{p^2} (\theta''_{-r} - \theta''_r) \right),$$

$$\chi_r^-(q, z) = \frac{1}{\Omega(q, z)} \left(\left(\frac{r^2}{4p^2} - \frac{1}{4} \right) (\theta_{p-r} - \theta_{p+r}) + \frac{r}{p^2} (\theta'_{p-r} + \theta'_{p+r}) + \frac{1}{p^2} (\theta''_{p-r} - \theta''_{p+r}) \right)$$

for $1 \leq r \leq p-1$ (with the theta-function arguments (q, z) omitted in the right-hand sides), and

$$\chi_p^+(q, z) = \frac{2\theta'_p(q, z)}{p\Omega(q, z)}, \quad \chi_p^-(q, z) = \frac{2\theta'_0(q, z)}{p\Omega(q, z)}.$$

This readily follows by direct calculation: for $1 \leq r \leq p-1$, the sum of characters in (3.31) and (3.41) yields

$$\begin{aligned} \chi_r^+(q, z) &= \sum_{m \geq 1} \chi^{\mathcal{K}_{m,m}[r]}(q, z) + \sum_{m \geq 0}^* \chi^{\mathcal{K}'_{m,m}[r]}(q, z) \\ &= \frac{1}{q^{\frac{1}{8}} \vartheta_{1,1}(q, z)} \sum_{a \in \mathbb{Z}} a^2 q^{p(\frac{r}{2p} + a)^2} (z^{-\frac{r+1}{2} - ap} - z^{\frac{r-1}{2} + ap}), \end{aligned}$$

where the asterisk affects the $m = 0$ term just as above. For \mathcal{Y}_r^- , no term is excluded at $m = 0$, and summation over half-integer values of m gives

$$\begin{aligned} \chi_r^-(q, z) &= \sum_{m \geq \frac{1}{2}} \chi^{\mathcal{K}_{m,m}[r]}(q, z) + \sum_{m \geq \frac{1}{2}} \chi^{\mathcal{K}'_{m,m}[r]}(q, z) = 2 \sum_{m \geq \frac{1}{2}} \chi^{\mathcal{K}'_{m,m}[r]}(q, z) = \\ &= \frac{1}{q^{\frac{1}{8}} \vartheta_{1,1}(q, z)} \sum_{a \in \mathbb{Z} + \frac{1}{2}} (a^2 - \frac{1}{4}) q^{p(\frac{r}{2p} + a)^2} (z^{-\frac{r+1}{2} - ap} - z^{\frac{r-1}{2} + ap}). \end{aligned}$$

In terms of theta-functions, this gives the above formulas. For $r = p$, we find from (3.45) and (3.46) that

$$\chi_p^+(q, z) = \sum_{m \geq 1} \chi^{\mathcal{K}_{m,m}[p]}(q, z) + \sum_{m \geq 0} \chi^{\mathcal{K}'_{m,m}[p]}(q, z) = \frac{2\theta'_p(q, z)}{p\Omega(q, z)},$$

and a similar calculation leads to $\chi_p^-(q, z)$.

4.2.2. Spectral flow transformation properties. We next study the spectral-flow orbit of the above characters $\chi_r^\pm(q, z)$. It follows from (B.5)–(B.8) that their spectral flow transformation properties are given by

$$\begin{aligned} \chi_{r;1}^+(q, z) &= -\chi_r^-(q, z) - \omega_r^-(q, z) - \frac{1}{2}\chi_{p-r}(q, z), \\ \chi_{r;1}^-(q, z) &= -\chi_r^+(q, z) - \omega_r^+(q, z) \end{aligned}$$

for $1 \leq r \leq p-1$, and

$$\begin{aligned} \chi_{p;1}^+(q, z) &= -\chi_p^-(q, z) - \omega_p^-(q, z), \\ \chi_{p;1}^-(q, z) &= -\chi_p^+(q, z) - \omega_p^+(q, z), \end{aligned}$$

where $\chi_r(q, z)$ are the integrable representation characters and

$$(4.4) \quad \begin{aligned} \omega_r^+(q, z) &= \frac{1}{\Omega(q, z)} \left(\frac{r}{2p} (\theta_r(q, z) + \theta_{-r}(q, z)) - \frac{1}{p} (\theta'_r(q, z) - \theta'_{-r}(q, z)) \right), \\ \omega_r^-(q, z) &= \frac{1}{\Omega(q, z)} \left(\frac{r}{2p} (\theta_{p-r}(q, z) + \theta_{-p}(q, z)) - \frac{1}{p} (\theta'_{r-p}(q, z) - \theta'_{p-r}(q, z)) \right) \end{aligned}$$

for $1 \leq r \leq p-1$, and

$$(4.5) \quad \omega_p^+(q, z) = \frac{\theta_p(q, z)}{\Omega(q, z)}, \quad \omega_p^-(q, z) = \frac{\theta_0(q, z)}{\Omega(q, z)}.$$

Defining the spectral-flow transformation rule for ω_r^\pm as for characters (see (2.5)), we further calculate the transformation laws

$$\begin{aligned} \omega_{r;1}^+(q, z) &= -\omega_r^-(q, z) - \frac{1}{2}\chi_{p-r}(q, z), \\ \omega_{r;1}^-(q, z) &= -\omega_r^+(q, z) + \frac{1}{2}\chi_r(q, z) \end{aligned}$$

for $1 \leq r \leq p-1$, while $\omega_{p;1}^\pm(q, z) = \omega_p^\pm(q, z)$. We do not detail the representation-theory interpretation of the ω_r^\pm in this paper.

4.3. Modular transformation properties. We now evaluate the modular transformation properties of the $5p-1$ functions given by the above W -algebra characters χ_r^\pm , $1 \leq r \leq p$, the integrable representation characters χ_r , $1 \leq r \leq p-1$, and the ω_r^\pm , $1 \leq r \leq p$.

4.3.1. The “minimal” $SL(2, \mathbb{Z})$ -representation $\mathcal{R}_{\text{int}}(p)$. We first recall the well-known transformation formulas

$$(4.6) \quad \chi_r(\tau+1, \nu) = \lambda_{r,p} \chi_r(\tau, \nu), \quad \lambda_{r,p} = e^{i\pi(\frac{r^2}{2p} - \frac{1}{4})},$$

$$(4.7) \quad \chi_r(-\frac{1}{\tau}, \frac{\nu}{\tau}) = \sqrt{\frac{2}{p}} e^{i\pi k \frac{\nu^2}{2\tau}} \sum_{s=1}^{p-1} \sin \frac{\pi r s}{p} \chi_s(\tau, \nu),$$

which just state that the integrable representation characters χ_r span a $(p-1)$ -dimensional $SL(2, \mathbb{Z})$ -representation $\mathcal{R}_{\text{int}}(p)$.

4.3.2. Remark. Strictly speaking, to define $\mathcal{R}_{\text{int}}(p)$, we have to “eliminate” the $e^{i\pi k \frac{\nu^2}{2\tau}}$ factor in (4.7); this then gives the $SL(2, \mathbb{Z})$ -representation uniquely defined by the T -transformation as in (4.6) and the S -transformation

$$(4.8) \quad S\chi_r = \sqrt{\frac{2}{p}} \sum_{s=1}^{p-1} \sin \frac{\pi r s}{p} \chi_s.$$

The relation between (4.7) and (4.8) can be understood in terms of an automorphy factor. The argument (with some details omitted, see [58] and also [10, Sec. 4.1]) is based on the fact that $j(\gamma; \tau, \nu)$ defined for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as

$$(4.9) \quad j(\gamma; \tau, \nu) = \zeta_{c,d}^{-1} (c\tau+d)^{-\frac{1}{2}} e^{-i\pi \frac{c\nu^2}{c\tau+d}},$$

where $\zeta_{c,d}$ is a certain eighth root of unity [58], satisfies the cocycle condition $j(\gamma\gamma'; \tau, \nu) = j(\gamma'; \tau, \nu) j(\gamma; \gamma'\tau, \gamma'\nu)$, $j(1; \tau, \nu) = 1$.

In what follows, we write the modular transformations as they follow from calculations for the characters, with the understanding that the $e^{i\pi k \frac{\nu^2}{2\tau}}$ factors are to be omitted when

we speak of finite-dimensional $SL(2, \mathbb{Z})$ representations. *Matrix* automorphy factors are also used to unravel the structure of representations derived below.

4.3.3. Lemma. *The functions*

$$(4.10) \quad \begin{aligned} \pi_0(\tau, \nu) &= \omega_p^-(\tau, \nu), \\ \pi_r(\tau, \nu) &= \omega_r^+(\tau, \nu) + \omega_{p-r}^-(\tau, \nu), \quad 1 \leq r \leq p-1, \\ \pi_p(\tau, \nu) &= \omega_p^+(\tau, \nu) \end{aligned}$$

span a $(p+1)$ -dimensional $SL(2, \mathbb{Z})$ -representation \mathcal{R}_{p+1} :

$$(4.11) \quad \begin{aligned} \pi_r(\tau+1, \nu) &= \lambda_{r,p} \pi_r(\tau, \nu), \\ \pi_r\left(-\frac{1}{\tau}, \frac{\nu}{\tau}\right) &= i \sqrt{\frac{2}{p}} e^{i\pi k \frac{\nu^2}{2\tau}} \left(\frac{1}{2} \pi_0(\tau, \nu) + \frac{(-1)^r}{2} \pi_p(\tau, \nu) + \sum_{s=1}^{p-1} \cos \frac{\pi r s}{p} \pi_s(\tau, \nu) \right) \end{aligned}$$

for $0 \leq r \leq p$. *The functions*

$$(4.12) \quad \begin{aligned} \bar{\omega}_r(\tau, \nu) &= (p-r) \omega_r^+(\tau, \nu) - r \omega_{p-r}^-(\tau, \nu), \\ \zeta_r(\tau, \nu) &= \tau \bar{\omega}_r(\tau, \nu), \end{aligned} \quad 1 \leq r \leq p-1,$$

transform as

$$(4.13) \quad \bar{\omega}_r(\tau+1, \nu) = \lambda_{r,p} \bar{\omega}_r(\tau, \nu), \quad \zeta_r(\tau+1, \nu) = \lambda_{r,p} (\zeta_r(\tau, \nu) + \bar{\omega}_r(\tau, \nu)),$$

$$(4.14) \quad \bar{\omega}_r\left(-\frac{1}{\tau}, \frac{\nu}{\tau}\right) = \sqrt{\frac{2}{p}} e^{i\pi k \frac{\nu^2}{2\tau}} \sum_{s=1}^{p-1} \sin \frac{\pi r s}{p} \left(\zeta_s(\tau, \nu) - \frac{p\nu}{2} \chi_s(\tau, \nu) \right),$$

$$(4.15) \quad \zeta_r\left(-\frac{1}{\tau}, \frac{\nu}{\tau}\right) = \sqrt{\frac{2}{p}} e^{i\pi k \frac{\nu^2}{2\tau}} \sum_{s=1}^{p-1} \sin \frac{\pi r s}{p} \left(-\bar{\omega}_s(\tau, \nu) + \frac{p\nu}{2\tau} \chi_s(\tau, \nu) \right).$$

This is shown by straightforward calculation based on (B.10)–(B.12) for the S -transformations. For T , the formulas are obvious.

For notational simplicity, we no longer use a special notation for functions like $\tau \bar{\omega}_r$ in (4.12). It must be clear from **C.2.1** how the occurrence of τ give rise to \mathbb{C}^2 tensor factors in $SL(2, \mathbb{Z})$ -representations.

4.3.4. $SL(2, \mathbb{Z})$ representation structure: a deformed $\mathbb{C}^2 \otimes \mathcal{R}_{\text{int}}(p)$. The admixture of ν times integrable representation characters in (4.14) fits into the representation structure described in **C.2.2**, with a direct sum of representations deformed by a matrix automorphy factor. The functions $\bar{\omega}_r$, $\tau \bar{\omega}_r$, and χ_r are combined into a column $\begin{pmatrix} f(\tau) \bar{\omega} \\ \chi \end{pmatrix}$, where $f(\tau)$ is a polynomial of degree ≤ 1 and we omit the indices, letting $\bar{\omega}$ and χ denote a vector in \mathbb{C}^{p-1} each. The column of the above form (read from bottom up) can therefore be considered an element of $\mathbb{C}^{p-1} \oplus \mathbb{C}^2 \otimes \mathbb{C}^{p-1}$. The $SL(2, \mathbb{Z})$ -action defined by (4.13)–(4.14) (in the version where this is a *right* action) differs from that on $\mathbb{C}^{p-1} \oplus \mathbb{C}^2 \otimes \mathbb{C}^{p-1} =$

$\mathcal{R}_{\text{int}}(p) \oplus \mathbb{C}^2 \otimes \mathcal{R}_{\text{int}}(p)$ by a matrix automorphy factor: the action is given by

$$(4.16) \quad \begin{pmatrix} f(\tau)\overline{\omega} \\ \chi \end{pmatrix} \cdot \gamma = \begin{pmatrix} (c\tau + d)f(\gamma\tau)\overline{\omega} \cdot \gamma + \beta v c f(\gamma\tau)\chi \cdot \gamma \\ \chi \cdot \gamma \end{pmatrix}, \quad \beta = -\frac{p}{2},$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the right-hand side acts on each \mathbb{C}^{p-1} as on the integrable representation characters and $\gamma\tau$ is defined in **C.1**. At $\beta = 0$ (or, formally, $v = 0$), obviously, the matrix automorphy factor becomes the identity matrix and the transformation law in (4.16) becomes that on $\mathcal{R}_{\text{int}}(p) \oplus \mathbb{C}^2 \otimes \mathcal{R}_{\text{int}}(p)$.

In the next lemma, we encounter a $\tau^2\varphi_r$ and hence a \mathbb{C}^3 tensor factor (cf. **C.3.1**).

4.3.5. Lemma. *The functions*

$$\begin{aligned} \rho_0(\tau, v) &= \chi_p^-(\tau, v), \\ \rho_r(\tau, v) &= \chi_r^+(\tau, v) + \chi_{p-r}^-(\tau, v) + \frac{r}{2p}\chi_r(\tau, v), \quad 1 \leq r \leq p-1, \\ \rho_p(\tau, v) &= \chi_p^+(\tau, v) \end{aligned}$$

transform as

$$\begin{aligned} \rho_r(\tau + 1, v) &= \lambda_{r,p}\rho_r(\tau, v), \\ \rho_r\left(-\frac{1}{\tau}, \frac{v}{\tau}\right) &= i\sqrt{\frac{2}{p}}e^{i\pi k\frac{v^2}{2\tau}} \left(\frac{1}{2}(\tau\rho_0(\tau, v) + v\pi_0(\tau, v)) + \frac{(-1)^r}{2}(\tau\rho_p(\tau, v) + v\pi_p(\tau, v)) \right. \\ &\quad \left. + \sum_{s=1}^{p-1} \cos\frac{\pi rs}{p} (\tau\rho_s(\tau, v) + v\pi_s(\tau, v)) \right). \end{aligned}$$

The functions

$$\varphi_r(\tau, v) = (p-r)\chi_r^+(\tau, v) - r\chi_{p-r}^-(\tau, v) - \left(\frac{r^2}{4p} + \frac{1}{8i\pi\tau}\right)\chi_r(\tau, v), \quad 1 \leq r \leq p-1,$$

transform as

$$\begin{aligned} \varphi_r(\tau + 1, v) &= \lambda_{r,p}\varphi_r(\tau, v), \\ \varphi_r\left(-\frac{1}{\tau}, \frac{v}{\tau}\right) &= \sqrt{\frac{2}{p}}e^{i\pi k\frac{v^2}{2\tau}} \sum_{s=1}^{p-1} \sin\frac{\pi rs}{p} \left(\tau^2\varphi_s(\tau, v) + v\tau\overline{\omega}_s(\tau, v) - \frac{pv^2}{4}\chi_s(\tau, v) \right). \end{aligned}$$

This also follows by a direct calculation based on (B.10)–(B.12) for S . For T , apart from the same eigenvalues $\lambda_{r,p}$, the transformations amount to substituting $\tau \mapsto \tau + 1$ in polynomials of degree not greater than 2, which is not a difficult calculation.

4.3.6. $SL(2, \mathbb{Z})$ -representation structure: deformed $\mathbb{C}^2 \otimes \mathcal{R}_{p+1}$ and $\mathbb{C}^3 \otimes \mathcal{R}_{\text{int}}(p)$. It follows from **4.3.5** and **C.2.2** that the ρ (with suppressed indices, i.e., viewed as a vector in \mathbb{C}^{p+1}) transform under $SL(2, \mathbb{Z})$ (again in the right-action version) as

$$(4.17) \quad \begin{pmatrix} f(\tau)\rho \\ \pi \end{pmatrix} \cdot \gamma = \begin{pmatrix} (c\tau + d)f(\gamma\tau)\rho \cdot \gamma + \alpha v c f(\gamma\tau)\pi \cdot \gamma \\ \pi \cdot \gamma \end{pmatrix}, \quad \alpha = 1,$$

where $f(\tau)$ is a polynomial of degree ≤ 1 and in the right-hand side the $SL(2, \mathbb{Z})$ action on each \mathbb{C}^{p+1} is as on \mathcal{R}_{p+1} , see (4.11). This representation is also a deformation of $\mathcal{R}_{p+1} \oplus \mathbb{C}^2 \otimes \mathcal{R}_{p+1}$ via a matrix automorphy factor.

Similarly, it follows from **4.3.5** and **C.3.2** that the $SL(2, \mathbb{Z})$ -action on the φ (now viewed as a vector from \mathbb{C}^{p-1} endowed with the $SL(2, \mathbb{Z})$ representation isomorphic to $\mathcal{R}_{\text{int}}(p)$) is a ‘‘composition’’ of the finite-dimensional representations and an ‘‘even larger’’ matrix automorphy factor: the transformations derived in the last lemma are equivalent to the $SL(2, \mathbb{Z})$ action given by

$$(4.18) \quad \begin{pmatrix} f(\tau)\varphi \\ \varpi \\ \chi \end{pmatrix} \cdot \gamma = \begin{pmatrix} (c\tau + d)^2 f(\gamma\tau)\varphi \cdot \gamma + \alpha v c (c\tau + d) f(\gamma\tau)\varpi \cdot \gamma + \frac{\alpha\beta}{2} v^2 c^2 f(\gamma\tau)\chi \cdot \gamma \\ \varpi \cdot \gamma \\ \chi \cdot \gamma \end{pmatrix},$$

where $f(\tau)$ is a polynomial of degree ≤ 2 , in the right-hand side $SL(2, \mathbb{Z})$ acts on each \mathbb{C}^{p-1} as on $\mathcal{R}_{\text{int}}(p)$, and

$$\alpha = 1, \quad \beta = -\frac{p}{2}$$

as above. At zero values of α and β we recover a direct sum of finite-dimensional representations, the ‘‘ φ ’’ one being $\mathbb{C}^3 \otimes \mathcal{R}_{\text{int}}(p)$.

4.3.7. Example: $k = 0$. For $k = 0$ (and $9p - 3 = 15$), the W -algebra generators are given in **4.1.3**. We here have a single integrable representation character $\chi_1 = 1$. The other $8 + 6 = 14$ generalized characters are a triplet (π_0, π_1, π_2) , a ‘‘ τ ’’-doublet $(\varpi_1, \tau\varpi_1)$, a \mathbb{C}^2 (due to τ) tensored with another triplet (ρ_0, ρ_1, ρ_2) , with $v\pi_r$ occurring in its S -transform, and, finally, a ‘‘ τ ’’-triplet $(\varphi_1, \tau\varphi_1, \tau^2\varphi_1)$, with both ϖ_1 and χ_1 occurring in its S -transform.

We also note that the $c = 0$ logarithmic model of $\widehat{sl}(2)_0$ is somewhat ‘‘smaller in size’’ than the celebrated $(3, 2)$ logarithmic model with the central charge $c = 0$ [59]. The logarithmic $(p = 3, q = 2)$ model involves $\frac{1}{2}(p-1)(q-1) + 2pq = 13$ irreducible W -algebra representations, while the space of torus amplitudes has (based on the modular-group argument) dimension $\frac{1}{2}(3p-1)(3q-1) = 20$ [12]. For $\widehat{sl}(2)_0$, these two numbers seem to be $5p - 1 = 9$ and $9p - 3 = 15$ respectively.

5. HAMILTONIAN REDUCTION TO THE W -ALGEBRA OF THE $(p, 1)$ MODEL

In this section, we apply the Hamiltonian reduction functor to the W -algebra of the logarithmic $\widehat{sl}(2)_k$ model. We show that the W -generators (4.2) and (4.3) reduce to gen-

erators of the triplet W -algebra [7, 2, 8] of the $(p, 1)$ logarithmic model, which were defined in [10] in terms of a Virasoro screening.

It is well-known that the result of Hamiltonian reduction of the $\widehat{sl}(2)_k$ algebra itself is the Virasoro algebra

$$(5.1) \quad T(z)T(w) = \frac{d/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

with central charge

$$(5.2) \quad d = 13 - \frac{6}{k+2} - 6(k+2).$$

5.1. Theorem. *Hamiltonian reduction of the W -algebra generated by the currents (4.2) and (4.3) is the triplet W -algebra of the $(p=k+2, 1)$ logarithmic model.*

5.2. Construction of the “inverse reduction”. To find how the W -algebra representations are reduced, it is useful to recall how the $\widehat{sl}(2)$ currents are *reconstructed* from the reduction result $T(z)$.

5.2.1. Lemma ([60]). *Let ϕ and φ be two scalar fields with the operator products*

$$\partial\varphi(z)\partial\varphi(w) = \frac{1}{(z-w)^2}, \quad \partial\phi(z)\partial\phi(w) = \frac{-1}{(z-w)^2}$$

and let $T(z)$ satisfy (5.1) with d given by (5.2). For $k \neq 0$, the currents

$$(5.3) \quad E(z) = e^{\sqrt{\frac{2}{k}}(\varphi(z)-\phi(z))},$$

$$(5.4) \quad H(z) = \sqrt{\frac{k}{2}}\partial\varphi(z),$$

$$(5.5) \quad F(z) = \left((k+2)T(z) - \frac{k}{2}\partial\phi\partial\phi(z) - \sqrt{\frac{k}{2}}(k+1)\partial^2\phi(z) \right) e^{-\sqrt{\frac{2}{k}}(\varphi(z)-\phi(z))}$$

then satisfy the $\widehat{sl}(2)$ OPEs in (2.2).

We emphasize that no free-field representation is required of $T(z)$.

This construction of $E(z)$, $H(z)$, and $F(z)$ can be considered an “inversion” of the Hamiltonian reduction starting with its result, the energy-momentum tensor $T(z)$. The reduction of any expression built out of the $\widehat{sl}(2)$ currents thus amounts to simply expressing the currents in terms of $T(z)$, $\varphi(z)$, and $\phi(z)$ (and, depending on one’s taste, eventually setting the two free fields equal to zero). In particular, the Sugawara energy-momentum tensor evaluates in accordance with this procedure as

$$T_{\text{Sug}}(z) = T(z) + \frac{1}{2}\partial\varphi\partial\varphi(z) - \frac{1}{2}\partial\phi\partial\phi(z) - \sqrt{\frac{k}{2}}\partial^2\phi(z).$$

5.2.2. Remark. The above formulas do not allow the limit $k \rightarrow 0$. But this can be considered an artifact of the choice of scalar fields in (5.3)–(5.5). Changing them as

$$\varphi = \frac{1}{\sqrt{2}} \left(\frac{2}{\sqrt{k}} \bar{X} + \frac{\sqrt{k}}{2} X \right), \quad \phi = \frac{1}{\sqrt{2}} \left(\frac{2}{\sqrt{k}} \bar{X} - \frac{\sqrt{k}}{2} X \right)$$

where the new fields X and \bar{X} have the OPEs

$$\partial X(z) \partial \bar{X}(w) = \frac{1}{(z-w)^2}, \quad \partial X(z) \partial X(w) = \text{reg}, \quad \partial \bar{X}(z) \partial \bar{X}(w) = \text{reg},$$

maps (5.3)–(5.5) into a form where k can be set equal to zero, with the result

$$\begin{aligned} E(z) &= e^{X(z)}, \\ H(z) &= \partial \bar{X}(z), \\ F(z) &= (2T(z) - \partial \bar{X} \partial \bar{X}(z) - \partial^2 \bar{X}(z)) e^{-X(z)} \end{aligned}$$

for the $\widehat{s\ell}(2)_0$ currents. We restrict ourself to the observation that this simple change of variables suffices to resolve the apparent $k = 0$ problem, and proceed with (5.3)–(5.5), which we prefer for essentially esthetical reasons.

5.2.3. Reducing the representations. The simple recipe to “invert” the reduction extends to representations. A spin- λ (nontwisted) $\widehat{s\ell}(2)$ highest-weight state/operator $U_\lambda(z) \doteq |\lambda\rangle$ is obtained by “dressing” the Virasoro primary $\mathcal{V}_\delta(z)$ of dimension⁹

$$(5.6) \quad \delta = \frac{\lambda(\lambda+1)}{k+2} - \lambda$$

as

$$U_\lambda(z) = \mathcal{V}_\delta(z) e^{\lambda \sqrt{\frac{2}{k}} (\varphi(z) - \phi(z))}.$$

Taking $\lambda = \lambda^+(r, s)$ (see 2.2) corresponds to the Virasoro dimension

$$\delta_{r,s} = \frac{\lambda^+(r,s)(\lambda^+(r,s)+1)}{k+2} - \lambda^+(r,s) = \frac{r^2-1}{4(k+2)} + (k+2) \frac{s^2-1}{4} + \frac{1-rs}{2}.$$

For example, the spin- $\frac{1}{2}$ highest-weight state is thus constructed as $e^{\frac{1}{2} \sqrt{\frac{2}{k}} (\varphi(z) - \phi(z))} \mathcal{V}_{[21]}(z)$, where $\mathcal{V}_{[21]}(z) \equiv \mathcal{V}_{\delta_{2,1}}(z)$ is the “21” vertex operator for the Virasoro algebra of $T(z)$. Moreover, the differential equation $(k+2)\partial^2 \mathcal{V}_{[21]}(z) - T(z)\mathcal{V}_{[21]}(z) = 0$ satisfied by $\mathcal{V}_{[21]}(z)$ then implies the singular vector vanishing $F_0 F_0 (e^{\frac{1}{2} \sqrt{\frac{2}{k}} (\varphi(z) - \phi(z))} \mathcal{V}_{[21]}(z)) = 0$.

5.2.4. Expressing the $W^\pm(z)$ currents. Hamiltonian reduction of the fields generating the W -algebra, Eqs. (4.2) and (4.3), is particularly simple for $W^+(z)$ because of the simple formula for the $|\text{MFF}^-(r', 1)\rangle$ singular vectors, which, moreover, immediately evaluate explicitly in realization (5.3). In accordance with the construction of nontwisted highest-weight states in 5.2.3, we have

$$L_{m,m}[r](z) \Big|_{(5.3)-(5.5)} = \mathcal{V}_{[r, 2m+1]}(z) e^{(j-(k+2)m) \sqrt{\frac{2}{k}} (\varphi(z) - \phi(z))}$$

(recall notation (3.13)); therefore, in evaluating $W^+(z) = (E_{-1})^{3p-1} L_{1,1}[1](z)$ (see (4.3)),

⁹Hereinafter, all Virasoro primaries and their dimensions are with respect to $T(z)$ in (5.1)–(5.2).

we have $L_{1,1}[1](z)$ expressed through $\mathcal{V}_{[1,3]}(z) = \mathcal{V}_{2k+3}(z)$; next, it follows from (5.3) that the action of E_{-1} affects only the free-field sector, and therefore

$$(5.7) \quad W^+(z) \Big|_{(5.3)-(5.5)} = \mathcal{V}_{2k+3}(z) e^{(2k+3)\sqrt{\frac{2}{k}}(\varphi(z)-\phi(z))}.$$

This identifies the Hamiltonian reduction of $W^+(z)$ as the dimension- $(2p-1)$ Virasoro primary.

The Hamiltonian reduction of $W^-(z)$ is evaluated explicitly to within the ‘‘explicitness’’ of the MFF construction for singular vectors. With the $n = m$ operators (3.34) represented in the inverse Hamiltonian reduction setting as

$$U'_{m,m}[r](z) \Big|_{(5.3)-(5.5)} = \mathcal{V}_{[r+2mp,1]}(z) e^{(j+(k+2)m)\sqrt{\frac{2}{k}}(\varphi(z)-\phi(z))},$$

we evaluate $W^-(z)$ as a free-field vertex times a nonvanishing Virasoro singular vector built on the dimension-1 primary $\mathcal{V}_{[2p+1,1]}(z) = \mathcal{V}_1(z)$:

$$(5.8) \quad W^-(z) \Big|_{(5.3)-(5.5)} = (D_{2k+2}(T)\mathcal{V}_1(z)) e^{-(2k+3)\sqrt{\frac{2}{k}}(\varphi(z)-\phi(z))},$$

where $D_{2k+2}(T)\mathcal{V}_1(z) = \alpha\partial^{2k+2}\mathcal{V}_1(z) + \dots + \gamma T(z)^{k+1}\mathcal{V}_1(z)$ is a normal-ordered differential polynomial in $T(z)$ and $\mathcal{V}_1(z)$, linear in $\mathcal{V}_1(z)$, of the total degree $2k+2$, if we set $\deg T(z) = 2$, $\deg \mathcal{V}_1(z) = 1$, and $\deg \partial = 1$. Clearly, $\oint \mathcal{V}_1$ is one of the two Virasoro screenings.

We thus see from (5.7) and (5.8) that the two currents

$$(5.9) \quad \begin{aligned} w^+(z) &= \mathcal{V}_{2p-1}(z), \\ w^-(z) &= D_{2p-2}(T)\mathcal{V}_1(z) \end{aligned}$$

are the result of the Hamiltonian reduction of $W^+(z)$ and $W^-(z)$. Moreover, their construction in terms of Virasoro generators and vertices shows that they are the dimension- $(2p-1)$ currents generating the W -algebra of the $(p, 1)$ logarithmic conformal field theory model, which were constructed in [10] using a Virasoro screening operator.

5.2.5. *If we further use the bosonization*

$$T(z) = \frac{1}{2}\partial f(z)\partial f(z) + \frac{p-1}{\sqrt{2p}}\partial^2 f(z)$$

for the energy-momentum tensor and

$$(5.10) \quad \mathcal{V}_{[r,s]}(z) = e^{\sqrt{\frac{2}{p}}\lambda^{+(r,s)}f(z)}$$

for the vertices in terms of a free field with the OPE $\partial f(z)\partial f(w) = \frac{1}{(z-w)^2}$, then the vertex operators in (5.9) become $\mathcal{V}_{[1,3]}(z) \equiv \mathcal{V}_{2p-1}(z) = e^{-\sqrt{2p}f(z)}$ and $\mathcal{V}_{[2p+1,1]}(z) \equiv \mathcal{V}_1(z) = e^{\sqrt{2p}f(z)}$, as in the free-field construction in [10], with the screening given by $\oint e^{\sqrt{2p}f}$.

5.2.6. Example. For $k = 1$, the differential polynomial in (5.8) is given by

$$(5.11) \quad D_4 \mathcal{V}_1(z) = 9(-60T(z)\partial^2 \mathcal{V}_1(z) - 6\partial T(z)\partial \mathcal{V}_1(z) - 18\partial^2 T(z)\mathcal{V}_1(z) + 64T(z)T(z)\mathcal{V}_1(z) + 9\partial^4 \mathcal{V}_1(z))$$

(with nested normal ordering from right to left, as usual). In terms of the bosonization in (5.10), we then find

$$w^+(z) = e^{-\sqrt{6}f(z)}$$

and a straightforward evaluation of (5.11) gives

$$w^-(z) = (3120\partial^2 f(z)\partial^2 f(z) - 1440\partial^3 f(z)\partial f(z) - 960\sqrt{6}\partial^2 f(z)\partial f(z)\partial f(z) + 1440\partial f(z)\partial f(z)\partial f(z)\partial f(z) + 40\sqrt{6}\partial^4 f(z))e^{\sqrt{6}f(z)},$$

which is -80 times the $(p=3, 1)$ -model operator $W^+(z)$ in [10, Example 2.2.1].¹⁰

5.3. “Reduction” of the characters. We recall from [61] that taking residues of integrable or admissible $\widehat{\mathfrak{sl}}(2)$ characters at $z = q^n$ gives either zero or Virasoro-representation characters (times a two-boson character), in agreement with the Hamiltonian reduction (under which *some* irreducible $\widehat{\mathfrak{sl}}(2)$ representations map into the trivial Virasoro representation). This can be considered a “Hamiltonian reduction” at the level of characters. For example, the integrable $\widehat{\mathfrak{sl}}(2)_k$ characters $\chi_r(q, z)$ are holomorphic functions of z , and hence have zero residues.

This observation extends to the logarithmic/ W -algebra realm as follows. We take the residues at $z = 1$. As just noted, the integrable representation characters have zero residue at this point in particular. Next, for the W -algebra characters in **4.2.1**, we also have

$$\operatorname{res}_{z=1} \chi_r^\pm(q, z) = 0, \quad 1 \leq r \leq p$$

(although these characters do have poles elsewhere, as is easy to see). On the other hand, the $\omega_r^\pm(q, z)$ in (4.4)–(4.5) have nonvanishing residues at $z = 1$, yielding the $2p$ characters of the $(p, 1)$ -model triplet W -algebra in [10], times the factor $1/\eta(q)^2$ that accounts for the character of two free bosons:

$$(5.12) \quad \begin{aligned} \operatorname{res}_{z=1} \omega_r^+(q, z) &= \frac{1}{\eta(q)^2} \frac{r\theta_r(q) - 2\theta_r'(q)}{p\eta(q)}, \\ \operatorname{res}_{z=1} \omega_r^-(q, z) &= \frac{1}{\eta(q)^2} \frac{r\theta_{p-r}(q) + 2\theta_{p-r}'(q)}{p\eta(q)}, \end{aligned}$$

where the theta-constants in the right-hand sides are the corresponding theta-functions at $z = 1$. The resulting characters are indeed those in [10] (identification requires readjusting the conventions for the theta-derivative and for the \pm labeling).

¹⁰The \pm conventions seem to be particularly difficult to match.

Another interpretation of the above residue formulas can be given in terms of *specialization* of the W -algebra characters to $z = 1$.¹¹ That is (switching to the ν -language), setting $\nu = 0$ in the modular transformation properties in **4.3.3** and **4.3.5**, we have the well-defined specialized characters

$$\begin{aligned}\chi_r(\tau) &= \lim_{\nu \rightarrow 0} \chi_r(\tau, \nu), & 1 \leq r \leq p-1, \\ \chi_r^\pm(\tau) &= \lim_{\nu \rightarrow 0} \chi_r^\pm(\tau, \nu), & 1 \leq r \leq p.\end{aligned}$$

Taking the limit in the definitions in **4.3.5**, we obtain the corresponding $\rho_r(\tau)$, $1 \leq r \leq p+1$, and $\varphi_r(\tau)$, $1 \leq r \leq p-1$. In addition, we define

$$\widehat{\omega}_r^\pm(\tau) = \lim_{\nu \rightarrow 0} (\nu \omega_r^\pm(\tau, \nu)), \quad 1 \leq r \leq p,$$

which are just the residues in (5.12) times inessential constant factors. In accordance with the definitions in **4.3.3**, this then gives the corresponding $\widehat{\pi}_r(\tau)$, $1 \leq r \leq p+1$, and $\widehat{\omega}_r(\tau)$, $1 \leq r \leq p-1$.

Remarkably, these definitions allow taking $\nu \rightarrow 0$ in the modular transformation formulas in **4.3.3** and **4.3.5** (because the $\overline{\omega}_r$ enter the right-hand sides of the transformations in **4.3.5** only in the combination $\nu \overline{\omega}_r(\tau, \nu)$). But the S -transform formulas for the thus “specialized” characters acquire negative powers of τ , for example

$$\widehat{\pi}_r(-\frac{1}{\tau}) = i\sqrt{\frac{2}{p}} \left(\frac{1}{2\tau} \widehat{\pi}_0(\tau) + \frac{(-1)^r}{2\tau} \widehat{\pi}_p(\tau) + \frac{1}{\tau} \sum_{s=1}^{p-1} \cos \frac{\pi rs}{p} \widehat{\pi}_s(\tau) \right),$$

as well as implicit negative powers of τ in

$$\varphi_r(-\frac{1}{\tau}) = \sqrt{\frac{2}{p}} \sum_{s=1}^{p-1} \sin \frac{\pi rs}{p} \left(\tau^2 \varphi_s(\tau) + \tau \widehat{\omega}_s(\tau) \right),$$

because S -transforming $\tau^2 \varphi_r(\tau)$ gives rise to $\tau^{-1} \widehat{\omega}_s(\tau)$ in the right-hand side.

Rewriting the modular transformation formulas in terms of the $(p, 1)$ -model characters $\widetilde{\omega}_s(\tau) = \eta(\tau)^2 \widehat{\omega}_s(\tau)$ and $\widetilde{\pi}_s(\tau) = \eta(\tau)^2 \widehat{\pi}_s(\tau)$, we, in particular, reproduce the $SL(2, \mathbb{Z})$ representation $\mathcal{R}_{p+1} \oplus \mathbb{C}^2 \otimes \mathcal{R}_{\text{int}}(p)$ on these characters in [10]:

$$\begin{aligned}\widetilde{\pi}_r(-\frac{1}{\tau}) &= i\sqrt{\frac{2}{p}} \left(\frac{1}{2} \widetilde{\pi}_0(\tau) + \frac{(-1)^r}{2} \widetilde{\pi}_p(\tau) + \sum_{s=1}^{p-1} \cos \frac{\pi rs}{p} \widetilde{\pi}_s(\tau) \right), \\ \widetilde{\omega}_r(-\frac{1}{\tau}) &= \sqrt{\frac{2}{p}} \sum_{s=1}^{p-1} \sin \frac{\pi rs}{p} \tau \widetilde{\omega}_s(\tau).\end{aligned}$$

Transformations of the remaining specialized characters involve both the $(p, 1)$ characters and the free-boson characters $\eta(\tau)^{-2}$ in the right-hand sides:

¹¹I thank I. Tipunin for the suggestion.

$$\begin{aligned} \rho_r(-\frac{1}{\tau}) &= i\sqrt{\frac{2}{p}} \left(\frac{1}{2} \left(\tau\rho_0(\tau) + \frac{\tilde{\pi}_0(\tau)}{\eta(\tau)^2} \right) + \frac{(-1)^r}{2} \left(\tau\rho_p(\tau) + \frac{\tilde{\pi}_p(\tau)}{\eta(\tau)^2} \right) \right. \\ &\quad \left. + \sum_{s=1}^{p-1} \cos \frac{\pi rs}{p} \left(\tau\rho_s(\tau) + \frac{\tilde{\pi}_s(\tau)}{\eta(\tau)^2} \right) \right), \\ \varphi_r(-\frac{1}{\tau}) &= \sqrt{\frac{2}{p}} \sum_{s=1}^{p-1} \sin \frac{\pi rs}{p} \left(\tau^2 \varphi_s(\tau) + \frac{\tau \tilde{\omega}_s(\tau)}{\eta(\tau)^2} \right). \end{aligned}$$

6. FURTHER PROSPECTS

This section is essentially a todo list or, depending on one's standpoint, a list of things that the author has failed to accomplish. It may nevertheless be helpful in setting logarithmic $\widehat{sl}(2)$ models in some new perspectives.

6.1. Three fermionic screenings. It would be exceptionally interesting to see how the present construction generalizes to the case with three fermionic screenings, which is indeed exceptional because the W -algebra that commutes with three fermionic screenings is a Hamiltonian reduction of the exceptional affine Lie superalgebra $\widehat{D}(2|1;\alpha)$. It is at the same time (see [62]) the algebra of the conformal field theory of the coset

$$\frac{\widehat{sl}(2)_{k_1} \oplus \widehat{sl}(2)_{k_2}}{\widehat{sl}(2)_{k_1+k_2}},$$

which makes it particularly interesting in applications.

6.2. Kazhdan–Lusztig correspondence, the dual quantum group, and fusion. We recall that the property of a chiral algebra and a quantum group associated with the screenings to be each other's centralizers underlies the Kazhdan–Lusztig correspondence [31] between representation categories of the chiral algebra and the quantum group.

Logarithmic conformal field theory models have nice quantum-group counterparts, which capture at least part of the structure of logarithmic models and are therefore quite useful in investigating them [11, 32, 33]. On the quantum-group side, the central role is played by two objects, the *center* and the Grothendieck ring. In the known examples, the center carries the same modular group representation as is realized on generalized characters of the logarithmic model.¹² Elements of the quantum-group center also translate into boundary states in boundary conformal field theories [15]. The Kazhdan–Lusztig-dual quantum group in the $\widehat{sl}(2)_k$ context — a quantum supergroup $\overline{U}_q sl(2|1)$ at a root of unity — may actually correspond to the logarithmic $\widehat{sl}(2)_k/u(1)$ theory, because the number of screenings is then equal to the number of free fields. Indications of a complicated structure of its center can already be found in [64]. For the center to carry a modular group action at all, the quantum group must be ribbon and factorizable (cf. [11, 33]);

¹²A decomposition of the $SL(2, \mathbb{Z})$ -representation on a quantum group center involving \mathbb{C}^n tensor factors (actually, \mathbb{C}^2) was first, to our knowledge, observed in [63].

finding these structures, or at least some analogues thereof, for a quantum *supergroup* is a separate, quite interesting task. Hamiltonian reduction of logarithmic conformal field theories in Sec. 5 must also have a counterpart for the respective Kazhdan–Lusztig-dual quantum groups, or at least for their centers and the modular group actions on them.

The previous experience with Kazhdan–Lusztig-dual quantum groups shows that their Grothendieck rings give (a “ K_0 -version” of) the fusion for the corresponding W -algebras [10, 11, 12, 29]. Although this “experimental law” might require modification in the current case of the dual pair given by the above W -algebra and $\overline{U}_q sl(2|1)$ at a root of unity, the Grothendieck ring of the dual quantum group is certainly related to the W -algebra representation theory, being at the same time an object that is much easier to evaluate.¹³

Another, rather speculative, inference from the (so far hypothetical) Kazhdan–Lusztig correspondence with a quantum $sl(2|1)$ is that because $sl(2|1)$ has typical (“wide”) and atypical (“narrow”) representations, a similar picture, inasmuch as it survives specializing to a root of unity and imposing constraints in the quantum group, is to be expected for the W -algebra representations. The y_r^\pm representations constructed above are then certainly the “narrow” ones. The problem of “typical/wide/massive” representations and of the role they may play is left for the future (their $\widehat{sl}(2)$ “building blocks” may have the extremal diagrams shaped like those of admissible or even relaxed representations, cf. [5]).

6.3. More general models: “triplet” W -algebras instead of $\widehat{sl}(2)$. For integer k , there must exist logarithmic models generalizing the logarithmic $\widehat{sl}(2)_k$ differently than $\widehat{sl}(2) \rightarrow \widehat{sl}(n)$, which first suggests itself. The bosonization of $\widehat{sl}(2)_k$ in 2.4 is the $n = 2$ case of a general pattern of algebras $\mathcal{W}_n^{(2)}(k)$, $n \geq 1$, constructed very similarly [53] (the $n = 1$ case is merely the $\beta\gamma$ system and the $n = 3$ case is the Bershadsky–Polyakov algebra [66, 67]). For $n > 2$, the set of $n + 1$ vectors in \mathbb{C}^{n+1} generalizing the data ξ, ψ_-, ψ_+ in 2.4.1 is $\xi, a_{n-2}, \dots, a_1, \psi_-, \psi_+$ with the Gram matrix generalizing (2.16) as

$$\begin{array}{c|c} \xi & \left(\begin{array}{cccccccc} 0 & 0 & \dots\dots\dots & 0 & 1 & -1 & & \\ 0 & 2(k+n) & -k-n & 0 & \dots\dots\dots & 0 & & \\ 0 & -k-n & 2(k+n) & -k-n & 0 & \dots\dots\dots & 0 & \\ \dots\dots\dots & & & & & & & \\ \dots\dots\dots & & & & & & & \\ a_1 & 0 & 0 & \dots\dots & 0 & -k-n & 2(k+n) & -k-n & 0 \\ \psi_- & 1 & 0 & \dots\dots\dots & 0 & -k-n & 1 & k+n-1 & \\ \psi_+ & -1 & 0 & \dots\dots\dots & 0 & k+n-1 & 1 & & \end{array} \right) \end{array}$$

¹³We recently became aware of the results in [65], which go far beyond the Grothendieck ring for the quantum groups closely related to those dual to the $(p, 1)$ logarithmic models: tensor products of the indecomposable quantum group representations are evaluated there. We thank K. Erdmann for the communication.

(the determinant is given by $-n(k+n)^{n-1}$). These vectors and an $(n+1)$ -tuple of scalar fields φ are used to construct the screenings (with the dot denoting the Euclidean scalar product in \mathbb{C}^{n+1})

$$E_i = \oint e^{a_i \cdot \varphi}, \quad i = 1, \dots, n-2, \quad Q_- = \oint e^{\psi \cdot \varphi}, \quad Q_+ = \oint e^{\psi_+ \cdot \varphi}$$

representing the nilpotent subalgebra of $\mathcal{U}_q s\ell(n|1)$. The centralizer of the screenings is a W -algebra $\mathcal{W}_n^{(2)}(k)$ generated by two currents $E(z)$ and $F(z)$ with the OPEs

$$E(z)F(w) = \frac{\lambda_{n-1}(n, k)}{(z-w)^n} + \frac{n\lambda_{n-2}(n, k)H(w)}{(z-w)^{n-1}} + \dots,$$

where $\lambda_m(n, k) = \prod_{i=1}^m (i(k+n-1) - 1)$, and, further,

$$H(z)E(w) = \frac{E(w)}{z-w}, \quad H(z)F(w) = -\frac{F(w)}{z-w}, \quad H(z)H(w) = \frac{\frac{n-1}{n}k+n-2}{(z-w)^2}.$$

Quite an explicit construction of $E(z)$ and $F(z)$ is available [53].

The entries of the above Gram matrix are integer for integer k and the fermionic screenings commute for $k+n \geq 1$; “integrable” representations of $\mathcal{W}_n^{(2)}(k)$, mentioned in [53] (also see [49]), are then a good starting point for the construction of logarithmic models.

6.4. Other logarithmic $\widehat{s\ell}(2)_k$? There are two aspects of “other” logarithmic $\widehat{s\ell}(2)_k$ -models with nonnegative integer k . First, there are various possibilities of constructing “essentially larger” models, e.g., by taking the kernel of only one screening. Second, with just two screenings, an a priori different logarithmic extension of $\widehat{s\ell}(2)_k$ conformal models with integer $k \geq -1$ is possible based on the “nonsymmetric” bosonization of $\widehat{s\ell}(2)_k$ (with a fermionic and a bosonic screening). It would be not entirely trivial if the results actually coincide with those in this paper.

One more possibility is to “bosonize” $\widehat{s\ell}(2)_k$ just by the construction in (5.4) in terms of two free fields and an energy-momentum tensor $T(z)$ that is *not* represented through a free field. The screening/kernel machinery then involves the screening S constructed in terms of the fields in 5.2.1 as [24]

$$(6.1) \quad S = \oint e^{\sqrt{\frac{k}{2}}\phi} \mathcal{V}_{[12]},$$

where $\mathcal{V}_{[12]}(z)$ is the “12” vertex operator for $T(z)$, of dimension $\delta_{12} = \frac{3k}{4} + 1$. The proof that S is a screening uses the differential equation $\partial^2 \mathcal{V}_{[12]}(z) - (k+2)T(z)\mathcal{V}_{[12]}(z) = 0$ satisfied by $\mathcal{V}_{[12]}(z)$. It is readily seen that S is a *fermionic* screening. Moreover, like the Virasoro vertex operator $\mathcal{V}_{[12]}$, it has two components that map differently between Virasoro modules. The construction of the logarithmic $\widehat{s\ell}(2)_k$ model may then be repeated with this screening action.

6.5. Rational k . Constructing a logarithmic $\widehat{s\ell}(2)_k$ theory for rational k appears to be a more complicated (and certainly bulkier) problem. The bosonization in (2.17) is still

applicable for rational k , and the two fermionic screenings given by (2.18) exist, but the corresponding vertex operators $e^{\psi_{\pm} \cdot \varphi(z)}$ are no longer local with respect to each other. The extension to rational k may turn out to be easier to achieve in the “nonsymmetric” bosonization of $\widehat{sl}(2)$, with one bosonic and one fermionic screenings, but this task seems to be rather involved anyway. The ultimate logarithmic theory is then to include the elegant constructions in [3, 5].

6.6. Projective modules. Taking the kernel of screenings and identifying the relevant W -algebra and its irreducible representations is only the first step in constructing (the chiral sector of) a logarithmic conformal field theory model because *extensions* of these representations must be taken in order to obtain modules where the relevant generators (e.g., L_0) act nonsemisimply. The extensions are to be taken “up to the limit,” which means constructing projective covers of irreducible W -algebra modules. The full space of states in a given chiral sector is then the sum of all nonisomorphic indecomposable projective modules. This is just another major difference from the semisimple/rational case, where the chiral space of states is exhausted by irreducible representations. At chosen fractional values of k , constructions of some indecomposable $\widehat{sl}(2)$ -modules were given in [3, 5]. A complication to be encountered with the W -algebra in this paper may be expected from the fact that the modular group representation is infinite-dimensional before the matrix automorphy factors are isolated, which may suggest some pathologies in the W -algebra projective modules.

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APPENDIX A. EMBEDDING STRUCTURE OF VERMA $\widehat{sl}(2)_k$ -MODULES

We recall the embedding structure of some $\widehat{sl}(2)_k$ Verma modules with positive integer $k + 2$. It is of course well known, and is given here for convenience of reference; the reader may also find the picture of a twisted Verma module useful in deciphering the figures in the main body of the paper.

We use the notation \curvearrowright for the highest-weight state of a Verma module, to make it reminiscent of the Verma module extremal diagram: in accordance with the annihilation conditions in (2.6) for $\theta = 0$, the $\widehat{s\ell}(2)$ operators act on the highest-weight vector as

$$(A.1) \quad \begin{array}{ccccccc} & & & & \curvearrowright & & \\ & & & & \leftarrow F_0 & & \\ & & & & & & \searrow E_{-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

with the occupied states, shown with dots, extending from west to south-east, but not east or north. This notational convention is naturally extended to $(\theta = 1)$ -twisted Verma modules, whose extremal diagram has the form

$$(A.2) \quad \begin{array}{ccccccc} & & & & \curvearrowright & \cdot & \cdot \\ & & & & \xrightarrow{E_0} & \cdot & \cdot \\ & & & & \swarrow F_{-1} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

and we therefore let \curvearrowright denote the twisted highest-weight vector with twist 1.

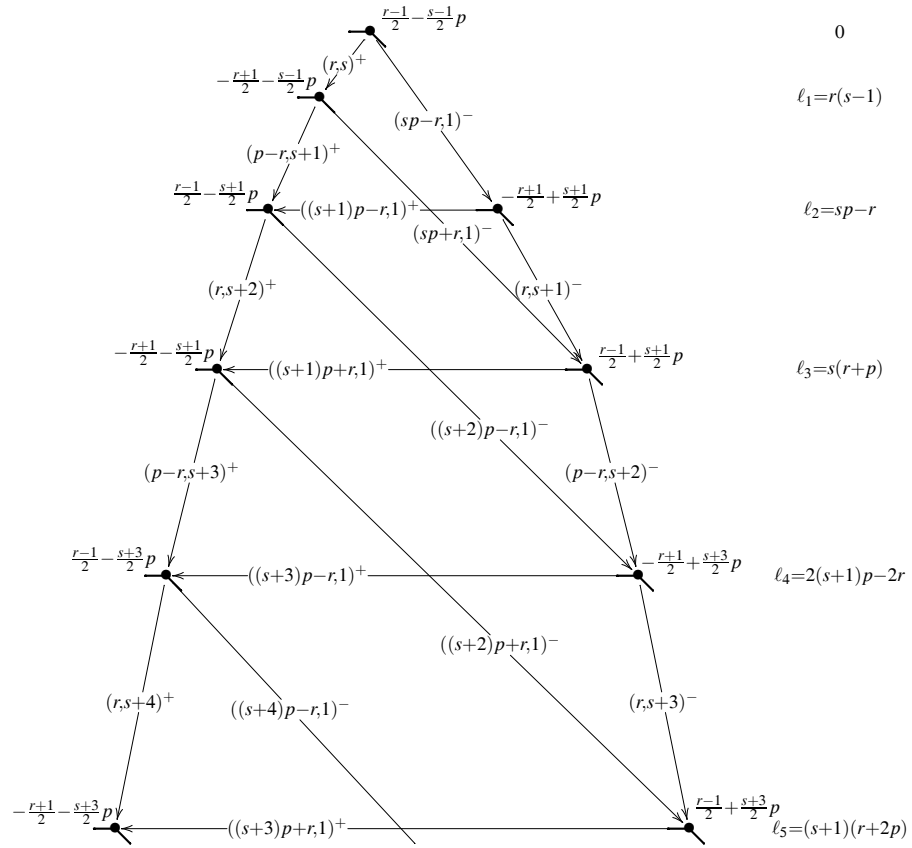
Let $p = k + 2 \in \{2, 3, \dots\}$ and $\lambda = \lambda^+(r, s) = \frac{r-1}{2} - p\frac{s-1}{2}$. Submodules in the Verma module \mathcal{M}_λ are then arranged as in Fig. A.1. Arrows labeled $(r', s')^\pm$ denote $|\text{MFF}^\pm(r', s')\rangle$ singular vectors (see 2.2). The labels at the Verma submodules (which are represented by dots) show the spin of the respective highest-weight vector. The right column gives the relative level with respect to the highest-weight vector of the module:

$$(A.3) \quad \ell_i = \begin{cases} \frac{i}{2} \left((s + \frac{i}{2} - 1)p - r \right), & \text{even } i, \\ (s + \frac{i-3}{2}) \left(r + \frac{i-1}{2} p \right), & \text{odd } i. \end{cases}$$

It may be instructive to consider the picture in Fig. A.1 transformed by the spectral flow with $\theta = 1$. More precisely, we take the twisted Verma module $\mathcal{M}_{\lambda+1+\frac{k}{2};1}$, whose highest-weight vector is in the grade $(\lambda + 1, \ell)$, where (λ, ℓ) is the grade of the highest-weight vector of \mathcal{M}_λ (see (2.6)). For $\lambda = \frac{r-1}{2} - p\frac{s-1}{2}$ as above, the embedding diagram of $\mathcal{M}_{\lambda+1+\frac{k}{2};1}$ is shown in Fig. A.2, where an arrow labeled $(a, b)_{\pm 1}^\pm$ denotes the $|\text{MFF}^\pm(a, b; 1)\rangle$ singular vector. The labels at the twisted highest-weight vectors *indicate their charge, not spin* (i.e., the eigenvalue of H_0 , not λ in (2.6)).

We leave it to the reader to place the two diagrams, of \mathcal{M}_λ and $\mathcal{M}_{\lambda+\frac{k}{2};1}$, into the corresponding grades in the same picture and see how an extension of one module by the other can then be constructed (the key is the matching lengths of the horizontal arrows in the two diagrams).

We next recall the characters of some irreducible subquotients occurring in Fig. A.1. Let \mathcal{N}_i denote the *right-hand* irreducible subquotient at the i th embedding level (i.e., on


 FIGURE A.1. Embedding structure of the Verma module $\mathcal{M}_{\lambda+(r,s)}$.

the level ℓ_i relative to the top of the diagram). From the BGG resolution, its character follows as

$$(A.4) \quad \chi^{\mathcal{N}_i}(q, z) = \frac{q^{\frac{r^2}{4p} - \frac{s-1}{2}r + \frac{(s-1)^2}{4}p}}{q^{\frac{1}{8}} \mathfrak{D}_{1,1}(q, z)} \times \left(\sum_{a \geq 0} + \sum_{a \leq -s-i-1} \right) q^{\frac{i+2a}{2}((s-1+\frac{i+2a}{2})p-r)} \left(z^{-\frac{r+1}{2} + \frac{s+i+2a-1}{2}p} - z^{\frac{r-1}{2} - \frac{s+i+2a-1}{2}p} \right)$$

for even i and

$$(A.5) \quad \chi^{\mathcal{N}_i}(q, z) = \frac{q^{\frac{r^2}{4p} - \frac{s-1}{2}r + \frac{(s-1)^2}{4}p}}{q^{\frac{1}{8}} \mathfrak{D}_{1,1}(q, z)} \times \left(\sum_{a \geq 1} + \sum_{a \leq 1-s-i} \right) q^{\frac{i+1+2a}{2}((s+\frac{i+2a-1}{2})p-r)} \left(z^{\frac{r-1}{2} - \frac{s+i+2a}{2}p} - z^{-\frac{r+1}{2} + \frac{s+i+2a}{2}p} \right)$$

for odd i .

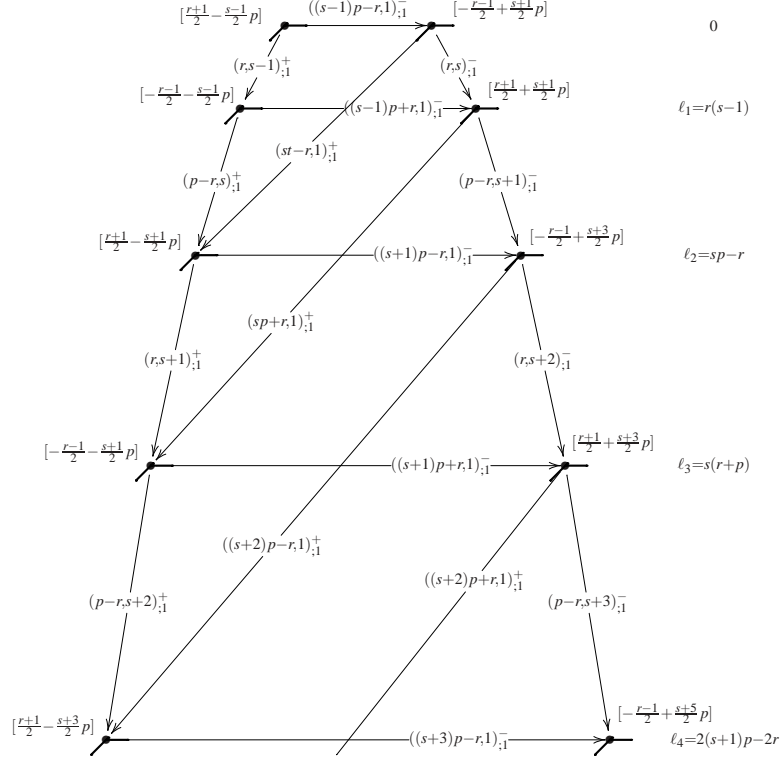


FIGURE A.2. Embedding structure of the $(\theta = 1)$ -twisted Verma module $\mathcal{M}_{\lambda^+(r,s);1}$. Labels in square brackets indicate *charges* of the corresponding twisted highest-weight vectors.

APPENDIX B. THETA-FUNCTION CONVENTIONS

The higher-level theta-functions are defined as

$$(B.1) \quad \theta_{r,p}(q, z) = \sum_{\iota \in \mathbb{Z} + \frac{r}{2p}} q^{p\iota^2} z^{p\iota}.$$

We set

$$(B.2) \quad \theta'_{r,p}(q, z) = z \frac{\partial}{\partial z} \theta_{r,p}(q, z), \quad \theta''_{r,p}(q, z) = \left(z \frac{\partial}{\partial z} \right)^2 \theta_{r,p}(q, z).$$

We also use the classic theta-functions

$$(B.3) \quad \vartheta_{1,1}(q, z) = \sum_{m \in \mathbb{Z}} q^{\frac{1}{2}(m^2 - m)} (-z)^{-m} = \prod_{m \geq 0} (1 - z^{-1} q^m) \prod_{m \geq 1} (1 - z q^m) \prod_{m \geq 1} (1 - q^m),$$

$$(B.4) \quad \vartheta(q, z) = \sum_{m \in \mathbb{Z}} q^{\frac{m^2}{2}} z^m$$

related to (B.1) as

$$\theta_{r,p}(q, z) = z^{\frac{r}{2}} q^{\frac{r^2}{4p}} \vartheta(q^{2p}, z^p q^r).$$

The quasiperiodicity properties of theta-functions are expressed as

$$(B.5) \quad \theta_r(q, zq^n) = q^{-p\frac{n^2}{4}} z^{-p\frac{n}{2}} \theta_{r+pn}(q, z),$$

with $\theta_{r+pn,p}(q, z) = \theta_r(q, z)$ for *even* n , and

$$(B.6) \quad \vartheta_{1,1}(q, zq^n) = (-1)^n q^{-\frac{1}{2}(n^2+n)} z^{-n} \vartheta_{1,1}(q, z), \quad n \in \mathbb{Z}.$$

It then follows that

$$(B.7) \quad \theta'_r(q, zq^n) = q^{-p\frac{n^2}{4}} z^{-p\frac{n}{2}} \left(\theta'_{r+pn,p}(q, z) - \frac{pn}{2} \theta_{r+pn}(q, z) \right),$$

$$(B.8) \quad \theta''_r(q, zq^n) = q^{-p\frac{n^2}{4}} z^{-p\frac{n}{2}} \left(\theta''_{r+pn,p}(q, z) - pn\theta'_{r+pn}(q, z) + \frac{p^2n^2}{4} \theta_{r+pn}(q, z) \right).$$

We resort to the standard abuse by writing $f(\tau, \nu)$ for $f(e^{2i\pi\tau}, e^{2i\pi\nu})$; it is tacitly assumed that $q = e^{2i\pi\tau}$ (with τ in the upper complex half-plane) and $z = e^{2i\pi\nu}$.

The modular T -transform of the theta-function is expressed as

$$(B.9) \quad \theta_{r,p}(\tau + 1, \nu) = e^{i\pi\frac{r^2}{2p}} \theta_{r,p}(\tau, \nu)$$

and the S -transform as

$$(B.10) \quad \theta_{r,p}\left(-\frac{1}{\tau}, \frac{\nu}{\tau}\right) = e^{i\pi\frac{p\nu^2}{2\tau}} \sqrt{\frac{-i\tau}{2p}} \sum_{s=0}^{2p-1} e^{-i\pi\frac{rs}{p}} \theta_s(\tau, \nu).$$

Therefore,

$$(B.11) \quad \theta'_{r,p}\left(-\frac{1}{\tau}, \frac{\nu}{\tau}\right) = e^{i\pi\frac{p\nu^2}{2\tau}} \sqrt{\frac{-i\tau}{2p}} \sum_{s=0}^{2p-1} e^{-i\pi\frac{rs}{p}} \left(\tau\theta'_s(\tau, \nu) + \frac{p\nu}{2} \theta_s(\tau, \nu) \right)$$

(the price paid for abusing notation is that $\theta'_r(\tau, \nu) = \frac{1}{2i\pi} \frac{\partial}{\partial \nu} \theta_r(\tau, \nu)$) and

$$(B.12) \quad \theta''_{r,p}\left(-\frac{1}{\tau}, \frac{\nu}{\tau}\right) = e^{i\pi\frac{p\nu^2}{2\tau}} \sqrt{\frac{-i\tau}{2p}} \sum_{s=0}^{2p-1} e^{-i\pi\frac{rs}{p}} \left(\tau^2\theta''_s(\tau, \nu) + p\nu\tau\theta'_s(\tau, \nu) + \left(\frac{p^2\nu^2}{4} + \frac{p\tau}{4i\pi}\right) \theta_s(\tau, \nu) \right).$$

We also note the formula

$$\Omega\left(-\frac{1}{\tau}, \frac{\nu}{\tau}\right) = -i\sqrt{-i\tau} e^{i\pi\frac{\nu^2}{2\tau}} \Omega(\tau, \nu)$$

for the function $\Omega(q, z) = q^{\frac{1}{8}} z^{\frac{1}{2}} \vartheta_{1,1}(q, z)$.

The eta function

$$(B.13) \quad \eta(q) = q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m)$$

transforms as

$$(B.14) \quad \eta(\tau + 1) = e^{\frac{i\pi}{12}} \eta(\tau), \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau} \eta(\tau).$$

In calculating residues in **5.3**, we also need the formula

$$\left. \frac{\partial \vartheta_{1,1}(q, z)}{\partial z} \right|_{z=q^n} = (-1)^n q^{-\frac{1}{8}} \eta(q)^3 q^{-\frac{n^2}{2} - \frac{3n}{2}}, \quad n \in \mathbb{Z}.$$

APPENDIX C. SOME ELEMENTARY TRICKS WITH $SL(2, \mathbb{Z})$ REPRESENTATIONS

C.1. $SL(2, \mathbb{Z}) \curvearrowright \mathfrak{h} \times \mathbb{C}$. We first recall the standard $SL(2, \mathbb{Z})$ -action on $\mathfrak{h} \times \mathbb{C}$ (where \mathfrak{h} is the upper half-plane),

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}: (\tau, v) \mapsto \left(\frac{a\tau + b}{c\tau + d}, \frac{v}{c\tau + d} \right).$$

The space \mathcal{F} of suitable (e.g., meromorphic or just fractional-linear in τ) functions on $\mathfrak{h} \times \mathbb{C}$ is then endowed with an $SL(2, \mathbb{Z})$ -action.

C.2. \mathbb{C}^2 .

C.2.1. The defining two-dimensional representation — *the doublet* — of $SL(2, \mathbb{Z})$ is the representation where

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

act on \mathbb{C}^2 just by these matrices. We choose a basis $\underline{1}, \underline{\tau}$ in \mathbb{C}^2 such that $S\underline{\tau} = -\underline{1}$, $S\underline{1} = \underline{\tau}$, $T\underline{\tau} = \underline{\tau} + \underline{1}$, and $T\underline{1} = \underline{1}$.

If π is any finite-dimensional $SL(2, \mathbb{Z})$ -representation, with $S_\pi = s$ and $T_\pi = t$ acting on vectors denoted by ω , then $\mathbb{C}^2 \otimes \pi$ is spanned by $\underline{1}\omega$ and $\underline{\tau}\omega$, with the action

$$S(\underline{\tau}\omega) = -s\omega, \quad S(\underline{1}\omega) = \underline{\tau}s\omega, \quad T(\underline{\tau}\omega) = \underline{\tau}t\omega + \underline{1}t\omega, \quad T(\underline{1}\omega) = \underline{1}t\omega.$$

This $\mathbb{C}^2 \otimes \pi$ representation can be realized using the $SL(2, \mathbb{Z})$ action on \mathcal{F} as follows: we identify $\underline{1} = 1$ and $\underline{\tau} = \tau$, view ω as a function of τ , which allows considering $f(\tau)\omega$ with $f \in \mathcal{F}$, and redefine the action of S and T as

$$(C.1) \quad S(f(\tau)\omega) = \tau f\left(-\frac{1}{\tau}\right)s\omega, \quad T(f(\tau)\omega) = f(\tau + 1)t\omega$$

(in other words, f is prescribed to transform with weight 1). Indeed, these formulas immediately imply, e.g., that $S(\tau\omega) = -\frac{1}{\tau}\tau s\omega = -s\omega$.¹⁴ Wishing to deal with the more

¹⁴By $f(-\frac{1}{\tau})$, we everywhere mean $f(\tau)\Big|_{\tau \rightarrow -\frac{1}{\tau}}$, and similarly for $f(\tau + 1)$. For example, the $\tau \rightarrow -\frac{1}{\tau}$ operation sends $f(\frac{\tau}{\tau+1})$ into $f(-\frac{1}{\tau-1})$.

standard $c\tau + d$ (rather than $-c\tau + a$ for the inverse matrix), we have to consider the *right* action, which is determined by (C.1) as

$$\gamma: f(\tau)\omega \mapsto (c\tau + d)f(\gamma\tau)\omega \cdot \gamma, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

Clearly, extending $f(\tau)$ beyond polynomials of degree 1 yields an infinite-dimensional $SL(2, \mathbb{Z})$ representation on the space of suitable functions times π , containing $\mathbb{C}^2 \otimes \pi$ as a subrepresentation.

C.2.2. If π' is another $SL(2, \mathbb{Z})$ -representation of the same dimension as π , with $S_{\pi'} = s'$ and $T_{\pi'} = t'$ acting on vectors denoted by χ , then the direct sum $\pi' \oplus \mathbb{C}^2 \otimes \pi$ admits a family of deformations achieved by introducing a (matrix) automorphy factor as follows. We start from the direct sum $\pi' \oplus \mathbb{C}^2 \otimes \pi$ realized as

$$(C.2) \quad \begin{pmatrix} f(\tau)\omega \\ \chi \end{pmatrix}$$

(where ω is from the representation space of π and χ is from π'), with $f(\tau)$ being a first- or zeroth-degree polynomial, in accordance with the above realization of the doublet. The entire representation space is then spanned by $\begin{pmatrix} f(\tau)\omega + \beta v g(\tau)\chi \\ \chi \end{pmatrix}$. The terms containing v in the top row are always linear in v and proportional to a chosen parameter β , because of their origin that becomes clear momentarily.

The $SL(2, \mathbb{Z})$ action is uniquely defined by the T and S actions

$$T \begin{pmatrix} f(\tau)\omega + \beta v g(\tau)\chi \\ \chi \end{pmatrix} = \begin{pmatrix} f(\tau + 1)t\omega + \beta v g(\tau + 1)t'\chi \\ t'\chi \end{pmatrix},$$

$$S \begin{pmatrix} f(\tau)\omega + \beta v g(\tau)\chi \\ \chi \end{pmatrix} = \begin{pmatrix} \tau f(-\frac{1}{\tau})s\omega + \beta \frac{v}{\tau} g(-\frac{1}{\tau})s'\chi + \beta v f(-\frac{1}{\tau})s'\chi \\ s'\chi \end{pmatrix}.$$

The third term in the top row of the S -transformation formula is the origin of terms proportional to βv . The prescription for the S -transformation rule is to act with and without an extra τ factor on terms without v and with it in the top row. That is, f in $f(\tau)\omega$ is assigned weight 1 as before, whereas any g in $v g(\tau)\chi$ is considered to have weight 0.

It is easy to see that the $SL(2, \mathbb{Z})$ -orbit (again in the right-action version) of elements (C.2) can then be written as in (4.16).

C.3. \mathbb{C}^3 .

C.3.1. Taking the symmetrized square of \mathbb{C}^2 gives the \mathbb{C}^3 representation; following **C.2.1**,

we can represent its basis as $\underline{1} \otimes \underline{1}$, $\underline{\tau} \otimes \underline{1} + \underline{1} \otimes \underline{\tau}$, and $\underline{\tau} \otimes \underline{\tau}$. A somewhat shorter notation for the same basis is $\underline{1}$, $\underline{2\tau}$, and $\underline{\tau^2}$, with the $SL(2, \mathbb{Z})$ -action immediately found as

$$\begin{aligned} S\underline{1} &= \underline{\tau^2}, & S\underline{2\tau} &= -\underline{2\tau}, & S\underline{\tau^2} &= \underline{1}, \\ T\underline{1} &= \underline{1}, & T\underline{2\tau} &= \underline{2\tau} + 2 \cdot \underline{1}, & T\underline{\tau^2} &= \underline{\tau^2} + \underline{2\tau} + \underline{1}. \end{aligned}$$

This form suggests a ‘‘functional realization’’ of $\mathbb{C}^3 \otimes \pi$ (for a finite-dimensional $SL(2, \mathbb{Z})$ representation π determined by $\varphi \mapsto s\varphi$ and $\varphi \mapsto t\varphi$) as the representation spanned by φ , $2\tau\varphi$, and $\tau^2\varphi$, with the $SL(2, \mathbb{Z})$ -action defined by

$$S(f(\tau)\varphi) = \tau^2 f(-\frac{1}{\tau})s\varphi, \quad T(f(\tau)\varphi) = f(\tau+1)t\varphi.$$

In other words, f is assigned weight 2 and $SL(2, \mathbb{Z})$ acts (in the right-action version) as

$$\gamma: f(\tau)\varphi \mapsto (c\tau + d)^2 f(\gamma\tau)\varphi \cdot \gamma.$$

Extending $f(\tau)$ beyond degree-2 polynomials gives an infinite-dimensional $SL(2, \mathbb{Z})$ representation in which $\mathbb{C}^3 \otimes \pi$ is a subrepresentation.

C.3.2. With $\mathbb{C}^3 \otimes \pi$ realized as in **C.3.1**, we construct a deformation of $\pi'' \oplus \pi' \oplus \mathbb{C}^3 \otimes \pi$ for three representations π'' , π' , and π of the same dimension. We write

$$(C.3) \quad \begin{pmatrix} f(\tau)\varphi \\ \omega \\ \chi \end{pmatrix}$$

for an arbitrary element of $\pi'' \oplus \pi' \oplus \mathbb{C}^3 \otimes \pi$, where $f(\tau)$ a polynomial of degree at most two. The ‘‘new’’ $SL(2, \mathbb{Z})$ -action on such elements gives rise to and is defined on elements of the form

$$\begin{pmatrix} f(\tau)\varphi + v g(\tau)\omega + v^2 h(\tau)\chi \\ \omega \\ \chi \end{pmatrix},$$

where the v -dependent terms are in fact proportional to the parameters α and β as becomes clear from the S -action formula below.

The T generator acts as (we omit the primes distinguishing the actions in π , π' , and π'')

$$\begin{pmatrix} f(\tau)\varphi + v g(\tau)\omega + v^2 h(\tau)\chi \\ \omega \\ \chi \end{pmatrix} \xrightarrow{T} \begin{pmatrix} f(\tau+1)t\varphi + v g(\tau+1)t\omega + v^2 h(\tau+1)t\chi \\ t\omega \\ t\chi \end{pmatrix}$$

and S as

$$\begin{pmatrix} f(\tau)\varphi + v g(\tau)\omega + v^2 h(\tau)\chi \\ \omega \\ \chi \end{pmatrix} \xrightarrow{S}$$

$$\begin{pmatrix} \tau^2 f(-\frac{1}{\tau})s\varphi + \nu g(-\frac{1}{\tau})s\omega + \frac{\nu^2}{\tau} h(-\frac{1}{\tau})s\chi + \alpha\nu\tau f(-\frac{1}{\tau})s\omega + \nu^2 \left(\frac{\alpha\beta}{2} f(-\frac{1}{\tau}) + \frac{\beta}{\tau} g(-\frac{1}{\tau}) \right) s\chi \\ s\omega \\ s\chi \end{pmatrix}.$$

The S -transformation rule in the top row can be described as follows. The ν -independent term transforms as a triplet (with a factor of τ^2 as in **C.3.1**); the term linear in ν transforms as “ ν times a doublet,” i.e., $S: \nu \cdot g(\tau)\omega \mapsto \frac{\nu}{\tau} \cdot \tau g(-\frac{1}{\tau})s\omega$ (with the extra τ factor as in **C.2.1**); the ν^2 -term involves no additional τ -dependent factors: it transforms in accordance with the action on functions on $\mathfrak{h} \times \mathbb{C}$; in addition, the pair $\begin{pmatrix} f(\tau)\varphi \\ \omega \end{pmatrix}$ gives rise to an extra term $\alpha\nu\tau f(-\frac{1}{\tau})s\omega$, the pair $\begin{pmatrix} \nu g(\tau)\omega \\ \chi \end{pmatrix}$ to an extra term $\nu^2 \frac{\beta}{\tau} g(-\frac{1}{\tau})s\chi$, and the pair $\begin{pmatrix} f(\tau)\varphi \\ \chi \end{pmatrix}$ to $\frac{\alpha\beta}{2} \nu^2 f(-\frac{1}{\tau})s\chi$.

It is not difficult to see that the (right-action) $SL(2, \mathbb{Z})$ -orbit of elements (C.3) can be written as in (4.18).

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