

$SU(2)$ Yang-Mills quantum mechanics of spatially constant fields

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Abstract

As a first step towards a strong coupling expansion of Yang-Mills theory, the $SU(2)$ Yang-Mills quantum mechanics of spatially constant gauge fields is investigated in the symmetric gauge, with the six physical fields represented in terms of a positive definite symmetric 3×3 matrix S . Representing the eigenvalues of S in terms of elementary symmetric polynomials, the eigenstates of the corresponding harmonic oscillator problem can be calculated analytically and used as orthonormal basis of trial states for a variational calculation of the Yang-Mills quantum mechanics. In this way high precision results are obtained in a very effective way for the lowest eigenstates in the spin-0 sector as well as for higher spin. Furthermore I find, that practically all excitation energy of the eigenstates, independently of whether it is a vibrational or a rotational excitation, leads to an increase of the expectation value of the largest eigenvalue $\langle \phi_3 \rangle$, whereas the expectation values of the other two eigenvalues, $\langle \phi_1 \rangle$ and $\langle \phi_2 \rangle$, and also the component $\langle B_3 \rangle = g \langle \phi_1 \phi_2 \rangle$ of the magnetic field, remain at their vacuum values.

1 Introduction

The so-called Yang-Mills mechanics originates from Yang-Mills field theory under the supposition of the spatial homogeneity of the gauge fields [1]. For this case the Lagrangian of pure $SU(2)$ Yang-Mills theory reduces to ¹

$$L = \frac{1}{2} \left(\dot{A}_{ai} - g \epsilon_{abc} A_{b0} A_{ci} \right)^2 - \frac{1}{2} B_{ai}^2(A), \quad (1)$$

with the magnetic field $B_{ai}(A) = (1/2)g\epsilon_{abc}\epsilon_{ijk}A_{bj}A_{ck}$. The local $SU(2)$ gauge invariance and the rotational invariance of the original Yang-Mills action reduces to the symmetry under the $SO(3)$ transformations local in time

$$\begin{aligned} A_{a0}^\omega(t) &= O(\omega(t))_{ab} A_{b0}(t) - \frac{1}{2g} \epsilon_{abc} \left(O(\omega(t)) \dot{O}(\omega(t)) \right)_{bc}, \\ A_{ai}^\omega(t) &= O(\omega(t))_{ab} A_{bi}(t), \end{aligned} \quad (2)$$

¹Everywhere in the paper we put the spatial volume $V = 1$. As result the coupling constant g becomes dimensionful with $g^{2/3}$ having the dimension of energy. The volume dependence can be restored in the final results by replacing g^2 with g^2/V .

and the global spatial rotations $A_{ai}^X = A_{aj}R(\chi)_{ji}$.

In the constrained Hamiltonian formulation (see e.g.[2]) the time dependence of the gauge transformations (2) is exploited to put the Weyl gauge $A_{a0} = 0$, $a=1,2,3$, and the physical states Ψ have to satisfy both the Schrödinger equation and the three Gauss law constraints

$$H\Psi = \frac{1}{2} \sum_{a,i} \left[\left(\frac{\partial}{\partial A_{ai}} \right)^2 + B_{ai}^2(A) \right] \Psi = E\Psi, \quad (3)$$

$$G_a \Psi = -i\epsilon_{abc} A_{bi} \frac{\partial}{\partial A_{ci}} \Psi = 0, \quad a = 1, 2, 3. \quad (4)$$

The G_a are the generators of the residual time independent gauge transformations, satisfying $[G_a, H] = 0$ and $[G_a, G_b] = i\epsilon_{abc}G_c$. Furthermore H commutes with the angular momentum operators $J_i = -i\epsilon_{ijk}A_{aj}\partial/\partial A_{ak}$, $i = 1, 2, 3$. The matrix element of an operator O is given in the Cartesian form

$$\langle \Psi' | O | \Psi \rangle \propto \int dA \Psi'^*(A) O \Psi(A). \quad (5)$$

For carrying out quantum mechanical calculations it is desirable to have a corresponding unconstrained Schrödinger equation and to find its eigenstates in an effective way with high accuracy at least for the lowest states. The basic ideas and first results of such an gauge fixed approach to Yang-Mills quantum mechanics will be presented in the following.

2 Unconstrained Hamiltonian

2.1 The symmetric gauge

The local symmetry transformation (2) of the gauge potentials A_{ai} prompts us with the set of coordinates in terms of which the separation of the gauge degrees of freedom occurs. As in [3] I use the polar decomposition for arbitrary 3×3 quadratic matrices

$$A_{ai}(q, S) = O_{ak}(q) S_{ki}, \quad (6)$$

with the orthogonal matrix $O(q)$, parametrized by the three angles q_i , and the positive definite, symmetric 3×3

matrix S . The decomposition (6) is unique and corresponds to the symmetric gauge $\chi_i(A) = \epsilon_{ijk} A_{jk} = 0$. The Jacobian is $|\partial(A_{ai})/\partial(q, S)| \propto \det \Omega(q) \prod_{i < j} (\phi_i + \phi_j)$, where ϕ_1, ϕ_2, ϕ_3 are the eigenvalues of S and $\Omega_{jm}(q) \equiv (1/2)\epsilon_{mkl} [O^T(q)\partial O(q)/\partial q_j]_{kl}$. The variables S and $\partial/\partial S$ make no contribution to the Gauss law operators $G_a = -iO_{as}(q)\Omega_{sj}^{-1}(q)\partial/\partial q_j$. Hence, assuming the invertibility of the matrix Ω , the non-Abelian Gauss laws (4) are equivalent to the set of Abelian constraints $\partial\Psi/\partial q_i = 0, i = 1, 2, 3$, and the physical Hamiltonian of $SU(2)$ Yang-Mills quantum mechanics reads

$$H = \frac{1}{2} \sum_{m,n} \left[- \left(\frac{\partial}{\partial S_{mn}} \right)^2 + \frac{1}{2} \gamma_{mn}^{-2} J_m J_n + B_{mn}^2(S) - [\gamma_{mn}^{-1}(S) - \delta_{mn} \text{tr}(\gamma^{-1}(S))] \frac{\partial}{\partial S_{mn}} \right] \quad (7)$$

with $\gamma_{ik}(S) := S_{ik} - \delta_{ik} \text{tr} S$ and the angular momenta $J_i = -2i\epsilon_{ijk} S_{aj} \partial/\partial S_{ak}, i = 1, 2, 3$, in terms of the physical variables (note the factor 2). The matrix element of a physical operator O is given by

$$\langle \Psi' | O | \Psi \rangle \propto \int dS \left[\prod_{i < j} (\phi_i + \phi_j) \right] \Psi'^*(S) O \Psi(S). \quad (8)$$

2.2 Unconstrained Hamiltonian in terms of rotational and scalar degrees of freedom

In order to achieve a more transparent form for the reduced Yang-Mills system (7) I shall limit myself in this work to the principle orbit configurations

$$0 < \phi_1 < \phi_2 < \phi_3 < \infty, \quad (9)$$

for the eigenvalues $\phi_1, \phi_2, \phi_3 > 0$ of the positive definite symmetric matrix S (not considering singular orbits where two or more eigenvalues coincide) and perform a principal-axes transformation

$$S = R(\alpha, \beta, \gamma) \text{diag}(\phi_1, \phi_2, \phi_3) R^T(\alpha, \beta, \gamma), \quad (10)$$

with the $SO(3)$ matrix R parametrized by the three Euler angles. The Jacobian of (10) is $|\partial S/\partial(\alpha, \beta, \gamma, \phi)| \propto \sin \beta \prod_{i < j} (\phi_i - \phi_j)$. In terms of the principal-axes variables, the physical Hamiltonian reads

$$H = \frac{1}{2} \sum_{\text{cyclic}}^3 \left[- \frac{\partial^2}{\partial \phi_i^2} - \frac{2}{\phi_i^2 - \phi_j^2} \left(\phi_i \frac{\partial}{\partial \phi_i} - \phi_j \frac{\partial}{\partial \phi_j} \right) + \xi_i^2 \frac{\phi_j^2 + \phi_k^2}{(\phi_j^2 - \phi_k^2)^2} + g^2 \phi_j^2 \phi_k^2 \right]. \quad (11)$$

All the rotational variables in the Hamiltonian (53), which agrees with that of [4], are combined into the intrinsic angular momenta $\xi_i = R_{ij}^T J_j$ obeying $[\xi_i, \xi_j] = -i\epsilon_{ijk} \xi_k$. The matrix elements of an operator O are given as

$$\langle \Psi' | O | \Psi \rangle \propto \int d\alpha \sin \beta d\beta d\gamma \int \left[\prod_{\text{cyclic}} d\phi_i (\phi_j^2 - \phi_k^2) \right] \Psi'^* O \Psi. \quad (12)$$

The potential term in (19), has a flat valley of degenerate absolute minima at zero energy, $\phi_1 = \phi_2 = 0, \phi_3$ – arbitrary, at the edge of (9). Close to the bottom of the valley the potential is that of a harmonic oscillator with a width narrowing down for larger values of ϕ_3 .

2.3 Symmetries

As a relic of the rotational invariance of the initial gauge field theory the Hamiltonian (53) possesses the symmetry

$$[H, J_k] = 0, \quad (13)$$

with the angular momentum operators $J_i = R_{ij} \xi_j$ satisfying $[J_i, J_j] = i\epsilon_{ijk} J_k$ and $[J_i, \xi_j] = 0$. Hence the eigenstates of H can be characterized by the quantum numbers J and M . Furthermore H is invariant under arbitrary permutations σ_{ij} of any two of the three indices 1, 2, 3, time reflections T , and parity reflections $P: \phi_i \rightarrow -\phi_i$,

$$[H, \sigma_{ij}] = 0, \quad [H, T] = 0, \quad [H, P] = 0. \quad (14)$$

2.4 Boundary conditions

The requirement of Hermiticity of H in the region bounded by the three boundary planes $\phi_1 = 0, \phi_1 = \phi_2, \phi_2 = \phi_3$ and at positive infinity, leads to the conditions at ϕ_1

$$\partial_1 \Psi \Big|_{\phi_1=0} = 0 \quad \vee \quad \Psi \Big|_{\phi_1=0} = 0, \quad (15)$$

corresponding to even or odd parity states respectively. I shall consider in the work only the first alternative allowing only even parity states in all spin sectors. Furthermore, due to the vanishing of the measure (12) on the boundaries $\phi_1 = \phi_2$ and $\phi_2 = \phi_3$, we have to require only that

$$\Psi \Big|_{\phi_2=\phi_1} = \text{finite} \quad \wedge \quad \Psi \Big|_{\phi_3=\phi_2} = \text{finite}. \quad (16)$$

Finally, normalisability of the wave functions requires that the wave functions vanish sufficiently fast at infinity.

2.5 Virial theorem

Writing $H = \frac{1}{2}(E^2 + B^2)$ and denoting the eigenstates of H by $|n\rangle$, with energies E_n , one obtains the virial theorem

$$\langle n | E^2 | n \rangle = 2 \langle n | B^2 | n \rangle, \quad (17)$$

It proves to be a very useful tool to judge the quality of the approximate eigenstates obtained using the variational approach. From (17) also follows ($G^2 := 2(B^2 - E^2)$)

$$\langle n | B^2 | n \rangle = \frac{2}{3} E_n, \quad \langle n | E^2 | n \rangle = \frac{4}{3} E_n, \quad \langle n | G^2 | n \rangle = -\frac{4}{3} E_n. \quad (18)$$

3 Low energy spin-0 states

For $\mathbf{J}^2 = \xi^2 = 0$, the Hamiltonian (11) reduces to

$$H_0 = \frac{1}{2} \sum_{\text{cyclic}}^3 \left[-\frac{\partial^2}{\partial \phi_i^2} - \frac{2}{\phi_i^2 - \phi_j^2} \left(\phi_i \frac{\partial}{\partial \phi_i} - \phi_j \frac{\partial}{\partial \phi_j} \right) + g^2 \phi_j^2 \phi_k^2 \right] \quad (19)$$

Similar as in the Calogero model [5] one can prove that the eigenstates of (19) are completely symmetric in the arguments (ϕ_1, ϕ_2, ϕ_3) . We therefore can have only the two possible forms, the parity even

$$\Psi^{(+)}(\phi_1, \phi_2, \phi_3) = \tilde{\Psi}^{(+)}(\phi_1^2, \phi_2^2, \phi_3^2), \quad (20)$$

and the parity odd

$$\Psi^{(-)}(\phi_1, \phi_2, \phi_3) = \phi_1 \phi_2 \phi_3 \tilde{\Psi}^{(-)}(\phi_1^2, \phi_2^2, \phi_3^2), \quad (21)$$

where the functions $\tilde{\Psi}^{(\pm)}$ are completely symmetric in the arguments $(\phi_1^2, \phi_2^2, \phi_3^2)$. $\Psi^{(+)}$ satisfies the first and $\Psi^{(-)}$ the second of the boundary conditions (15). I shall consider here only the even case (20), lower in energy. The odd case (21) can be treated completely analogously.

3.1 Elementary symmetric polynomials

Here it is useful to change to the new coordinates (e_1, e_2, e_3) defined as the elementary symmetric combinations

$$e_1 = \phi_1^2 + \phi_2^2 + \phi_3^2, \quad e_2 = \phi_1^2 \phi_2^2 + \phi_2^2 \phi_3^2 + \phi_3^2 \phi_1^2, \quad e_3 = \phi_1^2 \phi_2^2 \phi_3^2. \quad (22)$$

The corresponding Jacobian

$$\left| \frac{\partial(\phi_1, \phi_2, \phi_3)}{\partial(e_1, e_2, e_3)} \right| = \frac{1}{\sqrt{e_3} \sqrt{\Delta}}, \quad (23)$$

with the square root of the discriminant

$$\Delta \equiv \prod_{i < j} (\phi_i^2 - \phi_j^2)^2 = -27e_3^2 + 18e_1e_2e_3 - 4e_1^3e_3 - 4e_2^3 + e_2^2e_1^2, \quad (24)$$

cancels the original measure $\prod_{i < j} (\phi_i^2 - \phi_j^2)$ in (12). Furthermore let us consider the scaling transformation

$$s_1 = e_1, \quad s_2 = 3e_2/e_1^2, \quad s_3 = 27e_3/e_1^3, \quad (25)$$

with the Jacobian $|\partial(e_1, e_2, e_3)/\partial(s_1, s_2, s_3)| \propto s_1^5$. Then the Schrödinger equation of (19) becomes

$$\left[\frac{1}{6} g^2 s_1^2 s_2 - 2s_1 \frac{\partial^2}{\partial s_1^2} - 9 \frac{\partial}{\partial s_1} + \frac{1}{2s_1} \left(D^{(0)} - \frac{49}{4} \right) \right] \Psi = E \Psi, \quad (26)$$

with

$$D^{(0)} := D_0^{(0)} + D_{-1}^{(0)} + D_{-2}^{(0)}, \quad (27)$$

where

$$D_0^{(0)} := \left(2 \left(2s_2 \frac{\partial}{\partial s_2} + 3s_3 \frac{\partial}{\partial s_3} \right) + \frac{7}{2} \right)^2, \quad (28)$$

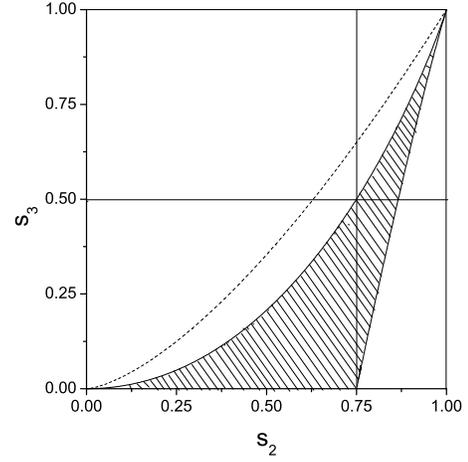


Figure 1: The (dashed) area of integration in (s_2, s_3) space. The dotted line is the limit $s_3 \leq s_2^{3/2}$ from algebraic inequalities.

$$D_{-1}^{(0)} := -4s_3 \frac{\partial^2}{\partial s_2^2} - 18 \left(2s_3 \frac{\partial}{\partial s_3} + 1 \right) s_2 \frac{\partial}{\partial s_3}, \quad (29)$$

$$D_{-2}^{(0)} := -12 \left(s_2 \frac{\partial}{\partial s_2} + 4s_3 \frac{\partial}{\partial s_3} + 2 \right) \frac{\partial}{\partial s_2}. \quad (30)$$

Note that the singularities in H_0 have disappeared via the transformation (22) to the elementary symmetric variables. The matrix elements become

$$\langle \Psi' | O | \Psi \rangle \propto \int_0^\infty ds_1 s_1^{7/2} \left[\int_0^{3/4} ds_2 \int_0^{s_3^{(\text{up})}(s_2)} \frac{ds_3}{\sqrt{s_3}} \Psi'^* O \Psi + \int_{3/4}^1 ds_2 \int_{s_3^{(\text{low})}(s_2)}^{s_3^{(\text{up})}(s_2)} \frac{ds_3}{\sqrt{s_3}} \Psi'^* O \Psi \right]. \quad (31)$$

Here the region of integration is given by the (s_2, s_3) satisfying the inequality, obtained from 24,

$$\tilde{\Delta} \equiv (-s_3^2 + 6s_2s_3 - 4s_3 - 4s_2^3 + 3s_2^2) > 0. \quad (32)$$

Denoting the two roots of the equation $\tilde{\Delta}(s_3, s_2) = 0$, quadratic in s_3 , by

$$s_3^{(\text{up})}(s_2) := (1 - \sqrt{1 - s_2})^2 (1 + 2\sqrt{1 - s_2}), \quad (33)$$

well defined and positive for all $0 \leq s_2 \leq 1/3$, and

$$s_3^{(\text{low})}(s_2) := (1 + \sqrt{1 - s_2})^2 (1 - 2\sqrt{1 - s_2}), \quad (34)$$

well defined for all $0 \leq s_2 \leq 1$, but positive only for $3/4 \leq s_2 \leq 1$. The region of positive $\tilde{\Delta}$ is the shaded region in Fig. 1.

3.2 Exact solution of the corresponding harmonic oscillator problem

Note that for the case when the Yang-Mills potential in (19) is replaced by the harmonic oscillator potential

$$g^2 (\phi_1^2 \phi_2^2 + \phi_2^2 \phi_3^2 + \phi_3^2 \phi_1^2) \longrightarrow \omega^2 (\phi_1^2 + \phi_2^2 + \phi_3^2), \quad (35)$$

that is $g^2 s_2 s_1^2 / 6$ replaced by $\omega s_1 / 2$, the above Schrödinger eigenvalue problem (26) separates into a density and a deformation problem² (k separation const., μ multiplicity),

$$\Phi_{nk\mu}(s) = R_{nk}(s_1) P_{k\mu}(s_2, s_3) . \quad (36)$$

The solutions of the density equations are given by

$$R_{nk}(s_1) = \sqrt{\frac{n!}{\Gamma(n+k+1)}} \omega^{\frac{9}{4}} (\omega s_1)^{\frac{1}{2}(k-\frac{7}{2})} e^{-\omega s_1/2} L_n^k(\omega s_1) , \quad (37)$$

satisfying the orthonormality relations

$$\int_0^\infty ds_1 s_1^{7/2} R_{nk}(s_1) R_{n'k}(s_1) = \delta_{nn'} , \quad (38)$$

with the energy eigenvalues

$$E_{nk} = (2n + 1 + k) \omega . \quad (39)$$

The values of k are determined by the corresponding deformation problem

$$D^{(0)} P_{k\mu}(s_2, s_3) = k^2 P_{k\mu}(s_2, s_3) . \quad (40)$$

In the space of monomials $s_2^p s_3^q$, $p, q \in N_0$ ordered by increasing $2p + 3q$, the operator $D^{(0)}$ has tridiagonal form, since the part $D_0^{(0)}$ is diagonal, and the parts $D_{-1}^{(0)}$ and $D_{-2}^{(0)}$ act as lowering operators, as can be easily seen from (28)-(30). Hence $D^{(0)}$ can easily be diagonalised with eigenvectors $P_{(p,q)}$ of the form

$$P_{(p,q)} = N_{(p,q)} \left[\begin{array}{c} s_2^p s_3^q + \sum_{2p'+3q' \leq 2p+3q} a(p, q; p', q') s_2^{p'} s_3^{q'} \end{array} \right] , \quad (41)$$

and eigenvalues

$$k = 2(2p + 3q) + \frac{7}{2} . \quad (42)$$

The first 3 eigenstates $P_{(p,q)}$, ordered by increasing eigenvalue k , are then

$$k = 7/2 \quad : \quad P_{(0,0)} = 1 , \quad (43)$$

$$k = 15/2 \quad : \quad P_{(1,0)} = \frac{11}{2} \sqrt{\frac{13}{15}} \left(s_2 - \frac{6}{11} \right) ,$$

$$k = 19/2 \quad : \quad P_{(0,1)} = \frac{221}{126} \sqrt{\frac{209}{10}} \left(s_3 - \frac{9}{17} s_2 + \frac{36}{221} \right) ,$$

and so on. Orthonormality of the states Φ with respect to the scalar product (31), leads to the corresponding orthonormality relations for the $P_{k\mu}$,

$$\int_0^{3/4} ds_2 \int_0^{s_3^{(\text{up})}(s_2)} \frac{ds_3}{\sqrt{s_3}} P_{k'\mu'} P_{k\mu} + \int_{3/4}^1 ds_2 \int_{s_3^{(\text{low})}(s_2)}^{s_3^{(\text{up})}(s_2)} \frac{ds_3}{\sqrt{s_3}} P_{k'\mu'} P_{k\mu} = \frac{6\sqrt{3}}{35} \delta_{k'k} \delta_{\mu'\mu} . \quad (44)$$

²Note that the variables s_1, s_2 and s_3 are related to the density and deformation variables ρ, β and γ used in [6] via $s_1 = \rho^2$, $s_2 = 1 - \beta^2$ and $s_3 = 1 - 3\beta^2 + 2\beta^3 \cos \gamma$.

We have chosen the overall normalisation constant such that the constant solution, $P_{(0,0)}$, corresponding to the lowest eigenvalue $k = 7/2$, is equal to one. Note that there is no eigenstate with $k = 11/2$. To the eigenvalue $k = 31/2$ correspond two states $P_{(3,0)}$ and $P_{(0,2)}$. One can easily check that states with different values of k are orthonormal to each other as it should be. To orthonormalise the states of equal k with respect to (45), one can use Gram-Schmidt orthonormalisation, $P_{(p,q)} \rightarrow P'_{(p,q)}$. In summary we obtain the eigenstates

$$\Phi_{npq}(s) = R_{nk}(s_1) P'_{(p,q)}(s_2, s_3) , \quad (45)$$

with the energy eigenvalues

$$E_{n,p,q} = \left(2(n + 2p + 3q) + \frac{9}{2} \right) \omega , \quad (46)$$

which are equidistant and depend only on the total number $2(n + 2p + 3q)$ of nodes. The degeneracy is therefore rapidly increasing with energy.

3.3 Low energy spin-0 spectrum from variational calculation

To obtain the low energy spectrum of the Hamiltonian I shall use the Rayleigh-Ritz variational method. Lead by the parabolic form of the Yang-Mills potential close to the bottom of the classical zero-energy valley, I shall use the orthonormal basis Φ_{npq} in (45) as trial functions. In order to have rapid convergence the frequency is fixed using the lowest eigenstate

$$\Phi_{000} = \sqrt{\Gamma(9/2)} \omega^{9/4} \exp[-\omega s_1/2] . \quad (47)$$

The stationarity conditions for the energy functional of this state,

$$E[\Phi_{000}] = \langle \Phi_{000} | H_0 | \Phi_{000} \rangle = \frac{9}{4} \omega + \frac{9}{4} \frac{g^2}{\omega^2} ,$$

leads to the optimal choice

$$\omega = \sqrt[3]{2} g^{2/3} \simeq 1.259921 g^{2/3} . \quad (48)$$

As a first upper bound for the groundstate energy of the Hamiltonian one therefore finds

$$E_0 \leq E[\Phi_{000}] \simeq 4.25223 g^{2/3} . \quad (49)$$

In order to improve this upper bound, one truncates the Fock space at a certain number of nodes of the wave functions and diagonalizes the corresponding truncated Hamiltonian H_{trunk} to find its eigenvalues and eigenstates. Extending to higher and higher numbers of nodes I obtain the low energy spectrum in the spin-0 sector to high numerical accuracy. Including all 174(1041) trial states up to 30(60) nodes, I obtain for the lowest state S_0 ,

$$E[S_0] = 4.116719740(35) g^{2/3} , \quad (50)$$

(i.e. the last two digits have to be replaced by the ones in brackets when going from 30 to 60 nodes). The state S_0 , given explicitly as (up to coefficients < 0.01)

$$S_0 = 0.98 \Phi_{000} - 0.07 \Phi_{100} - 0.04 \Phi_{200} - 0.16 \Phi_{010} + 0.02 \Phi_{001} + 0.03 \Phi_{020}, \quad (51)$$

nearly coincides with the state Ψ_{000} , The energies of the groundstate and the first excitations together with the corresponding deviation from the virial theorem (17)

$$\Delta E_{\text{virial}} := 2\langle B^2 \rangle - \langle E^2 \rangle, \quad (52)$$

are shown in Tab. 1.

The energy spectrum of the first spin-0 eigenstates (with trial states up to 30 nodes) are shown in Fig. 2. The errorbars are inside the lines. The energies obtained here are in good agreement with those obtained by Lüscher and Münster [7] and by Koller and van Baal [8], using gauge invariant wave functions as trial states.

4 Higher spin states

For non-vanishing spin we can write the Hamiltonian (11)

$$H = H_0 + \frac{1}{2} \sum_{i=1}^3 \xi_i^2 V_i, \quad V_i = \frac{\phi_j^2 + \phi_k^2}{(\phi_j^2 - \phi_k^2)^2}, \quad i, j, k \text{ cyclic}$$

with the spin-0 Hamiltonian (19). The eigenstates can be classified according to their \mathbf{J}^2 and J_3 quantum numbers and are superpositions of simultaneous eigenstates of \mathbf{J}^2, J_3 and ξ_3

$$|JM\rangle = \sum_{M'=0,1\pm,\dots,J\pm} \Psi_{MM'}^{(J)}(\phi_1, \phi_2, \phi_3) |JMM'\rangle, \quad (53)$$

with the combinations (for $M' > 0$)

$$|JMM'\pm\rangle := \frac{1}{\sqrt{2}} (|JMM'\rangle \pm |2M - M'\rangle), \quad (54)$$

of

$$|JMM'\rangle := i^J \sqrt{\frac{2J+1}{8\pi^2}} D_{MM'}^{(J)}(\alpha, \beta, \gamma). \quad (55)$$

Table 1: The energies and the deviation from the virial theorem for the lowest spin-0 eigenstates (using trial states up to 30 (60) nodes).

0^+	$E[g^{2/3}]$	$\Delta E_{\text{virial}}[g^{2/3}]$
0	4.116719740(35)	$7.6 \cdot 10^{-8} (1.2 \cdot 10^{-12})$
1	6.386361(58)	$2.4 \cdot 10^{-7} (1.3 \cdot 10^{-10})$
2	7.97348(34)	$4.8 \cdot 10^{-4} (1.3 \cdot 10^{-8})$
3	9.2039(23)	$1.8 \cdot 10^{-3} (3.1 \cdot 10^{-7})$
4	10.092(86)	$7.5 \cdot 10^{-4} (1.9 \cdot 10^{-6})$
5	10.966(37)	$-1.8 \cdot 10^{-2} (1.8 \cdot 10^{-5})$
6	12.17(05)	$-0.22 (1.5 \cdot 10^{-4})$

4.1 Spin 1

The spin-1 Schrödinger equation decays into three equations, one for each member of the cyclic triplet $(\Psi_1^{(1)}, \Psi_2^{(1)}, \Psi_3^{(1)}) = (\Psi_{M1-}^{(1)}, \Psi_{M1+}^{(1)}, \Psi_{M0}^{(1)})$,

$$\left[H_0 - E + \frac{1}{2}(V_2 + V_3) \right] \Psi_1^{(1)} = 0, \quad \text{and cycl. perm.}$$

One can easily show that no solutions exist which satisfy the boundary conditions (16).

4.2 Spin 2

The spin-2 Schrödinger equation decays into and into three equations, one for each member of the cyclic triplet $(\Psi_1^{(2)}, \Psi_2^{(2)}, \Psi_3^{(2)}) = (\Psi_{M1+}^{(2)}, \Psi_{M1-}^{(2)}, \Psi_{M2-}^{(2)})$,

$$\left[H_0 - E + \frac{1}{2}(V_2 + V_3) + 2V_1 \right] \Psi_1^{(2)} = 0, \quad \text{and cycl. perm.}$$

for which no solutions exist which satisfy the boundary conditions (16), and the coupled system

$$\left[H_0 - E + \frac{3}{2}(V_1 + V_2) \right] \Psi_{M0}^{(2)} + \frac{\sqrt{3}}{2}(V_1 - V_2) \Psi_{M2+}^{(2)} = 0$$

$$\left[H_0 - E + 2V_3 + \frac{1}{2}(V_1 + V_2) \right] \Psi_{M2+}^{(2)} + \frac{\sqrt{3}}{2}(V_1 - V_2) \Psi_{M0}^{(2)} = 0$$

for $\Psi_{M0}^{(2)}$ and $\Psi_{M2+}^{(2)}$. The solution of this spin-2 singlet system (56) can be written in the form

$$|2M\rangle = \Psi_1(s_1, s_2, s_3) s_1^{-1} Y_M(\phi_1^2, \phi_2^2, \phi_3^2; \alpha, \beta, \gamma) + \Psi_2(s_1, s_2, s_3) s_1^{-2} \tilde{Y}_M(\phi_1^2, \phi_2^2, \phi_3^2; \alpha, \beta, \gamma), \quad (56)$$

with the cyclic symmetric functions $\Psi_{1,2}(s_1, s_2, s_3)$ and elementary spin-2 fields $Y_M(\phi_1, \phi_2, \phi_3; \alpha, \beta, \gamma)$

$$Y_M = \sqrt{\frac{2}{3}} \left[\left(\phi_3 - \frac{1}{2}(\phi_1 + \phi_2) \right) |2M0\rangle + \frac{\sqrt{3}}{2}(\phi_1 - \phi_2) |2M2+\rangle \right] \quad (57)$$

and its dual $\tilde{Y}_M := Y_M|_{\phi_1 \rightarrow \phi_2 \phi_3}$. The matrix element of an operator $O(s)$ can be written as

$$\langle 2M' | O | 2M \rangle = \left[\langle \Psi_1' | (1 - s_2) O | \Psi_1 \rangle + \frac{1}{6} \langle \Psi_1' | (s_3 - s_2) O | \Psi_2 \rangle + \frac{1}{6} \langle \Psi_2' | (s_3 - s_2) O | \Psi_1 \rangle + \frac{1}{9} \langle \Psi_2' | (s_2^2 - s_3) O | \Psi_2 \rangle \right].$$

in terms of the corresponding spin-0 matrix elements (31). Specifying again to parity even states, the vector $\Psi = (\Psi_1, \Psi_2)$ satisfies the Schrödinger equation of the same form as (26) only with the scalar $D^{(0)}$ replaced with the new matrix operators $D^{(2)}$ with

$$D_0^{(2)} = \begin{pmatrix} \left(2 \left(2s_2 \frac{\partial}{\partial s_2} + 3s_3 \frac{\partial}{\partial s_3} + 1 \right) + \frac{7}{2} \right)^2 & 0 \\ 0 & \left(2 \left(2s_2 \frac{\partial}{\partial s_2} + 3s_3 \frac{\partial}{\partial s_3} + 2 \right) + \frac{7}{2} \right)^2 \end{pmatrix},$$

$$D_{-1}^{(2)} = \begin{pmatrix} D_{-1}^{(0)} & 8s_3 \frac{\partial}{\partial s_3} + 4 \\ 24 \frac{\partial}{\partial s_2} & D_{-1}^{(0)} \end{pmatrix},$$

$$D_{-2}^{(2)} = \begin{pmatrix} D_{-2}^{(0)} & 0 \\ 0 & D_{-2}^{(0)} - 24 \frac{\partial}{\partial s_2} \end{pmatrix}.$$

Again the corresponding harmonic oscillator Schrödinger Equation separates

$$\Phi_{nk\mu} = R_{nk}(s_1) \mathbf{P}_{k\mu}(s_2, s_3). \quad (58)$$

The density equations for R_{nk} are the same as in the spin-0 problem. The values of k are determined by the corresponding deformation problem

$$D^{(2)} \mathbf{P}_{k\mu}(s_2, s_3) = k^2 \mathbf{P}_{k\mu}(s_2, s_3), \quad (59)$$

using the basis $(s_2^p s_3^q, 0)$ and $(0, s_2^{\bar{p}} s_3^{\bar{q}})$. The first 3 eigenstates $P_{(p,q)}$, ordered by increasing eigenvalue k , are then

$$k = 11/2 : \quad \mathbf{P}_{(0,0)} = \frac{2}{\sqrt{195}} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$k = 15/2 : \quad \mathbf{P}_{(\bar{0},\bar{0})} = \sqrt{\frac{26}{35}} \begin{pmatrix} 2/13 \\ 1 \end{pmatrix},$$

$$k = 19/2 : \quad \mathbf{P}_{(1,0)} = \frac{17}{7} \sqrt{\frac{19}{15}} \begin{pmatrix} s_2 - 6/17 \\ 12/17 \end{pmatrix},$$

and so on.

As for the spin-0 case one can use this orthonormal basis of spin-2 states of the harmonic oscillator problem as trial states for a variational calculation of the corresponding eigenstates of the spin-2 Yang-Mills Hamiltonian. The energies of the groundstate and the first excitations together with the corresponding deviation from the virial theorem (17) are shown in Tab. 2.

The energy spectrum of the first spin-2 eigenstates (including the 270 trial states with up to 30 nodes) is shown in Fig. 2. The errorbars are inside the lines. Also for spin-2 the energies obtained here are in good agreement with those obtained by Lüscher and Münster [7] and by Koller and van Baal [8]. In particular I confirm that the first spin-2 state is lower in energy than the first spin-0 excitation.

Table 2: The energies and the deviation from the virial theorem for the lowest spin-2 eigenstates (using trial states up to 30 nodes).

2^+	$E[g^{2/3}]$	$\Delta E_{\text{virial}}[g^{2/3}]$
1	6.014500	$6.3 \cdot 10^{-6}$
2	7.82062	$4.8 \cdot 10^{-4}$
3	9.3335	$4.2 \cdot 10^{-3}$
4	9.9285	$6.0 \cdot 10^{-4}$
5	10.812	$-3.0 \cdot 10^{-2}$
6	11.91	-0.10

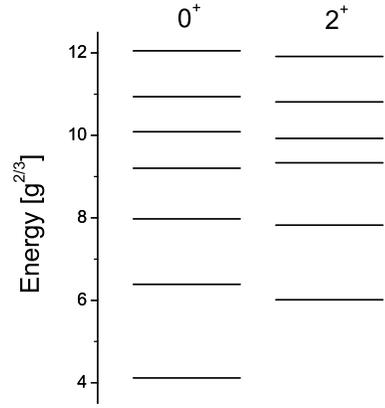


Figure 2: The energy spectrum of the first spin-0 and spin-2 eigenstates.

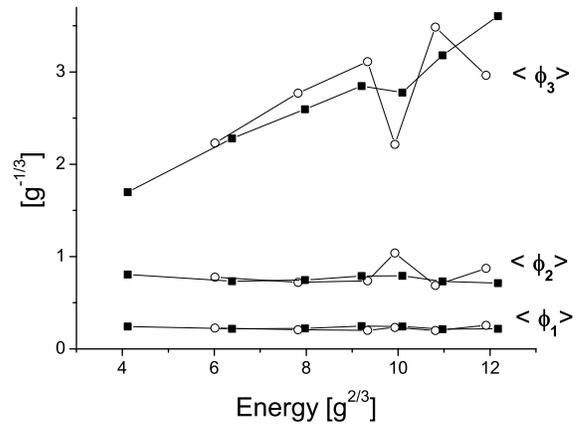


Figure 3: The expectation values $\langle \phi_1 \rangle$, $\langle \phi_2 \rangle$ and $\langle \phi_3 \rangle$ of the scalar fields along the principal axes as function of the energy for the lowest spin-0 (black boxes) and spin-2 (open circles) states.

5 Expectation values of permutation non-invariant operators

Expanding the obtained energy eigenstates in terms of the original ϕ_1, ϕ_2, ϕ_3 and analytically calculating the Gaussian integrals in the region (9), we obtain the expectation values $\langle \phi_1 \rangle, \langle \phi_2 \rangle$ and $\langle \phi_3 \rangle$ of the three fields along the principal axes. For the spin-0 groundstate e.g. we obtain

$$\begin{aligned} \langle 0 | \phi_1 | 0 \rangle_+ &= 0.242 g^{-1/3}, \\ \langle 0 | \phi_2 | 0 \rangle_+ &= 0.806 g^{-1/3}, \\ \langle 0 | \phi_3 | 0 \rangle_+ &= 1.699 g^{-1/3}. \end{aligned}$$

Fig. 3 shows the results for the lowest spin-0 (dark symbols) and spin-2 states (open white symbols) as function of the energy of the states. I find that $\langle \phi_3 \rangle$ raises universally with increasing energy, independent whether oscillator or spin excitation, whereas $\langle \phi_1 \rangle$ and $\langle \phi_2 \rangle$ stay practically constant.

The corresponding expectation values of the three in-

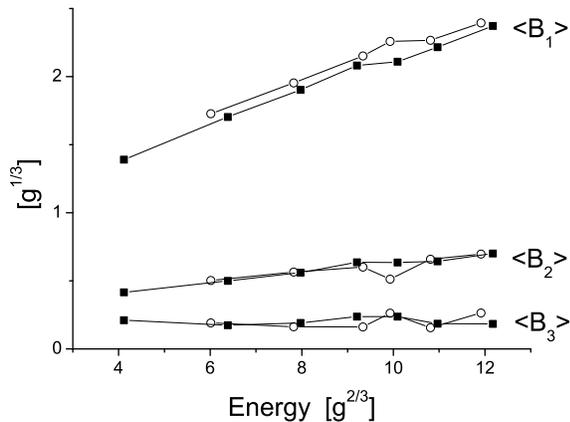


Figure 4: The expectation values $\langle B_1 \rangle$, $\langle B_2 \rangle$ and $\langle B_3 \rangle$ of the magnetic fields along the principal axes as function of the energy for the lowest spin-0 (black boxes) and spin-2 (open circles) states.

trinsic magnetic fields $B_1 = g\phi_2\phi_3$ etc.

$$\begin{aligned} \langle 0|B_1|0\rangle_+ &= 1.391 g^{1/3}, \\ \langle 0|B_2|0\rangle_+ &= 0.415 g^{1/3}, \\ \langle 0|B_3|0\rangle_+ &= 0.211 g^{1/3}, \end{aligned}$$

in accordance with the opposite ordering $B_1 > B_2 > B_3$, following from (9). The results for $\langle B_1 \rangle$, $\langle B_2 \rangle$ and $\langle B_3 \rangle$ as function of the energy for the lowest spin-0 and spin-2 states are shown in Fig. 4. $\langle B_3 \rangle$ stays practically constant with increasing energy, whereas $\langle B_1 \rangle$ and $\langle B_2 \rangle$ are raising.

Up to the considered number of 30 nodes the numerical errors are much smaller in both cases than the systematical error of the obtained states ($\leq 1\%$).

6 Summary and discussion

It has been shown that $SU(2)$ Yang-Mills quantum mechanics can be studied very effectively and with high accuracy, by choosing the symmetric gauge and performing a variational calculation using the orthonormal basis of exact eigenstates the corresponding harmonic oscillator problem as trial states in the spin-0 and the spin-2 sector. The virial theorem is shown to be fulfilled very well for the eigenstates obtained. The energies obtained here are in good agreement with those obtained by Lüscher and Münster [7] and by Koller and van Baal [8], using gauge invariant wave functions as trial states. In particular I confirm that the first spin-2 state is lower in energy than the first spin-0 excitation. Furthermore I have found that practically all excitation energy, independently whether it is due to a vibrational or a rotational excitation, goes into the increase of the field strength along one of the three principal axis fields, $\langle \phi_3 \rangle$, whereas $\langle \phi_1 \rangle$ and $\langle \phi_2 \rangle$, and also the component $\langle B_3 \rangle$ of the magnetic field along the intrinsic 3-direction, remain unchanged at their vacuum values.

Several techniques in my treatment, such as the separation of the harmonic oscillator problem into a density and a deformation problem and the diagonalization of the deformation operator in terms of symmetric polynomials, have for the case of spin-0 to some extent already been developed in the context of the Calogero model [5] and then later also in the context of Yang-Mills quantum mechanics in [8]. In difference to [8], the range of integration (9) here is positive definite, and the corresponding range and measure (31) in terms of the elementary symmetric variables is derived explicitly from the original measure (12). In particular the relative normalization of the deformation states (43) is different to that given in [8]. Furthermore the generalization of the use of the elementary symmetric variables to spin-2 and the consideration of permutation non-invariant expectation values here is new to the best of my knowledge.

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