

ON THE DUISTERMAAT-HECKMAN FORMULA  
AND INTEGRABLE MODELS<sup>†</sup>

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In this article we review the Duistermaat-Heckman integration formula and the ensuing equivariant cohomology structure, in the finite dimensional case. In particular, we discuss the connection between equivariant cohomology and classical integrability. We also explain how the integration formula is derived, and explore some possible new directions that could eventually yield novel integration formulas for nontrivial integrable models.

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## 1. Introduction

The Duistermaat-Heckman integration formula [1],[2] and its generalizations to path integrals have been discussed extensively both in the Mathematics [3]-[5] and in the Physics [6]-[7] literature. In this article, we shall review the original integration formula and show how it can be derived. In particular we shall discuss the equivariant cohomology structure which underlies the integration formula, explore connections between equivariant cohomology and classical integrability, and investigate how the known integration formulas could be generalized.

We shall limit ourselves to finite dimensional integrals, and we refer to the original articles [5,6] where path integral generalizations have been discussed: The integration formulas for path integrals can be obtained by generalizing the present construction to the loop space. Even though there are some genuine loop space intricacies that are absent in finite dimensions, the insight gained in finite dimensions is in any case quite applicable also in the loop space.

## 2. Equivariance, Integrable Models And Localization

We shall consider a  $2n$  dimensional compact phase space  $M$ , with local coordinates  $z^i$  and Poisson bracket

$$\{z^i, z^j\} = \omega^{ij}(z) \quad (1)$$

Here  $\omega^{ij}$  is the inverse matrix to the symplectic two-form on  $M$ ,

$$\omega = \frac{1}{2}\omega_{ij}dz^i dz^j \quad (2)$$

This is closed,

$$d\omega = 0 \quad (3)$$

so that locally we can introduce the one-form  $\vartheta$  called the symplectic potential such that

$$\omega = d\vartheta \quad (4)$$

We are interested in the exact evaluation of the "classical" partition function

$$\mathcal{Z} = \int \omega^n e^{-\beta H} \quad (5)$$

where  $H$  is some hamiltonian on  $M$ , and  $\beta$  is a real parameter (inverse temperature in classical statistical physics). The integration formula by Duistermaat and Heckman states, that if  $H$  determines the symplectic action of  $U(1)$  on the phase space the integral (5) localizes to the critical points of  $H$ ,

$$\mathcal{Z} = \frac{1}{\beta^n} \sum_{dH=0} \frac{\sqrt{\det|\omega_{ij}|}}{\sqrt{\det|\partial_{ij}H|}} \exp\{-\beta H\} \quad (6)$$

In this article we shall explain how (6) is derived. We shall also discuss some possible approaches to generalize this integration formula to a wider class of Hamiltonians.

The integration formula (6) is derived using equivariant cohomology on the phase space  $M$ . For this we consider the exterior algebra  $\Omega(M)$  of  $M$  and introduce the contraction operator  $i_{\mathcal{X}}$  with respect to a general vector field  $\mathcal{X}$ . It is a nilpotent operator on the exterior algebra  $\Omega(M)$ . We also introduce the equivariant exterior derivative

$$d_{\mathcal{X}} = d + \phi i_{\mathcal{X}} \quad (7)$$

where  $\phi$  is a real parameter, and the Lie derivative along  $\mathcal{X}$

$$\mathcal{L}_{\mathcal{X}} = d_{\mathcal{X}}^2 = \phi(di_{\mathcal{X}} + i_{\mathcal{X}}d) \quad (8)$$

On the subcomplex  $\Omega_{\mathcal{X}}$  of  $\mathcal{X}$ -invariant exterior forms

$$\mathcal{L}_{\mathcal{X}}\Omega_{\mathcal{X}} = 0 \quad (9)$$

the exterior derivative (7) is then nilpotent and defines the  $\mathcal{X}$ -equivariant subcomplex  $\Omega_{\mathcal{X}}$  of  $\Omega(M)$ . The corresponding cohomology determines the equivariant cohomology on  $M$ .

We shall assume that the action of  $\mathcal{X}$  on  $M$  is symplectic,

$$\mathcal{L}_{\mathcal{X}}\omega = di_{\mathcal{X}}\omega = 0 \quad (10)$$

Provided the one-form  $i_{\mathcal{X}}\omega$  is exact (for this the triviality of  $H^1(M, R)$  is sufficient), we can introduce the corresponding Hamiltonian  $H(z)$

$$i_{\mathcal{X}}\omega = -dH \quad (11)$$

In local coordinates  $z^i$  on  $M$  this becomes

$$\mathcal{X} = \omega^{ab}\partial_a H \partial_b \quad (12)$$

In order to use this formalism to derive the integration formula (6) and its generalizations, we realize the various operations on the exterior algebra  $\Omega(M)$  canonically. For this we introduce a canonical conjugate variable  $p_i$  for  $z^i$ , identify  $dz^i$  with anti-commuting  $c^i$  and the contraction operator on  $c^i$  with  $\bar{c}_i$ , with Poisson brackets

$$\{p_i, z^j\} = \delta_i^j \quad (13)$$

$$\{\bar{c}_i, c^j\} = \delta_i^j \quad (14)$$

In terms of these variables the exterior derivative, contraction and Lie derivative can be realized by the Poisson bracket actions of

$$d = p_i c^i \quad (15)$$

$$i_H = \mathcal{X}^i \bar{c}_i \quad (16)$$

$$\mathcal{L}_H = \mathcal{X}^i p_i + c^i \partial_i \mathcal{X}^j \bar{c}_j \quad (17)$$

Since

$$d_H(\phi H + \omega) = \phi(dH + i_H \omega) = 0 \quad (18)$$

by (11), we conclude that  $\phi H + \omega$  is an element of  $H^*(M)$  and determines an equivariant cohomology class. This is an equivalence class consisting of elements in  $\Omega(M)$  which are linear combinations of zero- and two-forms that can be represented as

$$\phi H + \omega + d_H \psi \quad (19)$$

where  $\psi \in \Omega(M)$  satisfies

$$\mathcal{L}_H \psi = 0 \quad (20)$$

and is a linear combination of the form

$$\psi = \psi_0 + \psi_1 + \psi_2 + \dots + \psi_{2n} \quad (21)$$

where  $\psi_k$  is a  $k$ -form on  $\Omega(M)$ . In particular, due to linearity of  $\mathcal{L}_H$  these  $k$ -forms also satisfy

$$\mathcal{L}_H \psi_k = 0 \quad (22)$$

Suppose now that there exists a  $\psi$  which satisfies (20) and in addition  $d_H \psi$  is a linear combination of a zero- and two-form. Denoting the zero-form by  $K$  and the

two-form by  $\Omega$ , we then have the relations ( $\phi = 1$  for simpler notation)

$$i_H\psi_1 = K \quad (23)$$

$$d\psi_0 + i_H\psi_2 = 0 \quad (24)$$

$$d\psi_1 + i_H\psi_3 = \Omega \quad (25)$$

$$d\psi_2 + i_H\psi_4 = 0 \quad (26)$$

*etc...*

In particular, since

$$di_H\psi_3 = -i_Hd\psi_3 = i_Hi_H\psi_5 = 0$$

by (22), we conclude that  $\Omega$  is a *closed* two-form (but not necessarily nondegenerate).

Furthermore, since

$$dK = di_H\psi_1 = -i_Hd\psi_1 = -i_H\Omega \quad (27)$$

we get

$$\Omega^{ab}\partial_b K = \omega^{ab}\partial_b H \quad (28)$$

so that the classical equations of motion for the two Hamiltonian systems  $(H, \omega)$  and  $(K, \Omega)$  coincide. As a consequence these two systems determine a *bi-Hamiltonian structure*. (Here we assume that  $\Omega_{ab}$  is nondegenerate on  $M$  except possibly on submanifolds of  $M$  which have a co-dimension larger than or equal to two. On these submanifolds, the Hamiltonian  $K$  must then vanish to keep the equations of motion non-singular.)

On the other hand, if we have a bi-Hamiltonian structure so that (28) holds we get

$$d_H(\phi K + \Omega) = 0 \quad (29)$$

that is,  $\phi K + \Omega$  is equivariantly closed with respect to  $d_H$ . We can then apply an equivariant form of Poincaré's lemma to conclude that there exist a (locally defined) form  $\psi$  on  $\Omega(M)$  such that

$$\phi K + \Omega = d_H\psi \quad (30)$$

Indeed, for an *integrable* model this can be easily solved at least locally: If  $(H, \omega)$  and  $(K, \Omega)$  is an integrable bi-Hamiltonian pair, we can introduce action-angle variables  $(I_i, \theta_i)$  (almost) everywhere on  $M$  such that both  $H$  and  $K$  depend only on the action variables,  $H = H(I)$  and  $K = K(I)$ . We assume that these variables are selected so that  $\Omega$  admits the Darboux form

$$\Omega = \sum_i dI_i \wedge d\theta_i \quad (31)$$

while  $\omega_{ab}$  remains a nontrivial function of  $(I_i, \theta_i)$ . The corresponding symplectic potential for  $\Omega$  is

$$\vartheta = \sum_i I_i d\theta_i + dF \quad (32)$$

where  $F$  is some function.

In the complement of the critical point set of  $K$  we have an action variable  $I_k$  such that

$$\frac{\partial K(I)}{\partial I_k} \neq 0$$

With  $\theta_k$  the corresponding angle variable, we introduce the following function on  $M$

$$W(I, \theta) = \theta_k \cdot \left( \frac{\partial K}{\partial I_k} \right)^{-1} \quad (33)$$

We then consider (30) ( $\phi = 1$  here for simplicity):

$$K + \Omega = (d + i_H)(\vartheta + dF) = \omega^{ab} \partial_b H \vartheta_a + \omega^{ab} \partial_b H \partial_a F + \Omega \quad (34)$$

Using ( ), we get further

$$= \sum_i \frac{\partial K}{\partial I_i} I_i + \{K, F\} + \sum_i dI_i \wedge d\theta_i \quad (35)$$

Hence, if we select

$$F(I, \theta) = W \cdot (K - \sum_i \frac{\partial K}{\partial I_i} I_i) + G(I) \quad (36)$$

with  $G(I)$  an arbitrary function of the action variables, we conclude that the one-form  $\psi = \vartheta + dF$  satisfies (30),

$$K + \Omega = d_H(\vartheta + dF) \quad (37)$$

The global existence of such a form  $\psi$  is connected to the nontriviality of the equivariant cohomology associated with  $d_H$ .

The preceding discussion suggests an intimate relationship between equivariant cohomology and the existence of bi-Hamiltonian structures, which deserves further investigation.

We remind, that if the symplectic two-forms  $\omega$  and  $\Omega$  are such that the following rank-(1,1) tensor is *nontrivial*,

$$L = \Omega \omega^{-1} = \Omega_{ac} \omega^{cb} c^a p_b \quad (38)$$

we can establish the integrability of  $(H, \omega)$  under a certain condition: From (26),

$$\mathcal{L}_H \Omega = (di_H + i_H d)d\psi = di_H d\psi \quad (39)$$

and since  $\psi$  satisfies (20), we conclude that

$$\mathcal{L}_H \Omega = 0 \quad (40)$$

Using (10), we then get

$$\mathcal{L}_H L = \mathcal{L}_H(\Omega\omega^{-1}) = (\mathcal{L}_H \Omega)\omega^{-1} + \Omega(\mathcal{L}_H \omega^{-1}) = 0 \quad (41)$$

or in components

$$\mathcal{X}_H^c \partial_c L_a^b + L_c^a \partial_a \mathcal{X}_H^c - L_a^c \partial_c \mathcal{X}_H^b = 0 \quad (42)$$

This we can write as

$$\frac{dL_a^b}{dt} = L_a^c (\partial_c \mathcal{X}_H^b) - (\partial_a \mathcal{X}_H^c) L_c^b \quad (43)$$

or

$$\frac{dL}{dt} = [L, U] \quad (44)$$

with  $U_a^b = \partial_a \mathcal{X}_H^b$ . This is the Lax equation and  $L, U$  is the Lax pair.

In order to prove integrability, we introduce

$$Q_k = \frac{1}{k} \text{Tr} L^k \quad (45)$$

and define the Nijenhuis tensor  $N$ , with components

$$N_{ab}^c = L_a^d \partial_d L_b^c - L_b^d \partial_d L_a^c - L_d^c (\partial_a L_b^d - \partial_b L_a^d) \quad (46)$$

Then,

$$\begin{aligned} N_{ab}^c (L^{n-1})_c^b &= (L_a^d \partial_d L_b^c - L_b^d \partial_d L_a^c - L_d^c [\partial_a L_b^d - \partial_b L_a^d]) (L^{n-1})_c^b \\ &= L_a^d (\partial_d L_b^c) (L^{n-1})_c^b - (L^n)_c^d (\partial_d L_a^c) - (L^n)_d^b (\partial_a L_b^d - \partial_b L_a^d) \\ &= L_a^d (\partial_d L_b^c) (L^{n-1})_c^b - (L^n)_d^b \partial_a L_b^d \end{aligned} \quad (47)$$

On the other hand,

$$\partial_d \text{Tr} L^n = \sum_m \text{Tr} (L \dots \partial_d L^{(m)} \dots L) = n \text{Tr} (\partial_d L) L^{n-1} = n (\partial_d L_b^c) (L^{n-1})_c^b$$

so that using

$$\partial_d Q_n = (\partial_d L_b^c) (L^{n-1})_c^b$$

we conclude that

$$N_{ab}^c(L^{n-1})_c^b = L_a^d \partial_d Q_n - \partial_a Q_{n+1} \quad (48)$$

As a consequence, *if the Nijenhuis tensor vanishes* we obtain the recursion relation

$$L_a^d \partial_d Q_n = \partial_a Q_{n+1} \quad (49)$$

which we can also write as

$$\omega^{ab} \partial_b Q_n = \Omega^{ab} Q_{n+1} \quad (50)$$

Furthermore, since

$$\{Q_i, Q_j\}_\omega = \omega^{ab} \partial_a Q_i \partial_b Q_j = \Omega^{ab} \partial_a Q_i \partial_b Q_{j+1} = \omega^{ab} \partial_a Q_{i-1} \partial_b Q_{j+1} = \{Q_{i-1}, Q_{j+1}\}_\omega$$

and assuming  $i > j$  and iterating  $i - j$  times, we get

$$\{Q_i, Q_j\}_\omega = \{Q_j, Q_i\}_\omega$$

Hence

$$\{Q_i, Q_j\}_\omega = 0 \quad \text{for all } i, j \quad (51)$$

and we have constructed  $n$  quantities in involution. Furthermore, since

$$Tr\{L^k \frac{dL}{dt}\} = Tr\{L^k [L, U]\} = 0 \quad (52)$$

we conclude that these quantities are conserved *i.e.* commute with the Hamiltonian  $H$ . This establishes the integrability of the Hamiltonian system  $(H, \omega)$ , provided (45) are complete *i.e.* the number of functionally independent (45) coincides with half the phase space dimension. However, in general there is no guarantee that these integrals are complete, and this completeness must be established independently.

Consider now the integral

$$\mathcal{Z} = \phi^n \int \omega^n e^{-\phi H} \quad (53)$$

for some Hamiltonian  $H$  that admits a bi-Hamiltonian structure. If we introduce anticommuting variables  $c^a$  we can write  $\mathcal{Z}$  as

$$\mathcal{Z} = (-)^n n! \phi^n \int dz^i \sqrt{\det|\omega_{ij}|} e^{-\phi H} = (-)^n n! \phi^n \int dz^i dc^i e^{-\phi H - \omega} \quad (54)$$

We assume that  $\psi$  is a one-form such that

$$\mathcal{L}_H \psi = 0 \quad (55)$$



With  $\lambda$  a real parameter we then argue that the following one-parameter family of integrals

$$\mathcal{Z}_\lambda = \int dz^i dc^i \exp\{-\phi H - \omega - \lambda d_H \psi\} \quad (56)$$

does not depend on  $\lambda$ . This implies in particular, that the integral (53) *only* depends on the equivalence class determined by  $(H, \omega)$  in the  $d_H$  equivariant cohomology.

In order to prove this  $\lambda$ -independence, we consider an infinitesimal variation  $\lambda \rightarrow \lambda + \delta\lambda$ , and show that

$$\mathcal{Z}_\lambda = \mathcal{Z}_{\lambda+\delta\lambda} \quad (57)$$

For this, we introduce the following infinitesimal change of variables in (56):

$$z^i \rightarrow \tilde{z}^i = z^i + \delta z^i = z^i + \delta\psi \cdot d_H z^i = z^i + \delta\psi c^i \quad (58)$$

$$c^i \rightarrow \tilde{c}^i = c^i + \delta c^i = c^i + \delta\psi \cdot d_H c^i = c^i - \delta\psi \mathcal{X}^i \quad (59)$$

with

$$\delta\psi = \delta\lambda \cdot \psi \quad (60)$$

Since  $\psi$  satisfies (55), the exponential in (56) is invariant under the change of variables (59). However, the Jacobian is nontrivial:

$$\begin{aligned} dz^i d\tilde{c}^i &= Sdet \left( \begin{array}{cc} \frac{\partial \tilde{z}^i}{\partial z^i} & \tilde{z}^i \frac{\overleftarrow{\partial}}{\partial c^i} \\ \frac{\partial \tilde{c}^i}{\partial z^i} & \frac{\partial \tilde{c}^i}{\partial c^i} \end{array} \right) dz^i dc^i \\ &= Sdet \left( \begin{array}{cc} 1 + \frac{\partial \delta\psi}{\partial z^i} c^i & \star \\ \star & 1 - \frac{\partial \delta\psi}{\partial c^i} \mathcal{X}^i \end{array} \right) dz^i dc^i \\ &= \left( 1 + Str \left( \begin{array}{cc} \frac{\partial \delta\psi}{\partial z^i} c^i & \star \\ \star & -\frac{\partial \delta\psi}{\partial c^i} \mathcal{X}^i \end{array} \right) \right) dz^i dc^i \\ &= \left( 1 + \frac{\partial \delta\psi}{\partial z^i} c^i + \frac{\partial \delta\psi}{\partial c^i} \mathcal{X}^i \right) dz^i dc^i \\ &= \left( 1 - \left( c^i \frac{\partial}{\partial z^i} - \mathcal{X}^i \frac{\partial}{\partial c^i} \right) \delta\psi \right) dz^i dc^i \end{aligned}$$

$$= (1 - d_H \delta \psi) dz^i dc^i \sim \exp\{-d_H(\delta \psi)\} dz^i dc^i = \exp\{-\delta \lambda d_H \psi\}$$

Hence

$$\begin{aligned} \mathcal{Z} &= \int dz^i dc^i \exp\{-\phi H + \omega - \lambda d_H \psi - d_H(\delta \psi)\} \\ &= \int dz^i dc^i \exp\{-\phi H + \omega - (\lambda + \delta \lambda) d_H \psi\} = Z_{\lambda + \delta \lambda} \end{aligned} \quad (61)$$

and we have established that if the Hamiltonian system  $(H, \omega)$  admits a bi-Hamiltonian structure the classical partition function (53) depends *only* on the equivalence class that  $(H, \omega)$  determines in the  $d_H$  equivariant cohomology.

### 3. Duistermaat-Heckman Integration Formula

In order to construct examples we consider a compact Lie group  $G$  that acts on  $M$  by local diffeomorphisms which are generated by vector fields  $\mathcal{X}_a$ ,  $a = 1, \dots, m$ . Their commutation relations defines a representation of the Lie algebra  $\hat{\mathfrak{g}}$  of  $G$ ,

$$[\mathcal{X}_a, \mathcal{X}_b] = f_{abc} \mathcal{X}_c \quad (62)$$

where  $f_{abc}$  are the structure constants of  $\hat{\mathfrak{g}}$ .

We denote contraction *w.r.t.* the Lie algebra basis  $\{\mathcal{X}_a\}$  by  $i_a$ . The corresponding Lie derivatives

$$\mathcal{L}_a = di_a + i_a d \quad (63)$$

then generate the action of  $G$  on the exterior algebra  $\Omega(M)$  of  $M$ ,

$$[\mathcal{L}_a, \mathcal{L}_b] = f_{abc} \mathcal{L}_c \quad (64)$$

We assume that the action of  $G$  on  $M$  is symplectic,

$$\mathcal{L}_a \omega = di_a \omega = 0 \quad \text{for all } a \quad (65)$$

so that we can define the momentum map

$$H_G : M \mapsto \hat{\mathfrak{g}}^* \quad (66)$$

which gives a one-to-one correspondence between the vector fields  $\mathcal{X}_a$  and the components  $T_a$  of the momentum map,

$$H_G = \phi^a T_a \quad (67)$$

where  $\{\phi^a\}$  is a (symmetric) basis of  $\hat{\mathfrak{g}}$ . From the Jacobi identity for  $\hat{\mathfrak{g}}$  we then get the homomorphism

$$[\mathcal{X}_a, \mathcal{X}_b] = f_{abc} \mathcal{X}_c = \mathcal{X}_{\{T_a, T_b\}} \quad (68)$$

However, in general the Hamiltonian corresponding to the commutator of two generators may differ from the Poisson bracket of the corresponding Hamiltonians by a two-cocycle,

$$\{T_a, T_b\} = f_{abc} T_c + \kappa_{ab} \quad (69)$$

but here we shall assume that  $\kappa_{ab} = 0$ .

We again introduce the canonical realization (15)-(17) of the various operations.

The simplest example of the present construction is the canonical action of the circle  $G = U(1) \sim S^1$ . This action is generated by a vector field  $\mathcal{X}$ , the generator of the Lie-algebra  $u(1)$  of  $U(1)$ . The corresponding momentum map (67) is

$$H_{U(1)} = \phi H \quad (70)$$

and

$$\mathcal{X}^a = \omega^{ab} \partial_b H \quad (71)$$

and  $\phi$  is the generator of the dual basis of  $u(1)$ , a real parameter.

We wish to derive the integration formula (6) for the integral

$$\mathcal{Z} = \phi^n \int \omega^n e^{-\phi H} \quad (72)$$

with  $H$  the hamiltonian in (70). For this we introduce the corresponding equivariant exterior derivative

$$d_H = d + \phi i_H \quad (73)$$

and the Lie-derivative with respect to  $\mathcal{X}$  is

$$d_H^2 = \phi(di_H + i_H d) = \phi \mathcal{L}_H .$$

so that on the subcomplex  $\Omega_{U(1)}$  of  $U(1)$ -invariant exterior forms,  $d_H$  is nilpotent and defines an exterior differential operator. The pertinent cohomology coincides with the equivariant cohomology  $H_{U(1)}^*(M)$  of the manifold  $M$ .

In order to derive (6), we first need a one-form  $\psi$  that we can use in (56), to localize it onto (6). For this we first observe that since the group  $G \sim U(1)$  is compact, we may construct a metric tensor  $g_{ab}$  on  $M$  for which the canonical flow of  $H$  is an isometry,

$$\mathcal{L}_H g = 0 \quad (74)$$

Such a metric is obtained by first selecting an arbitrary Riemannian metric  $\tilde{g}$  on  $M$ , and averaging it over the group  $G$  using its Haar measure. A converse is also true: Since  $M$  is compact the isometry group of  $g$  must also be compact.

We select

$$\psi = i_H g = g_{ij} \mathcal{X}^i c^j \quad (75)$$

where  $g$  is the Riemannian metric (74). As a consequence,

$$\mathcal{L}_H \psi = 0 \quad (76)$$

and we obtain the bi-Hamiltonian structure with

$$K = g_{ij} \mathcal{X}^i \mathcal{X}^j \quad (77)$$

$$\Omega_{ij} = \partial_i (g_{jk} \mathcal{X}^k) - \partial_j (g_{ik} \mathcal{X}^k) \quad (78)$$

and the integral

$$\mathcal{Z} = \phi^n \int dz^i dc^i \exp\{-\phi H - \omega + \lambda(K + \Omega)\} \quad (79)$$

is independent of  $\lambda$ .

Explicitly,

$$\mathcal{Z} = \phi^n \int dz^i dc^i \exp\{-\phi H - \omega - \lambda g_{ij} \mathcal{X}^i \mathcal{X}^j - \frac{1}{2} \lambda \cdot \Omega_{ij} c^i c^j\} \quad (80)$$

In the  $\lambda \rightarrow \infty$  we can then use

$$\delta(z^i) = \lim_{\lambda \rightarrow \infty} \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \sqrt{\det \|S_{ij}\|} \cdot e^{-\frac{\lambda}{2} z^i S_{ij} z^j} \quad (81)$$

$$\delta(c^i) = \lim_{\lambda \rightarrow \infty} \lambda^{-\frac{n}{2}} \frac{1}{\sqrt{\det \|A_{ij}\|}} \cdot e^{\frac{\lambda}{2} c^i A_{ij} c^j}. \quad (82)$$

for a symmetric ( $S$ ) and antisymmetric ( $A$ ) matrix respectively, to localize (80) onto (6):

$$\begin{aligned}
\mathcal{Z} &= \int dz^i dc^i \frac{\sqrt{\det \|\Omega_{ij}\|}}{\sqrt{\det \|g_{ij}\|}} \delta(\mathcal{X}) \delta(c) e^{-\phi H - \omega} = \int dz^i \frac{\sqrt{\det \|\Omega_{ij}\|}}{\sqrt{\det \|g_{ij}\|}} \delta(\mathcal{X}) e^{-\phi H} \\
&= \int d\mathcal{X}^i \det \left\| \left( \frac{\partial z^i}{\partial \mathcal{X}^j} \right) \right\| \cdot \frac{\sqrt{\det \|\Omega_{ij}\|}}{\sqrt{\det \|g_{ij}\|}} \delta(\mathcal{X}) e^{-\phi H} \\
&= \sum_{dH=0} \frac{1}{\det \|\partial_i \mathcal{X}^j\|} \frac{\sqrt{\det \|\Omega_{ij}\|}}{\sqrt{\det \|g_{ij}\|}} e^{-\phi H}
\end{aligned}$$

Since  $dH = 0$ , we have  $\mathcal{X} = 0$  and

$$\Omega_{ij} = \partial_i(g_{jk} \mathcal{X}^k) - \partial_j(g_{ik} \mathcal{X}^k) = g_{jk} \partial_i \mathcal{X}^k - g_{ik} \partial_j \mathcal{X}^k$$

On the other hand, in terms of its components  $\mathcal{L}_{HG} = 0$  becomes

$$\mathcal{X}^k \partial_k g_{ij} + g_{ik} \partial_j \mathcal{X}^k + g_{jk} \partial_i \mathcal{X}^k = 0$$

As a consequence,

$$g_{ik} \partial_j \mathcal{X}^k + g_{jk} \partial_i \mathcal{X}^k = 0$$

that is

$$\Omega_{ij} = 2g_{jk} \partial_i \mathcal{X}^k$$

Consequently we get

$$\begin{aligned}
\mathcal{Z} &= \sum_{dH=0} \frac{1}{\det \|\partial_i \mathcal{X}^j\|} \cdot \frac{\sqrt{\det \|g_{ij}\|} \cdot \sqrt{\det \|\partial_i \mathcal{X}^k\|}}{\sqrt{\det \|g_{ij}\|}} \exp\{-\phi H\} \\
&= \sum_{dH=0} \frac{e^{-\phi H}}{\sqrt{\det \|\partial_i \mathcal{X}^j\|}} \tag{83}
\end{aligned}$$

Since

$$\partial_i \mathcal{X}^j = \omega^{jk} \partial_i \partial_k H$$

we get finally get the Duistermaat-Heckman integration formula (6),

$$\mathcal{Z} = \sum_{dH=0} \frac{\sqrt{\det \|\omega_{ij}\|}}{\sqrt{\det \|\partial_{ij} H\|}} \exp\{-\phi H\} \tag{84}$$

for a Hamiltonian  $H$  that generates the action of  $U(1)$  on the phase space.

#### 4. Generalizations:

Our discussion in section 2. suggests, that the localization techniques to evaluate integrals of the form (5) could be applied to quite general integrable Hamiltonians: The existence of a functional  $\psi$  which is Lie derived by the Hamiltonian seems to be generally connected to the concept of integrability. Since the integral (5) depends on the equivariant cohomology class determined by  $\phi H + \omega$  rather than its given representative, it is natural to expect that (6) is just an example of a much more general phenomenon. At the moment this is not yet well understood, and thus we shall here only discuss a few possible ways to generalize the integration formula (6).

First, we shall consider the general case of a compact nonabelian Lie group  $G$  which acts on  $M$ , and we are interested in a Hamiltonian  $H_2$  which is a quadratic Casimir for the nonabelian Lie group generators  $T_a$ ,

$$\mathcal{C}_2 = \eta^{ab} T_a T_b \quad (85)$$

$$\{T_a, \mathcal{C}_2\} = 0 \quad \text{for all } a \quad (86)$$

Here  $\eta^{ab}$  is positive definite and nondegenerate. We consider the path integral

$$\mathcal{Z} = \phi^n \int dz^i dc^i \exp\{-\phi \eta^{ab} T_a T_b - \omega\} \quad (87)$$

We multiply this by

$$1 = \left(\frac{\phi}{\pi}\right)^{\frac{n}{2}} \sqrt{\det ||\eta^{ab}||} \int_{-\infty}^{\infty} dq_a \exp\{-\phi \eta^{ab} q_a q_b\} \quad (88)$$

which yields

$$\mathcal{Z} = \left(\frac{\phi}{\pi}\right)^{\frac{n}{2}} \phi^n \sqrt{\det ||\eta^{ab}||} \int_{-\infty}^{\infty} dq_a \exp\{-\phi \eta^{ab} q_a q_b\} \int dz^i dc^i \exp\{-2\phi \eta^{ab} q_a T_b - \omega\} \quad (89)$$

Denoting

$$H_q = 2\eta^{ab} q_a T_b$$

which we can identify as the momentum map (67) with  $\phi^a = 2\eta^{ab}q_b$ , we then conclude that the second integral in (89) is of the form (54). Consequently (89) localizes to

$$\mathcal{Z} = \left(\frac{\phi}{\pi}\right)^{\frac{n}{2}} \sqrt{\det \|\eta^{ab}\|} \int_{-\infty}^{\infty} dq_a \exp\{-\phi\eta^{ab}q_aq_b\} \sum_{dH_q=0} \frac{\sqrt{\det \|\omega_{ij}\|}}{\sqrt{\det \|\partial_{ij}H_q\|}} \exp\{-\phi H_q\} \quad (90)$$

Generalizations to higher order Casimirs, and for more general functionals  $H[T_a]$  of the Lie algebra generators can also be considered [5], [6].

Next, we first consider a (not necessarily integrable) Hamiltonian  $H$  with a number of conserved quantities  $Q_\alpha$ ,

$$\{H, Q_\alpha\} = 0 \quad (91)$$

Using these conserved quantities, we introduce the following (*degenerate!*) symmetric matrix

$$G_{ij}(Q) = \sum_{\alpha} \partial_i Q_\alpha \partial_j Q_\alpha \quad (92)$$

We then consider the Lie derivative  $\mathcal{L}_H G$ . In components, this gives

$$\partial_i(\{H, Q_\alpha\})\partial_j Q_\alpha + \partial_j(\{H, Q_\alpha\})\partial_i Q_\alpha = 0 \quad (93)$$

as a consequence of (91), so that the matrix (92) satisfies the Lie derivative condition (74). However, for the same reason we find that the corresponding one-form (75)

$$\psi_Q = G_{ij} \mathcal{X}^i c^j = \omega^{ik} \partial_k H \partial_i Q_\alpha \partial_j Q_\alpha c^j = \{H, Q_\alpha\} dQ_\alpha = 0 \quad (94)$$

and consequently we do not get a localization formula.

In the loop space (92) can be used to derive localization formulas [6], and it would be interesting to see if a proper variant of the present could be used to derive a partial localization also for the pertinent integral (5). With  $Q_\alpha$  the full set of conserved quantities for an integrable model  $(H, \omega)$  this could then yield a localization formula for a quite general integrable system.

We shall conclude this article by investigating properties of the following [8] alternative geometrical condition to the Lie derivative condition (74): We shall assume that instead of (74) we have a metric tensor which satisfies

$$\nabla_{\mathcal{X}_H} \mathcal{X}_H = 0 \quad (95)$$

or in components

$$\mathcal{X}^l \partial_l \mathcal{X}^k + \Gamma_{ij}^k \mathcal{X}^i \mathcal{X}^j = 0 \quad (96)$$

for the Hamiltonian vector field

$$\mathcal{X}_H^i = \omega^{ij} \partial_j H \quad (97)$$

This condition means, that the Hamiltonian flow of  $H$  is geodetic to  $g_{ij}$ .

We introduce the following Hamiltonian

$$K = \frac{1}{2} g_{ij} \mathcal{X}_H^i \mathcal{X}_H^j \quad (98)$$

and the following symplectic two-form

$$\Omega_{ij} = \partial_i (g_{jk} \mathcal{X}^k) - \partial_j (g_{ik} \mathcal{X}^k) \quad (99)$$

and argue, that  $(H, \omega)$  and  $(K, \Omega)$  determines a bi-hamiltonian pair, *i.e.*

$$\partial_i K = \Omega_{ij} \mathcal{X}_H^j \quad (100)$$

In order to establish this we consider the *r.h.s.* of (100),

$$\begin{aligned} \Omega_{ij} \mathcal{X}^j &= [\partial_i (g_{jk} \mathcal{X}^k) - \partial_j (g_{ik} \mathcal{X}^k)] \mathcal{X}^j \\ &= (\partial_i g_{ij}) \mathcal{X}^j \mathcal{X}^k + g_{jk} (\partial_i \mathcal{X}^k) \mathcal{X}^j - (\partial_j g_{ik}) \mathcal{X}^k \mathcal{X}^j - g_{ik} (\partial_j \mathcal{X}^k) \mathcal{X}^j \end{aligned} \quad (101)$$

We then use the component form (100), *i.e.*

$$\mathcal{X}^l (\partial_l \mathcal{X}^k) + \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \mathcal{X}^i \mathcal{X}^j = 0$$

from which we get

$$g_{kl} \mathcal{X}^j \partial_j \mathcal{X}^l = -(\partial_i g_{jk}) \mathcal{X}^i \mathcal{X}^j + \frac{1}{2} (\partial_k g_{ij}) \mathcal{X}^i \mathcal{X}^j$$

Substituting this into then last term in (101), we then get

$$\begin{aligned} &= (\partial_i g_{jk}) \mathcal{X}^j \mathcal{X}^k + g_{jk} (\partial_i \mathcal{X}^k) \mathcal{X}^j - (\partial_j g_{ik}) \mathcal{X}^k \mathcal{X}^j + (\partial_k g_{ij}) \mathcal{X}^k \mathcal{X}^j - \frac{1}{2} (\partial_i g_{jk}) \mathcal{X}^k \mathcal{X}^j \\ &= \frac{1}{2} (\partial_i g_{jk}) \mathcal{X}^j \mathcal{X}^k + g_{jk} (\partial_i \mathcal{X}^k) \mathcal{X}^j = \frac{1}{2} (\partial_i g_{jk}) \mathcal{X}^j \mathcal{X}^k + \frac{1}{2} g_{jk} (\partial_i \mathcal{X}^j) \mathcal{X}^k + \frac{1}{2} g_{jk} \mathcal{X}^j \partial_i \mathcal{X}^k \\ &= \partial_i \left( \frac{1}{2} g_{jk} \mathcal{X}^j \mathcal{X}^k \right) \end{aligned} \quad (102)$$



This coincides with the *l.h.s.* of (100), and establishes the bi-hamiltonian structure. In fact, we have here shown that

$$d_H(K + \Omega) = 0 \Leftrightarrow \nabla_{\mathcal{X}_H} \mathcal{X}_H = 0 \quad (103)$$

Unfortunately, we also find that the symplectic two-form  $\Omega$  is *degenerate* on a compact phase space [9]: Consider the  $\Omega$  - volume form

$$\Omega^n = \sqrt{\det|\Omega_{ij}|} dz^1 \wedge \dots \wedge dz^{2n} \quad (104)$$

Since

$$\Omega = d(i_H g) \quad (105)$$

globally on  $M$  (in obvious notation), Stokes theorem yields

$$\int_M \Omega^n = \int_M d(i_H g \wedge \Omega^{n-1}) = 0 \quad (106)$$

which implies that  $\det|\Omega_{ij}| = 0$ . However, for integrability it appears to be sufficient, that degeneracies of  $\Omega$  which occur only on submanifolds of  $M$  with co-dimensions two or more are not necessarily fatal, provided the Hamiltonian vanish at these submanifolds to keep the equations of motion non-singular.

Finally, we argue that  $K + \Omega$  is equivariantly exact, *i.e.* that there exist a one-form  $\psi$  such that

$$d_H \psi = K + \Omega \quad (107)$$

For this, we first observe that if we denote

$$\eta = i_H g = g_{ab} \mathcal{X}_H^a c^b$$

we get

$$d_H \eta = 2K + \Omega \quad (108)$$

If we then subtract (107) from (108) and define

$$\theta = \eta - \psi$$

we get

$$d_H \theta = d_H(\eta - \psi) = K \quad (109)$$

that is

$$d\theta = 0 \quad (110)$$

and

$$i_H\theta = K \tag{111}$$

If we then assume that  $\theta$  is exact so that we can write  $\theta = dF$  for some function  $F$  (by Poincaré's lemma such a  $F$  exists *at least* locally), we obtain

$$i_H dF = \mathcal{X}_H^a \partial_a F = K \tag{112}$$

so that

$$\psi = i_H g - dF \tag{113}$$

is the desired one-form that satisfies (107). Consequently we have established that the condition (103) yields relations analogous to those that yield the Duistermaat-Heckman formula, and it would be interesting to see how (103) could be used to derive new localization formulas.

## 5. Conclusions

In conclusion, we have investigated the relations between equivariant cohomology and classical integrability. In particular, we have explained in detail how the localization formulas for the classical partition function (5) are derived from the formalism of equivariant cohomology, and as an example we have derived the Duistermaat-Heckman integration formula in detail. We have also discussed some generalizations of the equivariance structure which underlies the Duistermaat-Heckman integration formula, and it would be interesting to see if these generalizations yield new integration formulas. In particular, it would be very interesting to understand fully the relation between integrability and equivariant cohomology, and whether localization techniques could in fact be extended to evaluate the partition functions for quite generic *quantum* integrable models.

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