2 and 3-dimensional Hamiltonians with Shape Invariance Symmetry

M. A. Jafarizadeh a,b,c *, H. Panahi-Talemi a,† and E. Faizi a,‡

^aDepartment of Theoretical Physics and Astrophysics, Tabriz University, Tabriz 51664, Iran.

^bInstitute for Studies in Theoretical Physics and Mathematics, Teheran 19395-1795, Iran.

^cPure and Applied Science Research Center, Tabriz 51664, Iran.

November 7, 2018

Abstract

Via a special dimensional reduction, that is, Fourier transforming over one of the coordinates of Casimir operator of su(2) Lie algebra and 4-oscillator Hamiltonian, we have obtained 2 and 3 dimensional Hamiltonian with shape invariance symmetry. Using this symmetry we have obtained their eigenspectrum. In the mean time we show equivalence of shape invariance symmetry and Lie algebraic symmetry of these Hamiltonians.

^{*}E-mail:jafarzadeh@ark.tabrizu.ac.ir

[†]E-mail:t-panahi@ark.tabrizu.ac.ir

[‡]E-mail:msph@ark.tabrizu.ac.ir

Keywords: Lie Algebra, Exactly Solvable, Shape Invariance, Fourier Transformation .

PACs numbers: 05.45.Ra, 05.45.Jn, 05.45.Tp

I INTRODUCTION

Exactly solvable quantum Hamiltonians (ESQH) have always attracted a lot of interest in theoretical physics and mathematical physics. Hence, construction of exactly solvable models is of great interest [1, 2, 3, 4, 5]. Familiar solvable potentials (particularly onedimensional one) have the property of shape invariance, where this property has played an important role in calculation of their determinant by Heat Kernel method [6, 7, 8, 9, 10]. For these potentials, eigenvalues and eigenvectors can be derived using the well known methods of supersymmetric quantum mechanics together with shape invariant factorization. The majority of potentials have also been shown to possess a Lie algebraic symmetry and hence are also solvable by group theoretical techniques. Actually one can establish a connection between ESQH with shape invariance symmetry and ESQH with Lie algebraic symmetry and can show that they are indeed equivalent [11, 12]. One of the authors has introduced some 2 and 3-dimensional shape invariant Hamiltonians [13, 14, 15]. In theses article they have shown that the shape invariance symmetry of these models is due to the existence of some Lie algebraic symmetry. Hence, in this article we construct new 2 and 3-dimensional EQSH with shape invariance symmetry, where su(2) and Heisenberg algebra are responsible for the existence of shape invariance symmetry in them.

This paper is organized as follows: In section II, using the left and right invariant vector fields of su(2) Lie algebra we first construct its Casimir operator. Then via Fourier transformation over one of the coordinates we construct 2-dimensional Hamiltonian $H_q(\theta, \psi)$ which possess shape invariance symmetry. Using this symmetry we obtain its eigenspectrum analytically. In section III, starting with Hamiltonian of 4-oscillator and Fourier transforming over one of the coordinates, we obtain 3-dimensional Hamiltonian corresponding to motion of a charged particle in presence of an electric field. We show that this 3-dimensional Hamiltonian possess a shape invariance symmetry and using this symmetry we obtain its eigenspectrum. What is so important in both models is that both Hamiltonians factorize shape invariantly into a product of second order differential operators. These second order operators themselves consist of the product of first order differential operators. The paper ends with a brief conclusion.

II 2-dimensional Hamiltonian obtained from SU(2) manifold

II.1 Left and Right invariant vector field of SU(2)

Considering the following parametrization of su(2) group manifold [16]

$$\Lambda = \exp(i\vec{\sigma}.\vec{n}\psi) = A \begin{pmatrix} \exp(i\psi) & 0\\ 0 & \exp(-i\psi) \end{pmatrix} A^{-1}$$
$$= \begin{pmatrix} \cos(\psi) - i\cos(\theta)\sin(\psi) & -i\sin(\theta)\sin(\psi)\exp(-i\phi)\\ -i\sin(\theta)\sin(\psi)\exp(i\phi) & \cos(\psi) + i\cos(\theta)\sin(\psi) \end{pmatrix}, \qquad (2.1)$$

where σ_i , i= 1, 2 and 3 are Pauli matrices and \vec{n} is a unit vector defined as:

$$\vec{n} = \sin(\theta)\cos(\phi)\vec{i} + \sin(\theta)\sin(\phi)\vec{j} + \cos(\theta)\vec{k},$$

and matrix A corresponds to the coherent state representation of su(2) defined as [17]:

$$A = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \exp(\frac{\beta}{2}) & 0 \\ 0 & \exp(\frac{-\beta}{2}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\tau^* & 1 \end{pmatrix}$$

with $\tau = \tan(\frac{\theta}{2}) \exp(-i\phi)$ and $\beta = \ln(1 + \tau\tau^*)$.

In order to obtain the left and right invariant vector field su(2) manifold with the above parametrization, it is convenient first to calculate its left and right invariant one form defined as $\Lambda^{-1}d\Lambda$ and $d\Lambda\Lambda^{-1}$ respectively [18].

As an example, let us write left invariant one form

$$\Lambda^{-1}d\Lambda = e^a_\alpha d\xi^\alpha \sigma_a,$$

where e^a_{α} are 3-beins and $\xi^{\alpha} = (\theta, \phi, \psi)$ are coordinates of su(2)-manifold. Defining the inverse of 3-bein $e^{\alpha}_a = g^{\alpha\beta}\delta_{ab}e^b_{\beta}$ with $g^{\alpha\beta}$ as inverse of metric $g_{\alpha\beta}$:

$$g_{\alpha,\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2(\psi) & 0 \\ 0 & 0 & \sin^2(\psi)\sin^2(\theta) \end{pmatrix},$$

then the left invariant vector field is defined as:

$$L_a = e_a^{\alpha} \frac{\partial}{\partial \xi^{\alpha}}.$$

Using the above prescription we obtain the following expression for left and right invariant vector field of su(2), respectively,

$$L_{+} = \frac{i}{2}e^{i\phi} \left[\sin(\theta)\partial_{\psi} + (i + \cos(\theta)\cot(\psi))\partial_{\theta} + (-\cot(\theta) + i\frac{\cot(\psi)}{\sin(\theta)})\partial_{\phi} \right], \qquad (2.2)$$

$$L_{-} = \frac{i}{2}e^{-i\phi} \left[\sin(\theta)\partial_{\psi} + (-i + \cos(\theta)\cot(\psi))\partial_{\theta} + (-\cot(\theta) - i\frac{\cot(\psi)}{\sin(\theta)})\partial_{\phi} \right],$$
(2.3)

$$L_3 = \frac{i}{2} \left(-\cos(\theta)\partial_{\psi} + \sin(\theta)\cot(\psi)\partial_{\theta} - \partial_{\phi} \right), \qquad (2.4)$$

$$R_{+} = \frac{i}{2}e^{i\phi} \left[\sin(\theta)\partial_{\psi} + (-i + \cos(\theta)\cot(\psi))\partial_{\theta} + (\cot(\theta) + i\frac{\cot(\psi)}{\sin(\theta)})\partial_{\phi} \right], \qquad (2.5)$$

$$R_{-} = \frac{i}{2}e^{-i\phi} \left[\sin(\theta)\partial_{\psi} + (i + \cos(\theta)\cot(\psi))\partial_{\theta} + (\cot(\theta) - i\frac{\cot(\psi)}{\sin(\theta)})\partial_{\phi}\right], \quad (2.6)$$

$$R_3 = \frac{i}{2} \left(-\cos(\theta)\partial_{\psi} + \sin(\theta)\cot(\psi)\partial_{\theta} + \partial_{\phi} \right), \qquad (2.7)$$

where $L_{\pm} = L_1 \pm iL_2$ and $R_{\pm} = R_1 \pm iR_2$. It is straightforward to show that the left and right invariant vector field fulfill the following su(2) Lie algebra:

 $[L_+, L_-] = 2L_3, \quad [L_3, L_\pm] = \pm L_\pm,$ (2.8)

$$[R_+, R_-] = -2R_3, \quad [R_3, R_\pm] = \mp R_\pm, \tag{2.9}$$

also, the left and right invariant generators commute with each other

$$[\vec{L}, \vec{R}] = 0. \tag{2.10}$$

Considering the Casimir operators of su(2) defined as:

$$L^{2} = \frac{1}{2}(L_{+}L_{-} + L_{-}L_{+}) + L_{3}^{2},$$

and ignoring the scale 1/4, we obtain

$$L^{2} = -\frac{1}{\sin^{2}(\psi)}\partial_{\psi}\sin^{2}(\psi)\partial_{\psi} - \frac{1}{\sin^{2}(\psi)}\left(\frac{1}{\sin(\theta)}\partial_{\theta}\sin(\theta)\partial_{\theta} + \frac{1}{\sin^{2}(\theta)}\partial_{\phi}^{2}\right).$$
 (2.11)

In obtaining the above formula we have used the left invariant generators. It is straightforward to show that we can obtain the same result with right invariant generators too, that means the Casimir operator of left and right operator are the same.

II.2 $H_q(\theta, \psi)$ Hamiltonian

Here through dimensional reduction we show that the above Casimir operator reduces to a Hamiltonian of motion of a charged particle in the presence of electric field. Hence, first we make one-dimensional reduction (eliminate the coordinate ϕ) through the usual Fourier transformation defined as

$$\tilde{f}(q) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(\phi) \exp(-i\phi q) d\phi, \qquad (2.12)$$

over an arbitrary function $f(\phi)$. Obviously the Casimir operator (2.11) reduces to the following operator

$$L_q^2(\theta,\psi) = -\frac{1}{\sin^2(\psi)}\partial_\psi \sin^2(\psi)\partial_\psi - \frac{1}{\sin^2(\psi)}\left(\frac{1}{\sin(\theta)}\partial_\theta \sin(\theta)\partial_\theta - \frac{q^2}{\sin^2(\theta)}\right),$$
 (2.13)

in the Hilbert space of Fourier transformed wavefunctions. In general the non- relativistic Hamiltonian of a charged particle over a 2-dimensional manifold with metric $g_{\mu\nu}$ in the presence of magneto static field \vec{B} with vector potential \vec{A} and electro static field \vec{E} with scalar potential V can be written as [10, 19]

$$H = -\frac{1}{\sqrt{g}} (\partial_{\mu} - iA_{\mu}) (\sqrt{g} g^{\mu\nu} (\partial_{\nu} - iA_{\nu})) + V, \qquad (2.14)$$

where g is the determinant of metric $g_{\mu\nu}$. After similarity transformation of the Casimir operator (2.13) defined as:

$$\tilde{L}_q^2(\theta,\psi) = \sin^{\frac{1}{2}}(\psi)L_q^2(\theta,\psi)\sin^{-\frac{1}{2}}(\psi),$$

we have

$$\tilde{L}_{q}^{2}(\theta,\psi) = -\frac{1}{\sin(\psi)}\partial_{\psi}\sin(\psi)\partial_{\psi} - \frac{1}{\sin^{2}(\psi)}\left(\frac{1}{\sin(\theta)}\partial_{\theta}\sin(\theta)\partial_{\theta} - \frac{q^{2}}{\sin^{2}(\theta)}\right) + \frac{1}{4}\cot^{2}(\psi) - \frac{1}{2}.$$
(2.15)

Comparing the operator (2.15) with the Hamiltonian (2.14) we obtain

$$g_{\psi\psi} = 1, g_{\theta\theta} = \sin^2(\psi), g_{\psi\theta} = g_{\theta\psi} = 0$$

and

$$A_{\psi} = 0, \quad A_{\theta} = \frac{i}{2}\cot(\theta) = d(\frac{i}{2}\ln(\sin(\theta))). \tag{2.16}$$

It is trivial to see that the vector potential given in (2.16) corresponds to the pure u(1)gauge field, hence it can be eliminate through the gauge transform $A \longrightarrow A_{\mu} + \partial_{\mu}\chi$ with gauge function $\chi = \frac{i}{2} \ln(\sin(\theta))$. After the above gauge transformation the su(2)-Casimir Hamiltonian reduces to

$$H_{q}(\theta,\psi) \equiv e^{-\chi} \tilde{L}_{q}^{2}(\theta,\psi) e^{\chi} = -\frac{1}{\sin(\psi)} \partial_{\psi} \sin(\psi) \partial_{\psi} - \frac{1}{\sin^{2}(\psi)} \partial_{\theta}^{2} + \frac{q^{2} - \frac{1}{4}}{\sin^{2}(\psi) \sin^{2}(\theta)} - \frac{3}{4}, \quad (2.17)$$

which can be interpreted as a non-relativistic Hamiltonian of a point particle over 2-dimensional sphere with metric

$$g_{\mu,\nu} = \left(\begin{array}{cc} 1 & 0\\ 0 & \sin^2(\psi) \end{array}\right)$$

in the presence of electric field with scalar potential

$$V = \frac{q^2 - \frac{1}{4}}{\sin^2(\psi)\sin^2(\theta)} - \frac{3}{4}.$$

Similarly, the left and right invariant vector fields given in (2.2)-(2.7) take the following form after the above given operations, namely, dimensional reduction, similarity transformation and gauge transformation:

$$\tilde{L}'_{+}(q) = \tilde{L}_{+}(q) + g(\theta, \psi, q), \quad \tilde{R}'_{+}(q) = \tilde{R}_{+}(q) - g^{\star}(\theta, \psi, q), \\
\tilde{L}'_{-}(q) = \tilde{L}_{-}(q) - g^{\star}(\theta, \psi, q), \quad \tilde{R}'_{-}(q) = \tilde{R}_{-}(q) + g(\theta, \psi, q), \\
\tilde{L}'_{3}(q) = \tilde{L}_{3}(q), \quad \tilde{R}'_{3}(q) = \tilde{R}_{3}(q),$$

where

$$\tilde{L}_{+}(q) = \frac{i}{2} \left(\sin(\theta)\partial_{\psi} + (i + \cos(\theta)\cot(\psi))\partial_{\theta} + i(q-1)(-\cot(\theta) + i\frac{\cot(\psi)}{\sin(\theta)}) \right) e^{-\frac{\partial}{\partial q}}, \quad (2.18)$$

$$\tilde{L}_{-}(q) = \frac{i}{2} \left(\sin(\theta) \partial_{\psi} + (-i + \cos(\theta) \cot(\psi)) \partial_{\theta} + i(q+1)(-\cot(\theta) - i\frac{\cot(\psi)}{\sin(\theta)}) \right) e^{\frac{\partial}{\partial q}}, \quad (2.19)$$

$$\tilde{L}_3(q) = \frac{i}{2} \left(-\cos(\theta)\partial_{\psi} + \sin(\theta)\cot(\psi)\partial_{\theta} - iq \right), \qquad (2.20)$$

$$\tilde{R}_{+}(q) = \frac{i}{2} \left(\sin(\theta)\partial_{\psi} + (-i + \cos(\theta)\cot(\psi))\partial_{\theta} + i(q-1)(\cot(\theta) + i\frac{\cot(\psi)}{\sin(\theta)}) \right) e^{-\frac{\partial}{\partial q}}, \quad (2.21)$$

$$\tilde{R}_{-}(q) = \frac{i}{2} \left(\sin(\theta) \partial_{\psi} + (i + \cos(\theta) \cot(\psi)) \partial_{\theta} + i(q+1)(\cot(\theta) - i\frac{\cot(\psi)}{\sin(\theta)}) \right) e^{\frac{\partial}{\partial q}}, \quad (2.22)$$

$$\tilde{R}_3(q) = \frac{i}{2} \left(-\cos(\theta)\partial_\psi + \sin(\theta)\cot(\psi)\partial_\theta + iq \right)$$
(2.23)

with $g(\theta, \psi, q)$ is:

$$g(\theta, \psi, q) = \frac{1}{4} \left(\cot(\theta) - i \frac{\cot(\psi)}{\sin(\theta)} \right) e^{\frac{\partial}{\partial q}},$$

where * means the usual complex conjugation. With some calculation one can show that the above algebra, that is, the commutation relations is unchanged under the above mentioned transformation and the Hamiltonian $H_q(\theta, \psi)$ can be written in terms of generators (2.18)-(2.20) in the following form

$$H_q(\theta,\psi) = \frac{1}{2} \left(\tilde{L}'_+(q) \tilde{L}'_-(q) + \tilde{L}'_-(q) \tilde{L}'_+(q) \right) + \tilde{L}'_3(q)^2.$$

Hence, $H_q(\theta, \psi)$ is still Casimir operator su(2) Lie algebra with generator given in (2.18)-(2.23).

II.3 Algebraic Solution of $H_q(\theta, \psi)$ Hamiltonian

In order to obtain eigenspectrum of Hamiltonian (2.17) by algebraic method, first we obtain eigenspectrum of the Casimir operator (2.11). Since this operator commutes with left and right invariant generators given in (2.10), therefore, we can obtain representation of su(2) simply by finding simultaneous eigenfunctions of the set of commuting operators, (R_3, L_3, L^2) . Denoting their simultaneous eigenfunction by $\chi^l_{m_L,m_R}(\theta, \psi, \phi)$, we can write

$$L^{2}\chi_{m_{L},m_{R}}^{l}(\theta,\psi,\phi) = l(l+1)\chi_{m_{L},m_{R}}^{l}(\theta,\psi,\phi), \qquad (2.24)$$

$$L_{3}\chi_{m_{L},m_{R}}^{l}(\theta,\psi,\phi) = m_{L}\chi_{m_{L},m_{R}}^{l}(\theta,\psi,\phi), \qquad (2.25)$$

$$R_{3}\chi^{l}_{m_{L},m_{R}}(\theta,\psi,\phi) = m_{R}\chi^{l}_{m_{L},m_{R}}(\theta,\psi,\phi).$$
(2.26)

Now solving the difference of the first order differential equations (2.25) and (2.26) we deduce that $\chi_{m_L,m_R}^l(\theta,\psi,\phi)$ is proportional to $e^{-i(m_R-m_L)\phi}$, hence we have $\chi_{m_L,m_R}^l(\theta,\psi,\phi) = e^{-i(m_R-m_L)\phi}f(\theta,\psi)$, where $f(\theta,\psi)$ can be determined from the solution of the sum of the equations (2.25) and (2.26), that is

$$-i\cos(\theta)\partial_{\psi}f(\theta,\psi) + i\sin(\theta)\cot(\psi)\partial_{\theta}f(\theta,\psi) = (m_L + m_R)f(\theta,\psi).$$
(2.27)

Now considering the highest weight defined by $m_L = -m_R = l$. this happenes if the right hand side of the equation (2.27) vanishes, hence it can be solved by characteristic method which leads to the following results:

$$\chi_{l,-l}^{l}(\theta,\psi,\phi) = \exp(2il\phi)f^{max}(\sin(\psi)\sin(\theta))$$

where f^{max} is an arbitrary function which can be determined by solving the first order differential equation:

$$R_{+}\chi_{l,-l}^{l}(\theta,\psi,\phi) = 0, \quad L_{+}\chi_{l,-l}^{l}(\theta,\psi,\phi) = 0.$$

Since the highest weight $\chi_{l,-l}^{l}$ belongs to the kernel of raising operators R_{+} and L_{+} , therefore the sum of the equations (2.2) and (2.5) leads to

$$u\frac{df^{max}(u)}{du} = 2lf^{max}(u),$$

where $u = \sin(\psi)\sin(\theta)$. Therefore, solving the above equation we obtain $f^{max}(u) = u^{2l}$, hence $\chi_{l,-l}^{l}(\theta, \psi, \phi)$ has the following form

$$\chi_{l,-l}^{l}(\theta,\psi,\phi) = e^{2il\phi}(\sin(\psi)\sin(\theta))^{2l}.$$
(2.28)

The other eigenweights can be obtained through the operation of the lowering operator R_{-} and L_{-} over the highest eigenfunction, that is, we have

$$\chi_{m_L,m_R}^l(\theta,\psi,\phi) = (L_-)^{l-m_L} (R_-)^{l+m_R} (e^{2il\phi} (\sin(\psi)\sin(\theta))^{2l}).$$
(2.29)

In order to eliminate the coordinate ϕ , first we transfer the function $e^{2il\phi}$ to the left hand side of the operators R_{-} and L_{-} in (2.29), then we get:

$$\chi^{l}_{m_{L},m_{R}}(\theta,\psi,\phi) = e^{i(m_{L}-m_{R})\phi}L_{-}(m_{L}-m_{R}+1)L_{-}(m_{L}-m_{R}+2)...$$
$$...L_{-}(l-m_{R})R_{-}(l-m_{R}+1)...R_{-}(2l)(\sin(\psi)\sin(\theta))^{2l},$$
(2.30)

where the operators $L_{-}(m)$ and $R_{-}(m)$ are defined as:

$$L_{-}(m) = \frac{i}{2} \left(\sin(\theta) \partial_{\psi} + (-i + \cos(\theta) \cot(\psi)) \partial_{\theta} + im(-\cot(\theta) - i \frac{\cot(\psi)}{\sin(\theta)}) \right), \qquad (2.31)$$

$$R_{-}(m) = \frac{i}{2} \left(\sin(\theta)\partial_{\psi} + (i + \cos(\theta)\cot(\psi))\partial_{\theta} + im(\cot(\theta) - i\frac{\cot(\psi)}{\sin(\theta)}) \right).$$
(2.32)

Finally, the Fourier transformation of (2.30) leads to

$$\chi_{q,m}^{l}(\theta,\psi) = L_{-}(q+1)L_{-}(q+2)...L_{-}(l+\frac{q-m}{2})R_{-}(l+\frac{q-m}{2}+1)...$$
$$...R_{-}(2l)(\sin(\psi)\sin(\theta))^{2l},$$
(2.33)

where $q = m_L - m_R$ and $m = m_L + m_R$.

Since the left and right invariant generators commute with each other, we can exchange these operators in (2.29) before Fourier transformation, whereas after Fourier transformation, we can use only the relation $L_{-}(q)R_{-}(q-1) = R_{-}(q)L_{-}(q-1)$. Since the Hamiltonian $H_{q}(\theta, \psi)$ can be obtained from the relations (2.15) and (2.17) via similarity transformation and gauge transformation, we have:

$$H_{q}(\theta,\psi) = \exp(-\xi)L_{q}^{2}(\theta,\psi)\exp(\xi), \quad L_{q}^{2}(\theta,\psi)\chi_{q,m}^{l}(\theta,\psi) = l(l+1)\chi_{q,m}^{l}(\theta,\psi), \quad (2.34)$$

where $\xi = -\frac{1}{2} \ln(\sin(\psi) \sin(\theta))$. Hence eigenfunction of Hamiltonian $H_q(\theta, \psi)$ can be written as:

$$\tilde{\chi}_{q,m}^{l}(\theta,\psi) = \exp(-\xi)\chi_{q,m}^{l}(\theta,\psi).$$
(2.35)

II.4 Shape Invariance Symmetry of $H_q(\theta, \psi)$

Here in this section we show that the Hamiltonian $H_q(\theta, \psi)$ possess both degeneracy and shape invariance symmetry [1, 2, 3]. As it is shown in section (II.3), functions $\tilde{\chi}_{q,m}^l(\theta, \psi) =$ $(\sin(\psi)\sin(\theta))^{\frac{1}{2}}\chi_{q,m}^l(\theta, \psi)$ are eigenfunctions of Hamiltonian $H_q(\theta, \psi)$ with the corresponding eigenvalue l(l+1). Since $|m_R| \leq l$ and $|m_L| \leq l$, therefore, $|q| \leq 2l$ and for a given value of q the parameter m can take the following values:

$$m = \begin{cases} 0, \pm 2, \pm 4, \dots, \pm (2l - |q|) & for \ |q| = even, \\ \pm 1, \pm 3, \dots, \pm (2l - |q|) & for \ |q| = odd. \end{cases}$$
(2.36)

Since the eigenvalue of Hamiltonian $H_q(\theta, \psi)$ is independent of m, therefore it has (2l+1-|q|)degenerate states for a given l or given energy l(l+1). To see the shape invariance symmetry of Hamiltonian $H_q(\theta, \psi)$, first we consider the Fourier transformed left and right invariant vector fields:

$$\tilde{L}_{+}(q) \equiv L_{+}(q-1)e^{-\frac{\partial}{\partial q}}$$

$$= \frac{i}{2} \left(\sin(\theta)\partial_{\psi} + (i+\cos(\theta)\cot(\psi))\partial_{\theta} + i(q-1)(-\cot(\theta) + i\frac{\cot(\psi)}{\sin(\theta)}) \right) e^{-\frac{\partial}{\partial q}}, \qquad (2.37)$$

$$\tilde{L}_{-}(q) \equiv L_{-}(q+1)e^{\frac{\partial}{\partial q}}$$

$$= \frac{i}{2} \left(\sin(\theta)\partial_{\psi} + (-i+\cos(\theta)\cot(\psi))\partial_{\theta} + i(q+1)(-\cot(\theta) - i\frac{\cot(\psi)}{\sin(\theta)}) \right) e^{\frac{\partial}{\partial q}}, \qquad (2.38)$$

$$\tilde{L}_{3}(q) \equiv L_{3}(q) = \frac{i}{2}(-\cos(\theta)\partial_{\psi} + \sin(\theta)\cot(\psi)\partial_{\theta} - iq) \qquad (2.39)$$

and

$$\tilde{R}_{+}(q) \equiv R_{+}(q-1)e^{-\frac{\partial}{\partial q}}$$

$$= \frac{i}{2} \left(\sin(\theta)\partial_{\psi} + (-i+\cos(\theta)\cot(\psi))\partial_{\theta} + i(q-1)(\cot(\theta) + i\frac{\cot(\psi)}{\sin(\theta)}) \right) e^{-\frac{\partial}{\partial q}}, \quad (2.40)$$

$$\tilde{R}_{-}(q) \equiv R_{-}(q+1)e^{\frac{\partial}{\partial q}}$$

$$= \frac{i}{2} \left(\sin(\theta)\partial_{\psi} + (i+\cos(\theta)\cot(\psi))\partial_{\theta} + i(q+1)(\cot(\theta) - i\frac{\cot(\psi)}{\sin(\theta)}) \right) e^{\frac{\partial}{\partial q}}, \quad (2.41)$$

$$\tilde{R}_{3}(q) \equiv R_{3}(q) = \frac{i}{2}(-\cos(\theta)\partial_{\psi} + \sin(\theta)\cot(\psi)\partial_{\theta} + iq). \quad (2.42)$$

After some tedious algebraic calculation we can derive the following relation between the above operators

$$L_3(q \pm 1)L_{\pm}(q) - L_{\pm}(q)L_3(q) = \pm L_{\pm}(q)$$
(2.43)

$$R_3(q \pm 1)R_{\pm}(q) - R_{\pm}(q)R_3(q) = \mp R_{\pm}(q).$$
(2.44)

These relations indicate that Hamiltonian $H_q(\theta, \psi)$ possesses shape invariance symmetry. Since by acting the operators $R_{\pm}(q)$ and $L_{\pm}(q)$ on both sides of eigenvalue equations:

$$L_q^2(\theta,\psi)\chi_{q,m}^l(\theta,\psi) = l(l+1)\chi_{q,m}^l(\theta,\psi),$$
$$R_3(q)\chi_{q,m}^l(\theta,\psi) = \frac{m-q}{2}\chi_{q,m}^l(\theta,\psi),$$
$$L_3(q)\chi_{q,m}^l(\theta,\psi) = \frac{m+q}{2}\chi_{q,m}^l(\theta,\psi),$$

we get,

$$R_{\pm}(q)\chi_{q,m}^{l}(\theta,\psi) = A_{\pm}(q,m)\chi_{q\pm1,m\mp1}^{l}(\theta,\psi), \qquad (2.45)$$

$$L_{\pm}(q)\chi_{q,m}^{l}(\theta,\psi) = B_{\pm}(q,m)\chi_{q\pm 1,m\pm 1}^{l}(\theta,\psi), \qquad (2.46)$$

with

$$A_{\pm}(q,m) = \frac{1}{2}\sqrt{(2l \mp (m-q))(2l \pm (m-q) + 2)},$$
(2.47)

$$B_{\pm}(q,m) = \frac{1}{2}\sqrt{(2l \mp (m+q))(2l \pm (m+q) + 2)}.$$
(2.48)

The above relations imply that the pair of operators (L_-, R_+) $[(L_+, R_-)]$ map degenerate eigenstates of Hamiltonian $H_q(\theta, \psi)$ for a given value of q into each other, that is they decrease [increase] the quantum number m by 2 units as follows:

$$L_{-}(q+1)R_{+}(q)\chi_{q,m}^{l}(\theta,\psi) = A_{+}(q,m)B_{-}(q+1,m-1)\chi_{q,m-2}^{l}(\theta,\psi),$$
$$L_{+}(q-1)R_{-}(q)\chi_{q,m}^{l}(\theta,\psi) = A_{-}(q,m)B_{+}(q-1,m+1)\chi_{q,m+2}^{l}(\theta,\psi).$$

Now introducing the operator $Y_+(q) := L_+(q-1)R_-(q)$ and $Y_-(q) := L_-(q+1)R_+(q)$ as the raising and lowering operators of degenerates states of Hamiltonian $H_q(\theta, \psi)$, we have the following shape invariance like symmetry between the degenerate states of Hamiltonian $H_q(\theta, \psi)$:

$$Y_{-}(q)Y_{+}(q)\chi_{q,m}^{l}(\theta,\psi) = E(q,m)\chi_{q,m}^{l}(\theta,\psi)$$
$$Y_{+}(q)Y_{-}(q)\chi_{q,m+2}^{l}(\theta,\psi) = E(q,m)\chi_{q,m+2}^{l}(\theta,\psi)$$

where

$$E(q,m) = A_{-}(q,m)A_{+}(q,m+2)B_{-}(q+1,m+1)B_{+}(q-1,m+1).$$

Thus, for a given value of q, we can obtain eigenfunction of Hamiltonian $H_q(\theta, \psi)$ with eigenvalue l(l + 1), simply by acting the pairs of operators $(L_-, R_+)[(L_+, R_-)]$ over the highest weight [lowest weight], where here we have derived the eigenfunction $\chi_{q,m}^l(\theta, \psi)$ by acting the lowering operator over the highest eigenstate as follows:

$$\chi_{q,m}^{l}(\theta,\psi) = k^{-1}(Y_{-}(q))^{\frac{2l-|q|-m}{2}}\chi_{q,(2l-|q|)}^{l}(\theta,\psi), \qquad (2.49)$$

where

$$k = B_{-}(q+1, m+1)B_{-}(q+1, m+3)...$$

$$\times B_{-}(q+1,2l-|q|-1)A_{+}(q,m+2)A_{+}(q,m+4)...A_{+}(q,2l-|q|).$$

Using the relation (2.33) we can obtain the highest weight eigenstates for q > 0 and q < 0,

$$\chi_{q,(2l-|q|)}^{l}(\theta,\psi) = \begin{cases} L_{-}(q+1)L_{-}(q+2)...\\ \times L_{-}(0)R_{-}(1)R_{-}(2)...R_{-}(2l)(\sin(\theta)\sin(\psi))^{2l} & for \ q < 0\\ R_{-}(q+1)R_{-}(q+2)...R_{-}(2l)(\sin(\theta)\sin(\psi))^{2l} & for \ q > 0. \end{cases}$$

On the other hand pair operator $(L_+, R_+)[or(L_-, R_-)]$ leave the eigenvalue m and lunchanged while they increase [decrease] the parameter q by 2 units, that is they map eigenfunction of Hamiltonian corresponding to the same energy with different q into each other. That is, they map isospectral Hamiltonian into each other, which nothing but shape invariance. In order to show this shape invariance symmetry, we act the related operators over $\chi^l_{q,m}(\theta, \psi)$, we then obtain:

$$L_{+}(q+1)R_{+}(q)\chi_{q,m}^{l}(\theta,\psi) = A_{+}(q,m)B_{+}(q+1,m-1)\chi_{q+2,m}^{l}(\theta,\psi)$$
$$L_{-}(q-1)R_{-}(q)\chi_{q,m}^{l}(\theta,\psi) = A_{-}(q,m)B_{-}(q-1,m+1)\chi_{q-2,m}^{l}(\theta,\psi),$$

obviously, the combined action of above operators leave the eigenvalues m and l unchanged while changing the parameter q by 2-units. Hence we define the operator $X_+(q) := L_+(q + 1)R_+(q)$ and $X_-(q) := L_-(q + 1)R_-(q + 2)$ as raising and lowering operators of parameter q. Then the shape invariance symmetry means:

$$X_{-}(q)X_{+}(q)\chi_{q,m}^{l}(\theta,\psi) = N(q,m)\chi_{q,m}^{l}(\theta,\psi)$$
$$X_{+}(q)X_{-}(q)\chi_{q+2,m}^{l}(\theta,\psi) = N(q,m)\chi_{q+2,m}^{l}(\theta,\psi)$$

where

$$N(q,m) = A_{+}(q,m)A_{-}(q+2,m)B_{+}(q+1,m-1)B_{-}(q+1,m+1)$$

or

$$N(q,m) = \frac{1}{16}(2l - m - q)(2l + m + q + 2)$$

$$\times \sqrt{(2l - m + q)(2l - m + q + 4)(2l + m - q + 2)(2l + m - q - 2)}.$$

For fixed values of energy l(l + 1) and given values of m, the parameter q can take the following values

$$q = (2l - |m|), (2l - |m| - 2), \dots, -(2l - |m| - 2), -(2l - |m|).$$

Hence obtaining the highest eigenstates, by solving the following first order differential equation

$$X_{+}(2l - |m|)\chi_{(2l - |m|),m}^{l}(\theta, \psi) = 0$$

where its integral leads to

$$\chi_{(2l-|m|),m}^{l}(\theta,\psi) = \begin{cases} L_{-}(2l-m+1)L_{-}(2l-m+2)...L_{-}(2l)(\sin(\theta)\sin(\psi))^{2l} & for \ m<0, \\ R_{-}(2l-m+1)R_{-}(2l-m+2)...R_{-}(2l)(\sin(\theta)\sin(\psi))^{2l} & for \ m>0. \end{cases}$$

Therefore using the shape invariance relation, we can obtain the eigenstates of Hamiltonian $H_q(\theta, \psi)$ by consecutive action of q-lowering operator over q-highest weight eigenstate,

III 3-dimensional Hamiltonian obtained from 4-Oscillators

Here in this section using the su(2)-parametrization of previous section, we obtain a special 3-dimensional Hamiltonian from the Hamiltonian of 4-oscillator with the same frequency, where we obtain its spectrum via the corresponding spectrum of 4-oscillator Hamiltonian. We show that such a Hamiltonian possesses shape invariance symmetry. The Hamiltonian of 4-oscillator with same frequency can be written as:

$$H = -\frac{1}{2}\Sigma_{i=0}^{4}(P_{i}^{2} + \frac{1}{2}\omega^{2}x_{i}^{2})$$

Now making the following change of variable:

$$x_{1} = -r\sin(\psi)\sin(\theta)\sin(\phi),$$

$$x_{2} = r\sin(\psi)\sin(\theta)\cos(\phi),$$

$$x_{3} = r\sin(\psi)\cos(\theta),$$

$$x_{4} = r\cos(\psi),$$
(3.1)

where ψ, θ, ϕ are the same coordinates used in the parametrization su(2) manifold, the Hamiltonian takes the following form

$$H(r,\theta,\psi,\phi) = -\frac{1}{2} \left[\frac{1}{r^3} \partial_r r^3 \partial_r + \frac{1}{r} \left(\partial_{\psi}^2 + 2\cot(\psi) \partial_{\psi} + \frac{1}{\sin^2(\psi)} (\partial_{\theta}^2 + \cot(\theta) \partial_{\theta} + \frac{1}{\sin^2(\theta)} \partial_{\phi}^2) \right) \right] + \frac{1}{2} \omega^2 r^2.$$
(3.2)

Since angular part of the above Hamiltonian is the same, the one given in (2.11), therefore, its eigenspectrum can be obtained straightforwardly through routine separation variable into radial and angular part which we are not interested in it here in this work. Actually here we are concerned with special Hamiltonian which can be obtained from this 4-oscillator Hamiltonian, that is, those Hamiltonians which possess shape invariance symmetry.

In order to achieve this, we write the above Hamiltonian in terms of raising and lowering operators defined in the usual way:

$$H = \omega (a_1^{\dagger} a_1 + a_2^{\dagger} a_2 + a_3^{\dagger} a_3 + a_4^{\dagger} a_4 + 2), \qquad (3.3)$$

where $a_i(a_i^{\dagger})$ are defined as:

$$a_i = \sqrt{\frac{\omega}{2}}(x_i + \frac{1}{\omega}\frac{d}{dx_i}), \ a_i^{\dagger} = \sqrt{\frac{\omega}{2}}(x_i - \frac{1}{\omega}\frac{d}{dx_i}).$$

These operators have the following form in radial coordinate (3.1)

$$\begin{aligned} a_1(a_1^{\dagger}) &= \sqrt{\frac{\omega}{2}} \left[-r\sin(\psi)\sin(\theta)\sin(\phi) + (-)\frac{1}{\omega} \left(-\sin(\psi)\sin(\theta)\sin(\theta)\sin(\phi)\partial_r - \frac{1}{r}\cos(\psi)\sin(\theta)\cos(\phi)\partial_{\psi} - \frac{1}{r}\frac{\cos(\theta)\sin(\phi)}{\sin(\psi)}\partial_{\theta} - \frac{1}{r}\frac{\cos(\phi)}{\sin(\psi)\sin(\theta)}\partial_{\phi} \right) \right], \\ -\frac{1}{r}\cos(\psi)\sin(\theta)\cos(\phi)\partial_{\psi} - \frac{1}{r}\frac{\cos(\theta)\cos(\phi)}{\sin(\psi)}\partial_{\theta} - \frac{1}{r}\frac{\sin(\phi)}{\sin(\psi)\sin(\theta)}\partial_{\phi} \right) \right], \\ +\frac{1}{r}\cos(\psi)\sin(\theta)\cos(\phi)\partial_{\psi} + \frac{1}{r}\frac{\cos(\theta)\cos(\phi)}{\sin(\psi)}\partial_{\theta} - \frac{1}{r}\frac{\sin(\phi)}{\sin(\psi)\sin(\theta)}\partial_{\phi} \right) \right], \\ a_3(a_3^{\dagger}) &= \sqrt{\frac{\omega}{2}} \left[r\sin(\psi)\cos(\theta) + (-)\frac{1}{omega} \left(\sin(\psi)\cos(\theta)\partial_r + \frac{1}{r}\cos(\psi)\cos(\theta)\partial_{\psi} - \frac{1}{r}\frac{\sin(\theta)}{\sin(\psi)}\partial_{\theta} \right) \right], \\ a_4(a_4^{\dagger}) &= \sqrt{\frac{\omega}{2}} \left[r\cos(\psi) + (-)\frac{1}{\omega} \left(\cos(\psi)\partial_r - \frac{1}{r}\sin\psi\partial_{\psi} \right) \right]. \end{aligned}$$

Now let us define the set of new operators $A_i(A_i^{\dagger})$, i= 1, 2 in terms of $a_i(a_i^{\dagger})$:

$$A_{1} = \frac{1}{\sqrt{2}}(a_{1} + ia_{2}), \quad A_{1}^{\dagger} = \frac{1}{\sqrt{2}}(a_{1}^{\dagger} - ia_{2}^{\dagger}),$$
$$A_{2} = \frac{1}{\sqrt{2}}(a_{1} - ia_{2}), \quad A_{2}^{\dagger} = \frac{1}{\sqrt{2}}(a_{1}^{\dagger} + ia_{2}^{\dagger}),$$

where, these new operators have the following differential form in radial coordinates:

$$A_{1} = \frac{i}{\sqrt{2}} \sqrt{\frac{\omega}{2}} e^{i\phi} \left[r\sin(\psi)\sin(\theta) + \frac{1}{r} \frac{\sin(\psi)\sin(\theta)}{\sin(\psi)\sin(\theta)} \right], \quad (3.4)$$

$$+ \frac{1}{\omega} \left(\sin(\psi)\sin(\theta)\partial_{r} - \frac{1}{r}\cos(\psi)\sin(\theta)\partial_{\psi} + \frac{1}{r}\frac{\cos(\theta)}{\sin(\psi)}\partial_{\theta} + \frac{1}{r}\frac{i}{\sin(\psi)\sin(\theta)}\partial_{\phi} \right), \quad (3.4)$$

$$+ \frac{1}{\omega} \left(\sin(\psi)\sin(\theta)\partial_{r} + \frac{1}{r}\cos(\psi)\sin(\theta)\partial_{\psi} + \frac{1}{r}\frac{\cos(\theta)}{\sin(\psi)}\partial_{\theta} - \frac{1}{r}\frac{i}{\sin(\psi)\sin(\theta)}\partial_{\phi} \right), \quad (3.5)$$

$$A_{2} = -\frac{i}{\sqrt{2}} \sqrt{\frac{\omega}{2}} e^{-i\phi} \left[r\sin(\psi)\sin(\theta) + \frac{1}{r}\frac{\cos(\theta)}{\sin(\psi)}\partial_{\theta} - \frac{1}{r}\frac{i}{\sin(\psi)\sin(\theta)}\partial_{\phi} \right), \quad (3.6)$$

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$$A_{2}^{\dagger} = \frac{i}{\sqrt{2}} \sqrt{\frac{\omega}{2}} e^{i\phi} \left[r \sin(\psi) \sin(\theta) - \frac{1}{\omega} \left(\sin(\psi) \sin(\theta) \partial_r + \frac{1}{r} \cos(\psi) \sin(\theta) \partial_{\psi} + \frac{1}{r} \frac{\cos(\theta)}{\sin(\psi)} \partial_{\theta} + \frac{1}{r} \frac{i}{\sin(\psi) \sin(\theta)} \partial_{\phi} \right) \right].$$
(3.7)

It is also straightforward to show that they have the following commutator relations:

$$[A_i, A_j^{\dagger}] = \delta_{ij}$$
, $[A_i, A_j] = [A_i^{\dagger}, A_j^{\dagger}] = 0$, $i, j = 1, 2$.

The 4-oscillators Hamiltonian (3.3) can be written in terms of the new oscillators as follows:

$$H = \omega (A_1^{\dagger} A_1 + A_2^{\dagger} A_2 + a_3^{\dagger} a_3 + a_4^{\dagger} a_4 + 2).$$
(3.8)

Now its eigenspectrum can be obtained by solving the following eigenvalue equation

$$H\Psi_{(n_1,n_2,n_3,n_4)}(r,\theta,\phi,\psi) = E_{(n_1,n_2,n_3,n_4)}\Psi_{(n_1,n_2,n_3,n_4)}(r,\theta,\phi,\psi),$$
(3.9)

by the usual algebraic method. Hence its eigenfunction can be written as:

$$\Psi_{(n_1,n_2,n_3,n_4)}(r,\theta,\phi,\psi) = N(A_1^{\dagger})^{n_1} (A_2^{\dagger})^{n_2} (a_3^{\dagger})^{n_3} (a_4^{\dagger})^{n_4} \exp(-\frac{\omega}{2}r^2), \qquad (3.10)$$

with $N = \frac{\omega}{\pi \sqrt{n_1! n_2! n_3! n_4!}}$ as the normalization constant, and energy $E_{(n_1, n_2, n_3, n_4)} = (n_1 + n_2 + n_3 + n_4 + 2)\omega$. Using the differential representation of the operator, the wavefunction (3.10) can be written in the following form

$$\Psi_{(n_1,n_2,n_3,n_4)}(r,\theta,\phi,\psi) = N2^{(1/2)(n_1+n_2)}e^{i(n_2-n_1)\phi}e^{-(1/2)r^2}(r\sin(\psi)\sin(\theta))^{(n_1+n_2)}$$
$$\times \mathcal{H}_{n_3}(r\sin(\psi)\cos(\theta))\mathcal{H}_{n_4}(r\cos(\psi))\Sigma_{i=0}^{n_1}(-1)^i i! \binom{n_1}{i}\binom{n_2}{i}(r\sin(\psi)\sin(\theta))^{2i}, \quad (3.11)$$

where \mathcal{H}_n is the Hermit polynomial of degree n and $\binom{n}{r} = \frac{n!}{r!(n-r)!}$. Now with the same prescription used in the previous section, we can eliminate ϕ , by Fourier transforming over it. Hence, by the Fourier transformation over ϕ , the 4-oscillator Hamiltonian reduces to the following Hamiltonian:

$$H_m(r,\theta,\psi) = -\frac{1}{2} \left[\frac{1}{r^3} \partial_r r^3 \partial_r \right]$$

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$$+\frac{1}{r^2}\left(\partial_{\psi}^2 + 2\cot(\psi)\partial_{\psi} + \frac{1}{\sin^2(\psi)}(\partial_{\theta}^2 + \cot(\theta)\partial_{\theta} - \frac{m^2}{\sin^2(\theta)})\right) + \frac{1}{2}\omega^2 r^2, \quad (3.12)$$

where after similarity transformation through function $r^{1/2}$, it reduces to

$$\tilde{H}_{m}(r,\theta,\psi) = r^{1/2}H_{m}(r,\theta,\psi)r^{-1/2} = -\frac{1}{2}\left[\frac{1}{r^{2}}\partial_{r}r^{2}\partial_{r} + \frac{1}{r^{2}}\left(\partial_{\psi}^{2} + 2\cot(\psi)\partial_{\psi} + \frac{1}{\sin^{2}(\psi)}(\partial_{\theta}^{2} + \cot(\theta)\partial_{\theta} - \frac{m^{2}}{\sin^{2}(\theta)}\right)\right] + \frac{1}{2}\omega^{2}r^{2} + \frac{3}{8r^{2}}.$$
(3.13)

On the other hand, the Hamiltonian $H_m(r, \theta, \psi)$ given by (3.12) can be written in the following form

$$H_m(r,\theta,\psi) = \omega (A_1^{\dagger}(m+1)A_1(m) + A_2^{\dagger}(m-1)A_2(m) + a_3^{\dagger}a_3 + a_4^{\dagger}a_4 + 2), \qquad (3.14)$$

with

$$A_{1}(m) = \frac{i}{\sqrt{2}} \sqrt{\frac{\omega}{2}} \left[r \sin(\psi) \sin(\theta) + \frac{1}{r} \frac{1}{\sin(\psi)} \sin(\theta) + \frac{1}{r} \frac{1}{\sin(\psi)} \frac{1}{\sin(\psi)} \sin(\theta) \right], \quad (3.15)$$

$$A_{1}^{\dagger}(m) = -\frac{i}{\sqrt{2}} \sqrt{\frac{\omega}{2}} \left[r \sin(\psi) \sin(\theta) - \frac{1}{r} \frac{m}{\sin(\psi)} \sin(\theta) \right], \quad (3.16)$$

$$-\frac{1}{\omega} \left(\sin(\psi) \sin(\theta) \partial_{r} + \frac{1}{r} \cos(\psi) \sin(\theta) \partial_{\psi} + \frac{1}{r} \frac{\cos(\theta)}{\sin(\psi)} \partial_{\theta} + \frac{1}{r} \frac{m}{\sin(\psi)} \sin(\theta) \right) \right], \quad (3.16)$$

$$A_{2}(m) = -\frac{i}{\sqrt{2}} \sqrt{\frac{\omega}{2}} \left[r \sin(\psi) \sin(\theta) + \frac{1}{r} \frac{1}{\sin(\psi)} \sin(\theta) \right], \quad (3.17)$$

$$A_{2}^{\dagger}(m) = \frac{i}{\sqrt{2}} \sqrt{\frac{\omega}{2}} \left[r \sin(\psi) \sin(\theta) - \frac{1}{\omega} + \frac{1}{r} \cos(\psi) \sin(\theta) \partial_{\psi} + \frac{1}{r} \frac{\cos(\theta)}{\sin(\psi)} \partial_{\theta} - \frac{1}{r} \frac{m}{\sin(\psi)} \sin(\theta) \right] \right], \quad (3.18)$$

It is straightforward to derive the following relation between Hamiltonian (3.12) and operator $A_i(m)(A_i^{\dagger}(m)), i= 1, 2$:

$$H(m-1)A_{1}^{\dagger}(m) - A_{1}^{\dagger}(m)H(m) = \omega A_{1}^{\dagger}(m),$$

$$H(m+1)A_{2}^{\dagger}(m) - A_{2}^{\dagger}(m)H(m) = \omega A_{2}^{\dagger}(m),$$

$$H(m+1)A_{1}(m) - A_{1}(m)H(m) = -\omega A_{1}(m),$$

$$H(m-1)A_{2}(m) - A_{2}(m)H(m) = -\omega A_{2}(m),$$
(3.19)

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where $H(m) := H_m(r, \theta, \psi)$. The above relations indicate that Hamiltonian (3.12) possesses shape invariance symmetry. To see this, we consider the Fourier transformation of eigenvalue equation (3.9):

$$H(m)\Psi_{(n,m,n_3,n_4)}(r,\theta,\psi) = E_{(n,n_3,n_4)}\Psi_{(n,m,n_3,n_4)}(r,\theta,\psi),$$
(3.20)

where $n = n_1 + n_2$, $m = n_2 - n_1$ and $E_{(n,n_3,n_4)} = (n + n_3 + n_4 + 2)\omega$. Since n_1 and n_2 are positive integers, therefore n is also a positive integer but m is an integer. For a given value of m, the quantum number n can be either even or odd integer, since, quantum numbers n_1 and n_2 vary by the same amount, so that m remains constant. Actually for some given value of m, the quantum number n can take the following values

$$n = |m|, |m| + 2, |m| + 4, \dots$$

On the other hand, in terms of n, the quantum number m can take the following values

$$m = -n, -n + 2, ..., n - 2, n$$

It is interesting to see that energy of Hamiltonian $H_m(r, \theta, \psi)$ is independent of m, hence these Hamiltonians are isospectral which is due to the existence of shape invariance symmetry as we show below.

Operating the operator $A_1^{\dagger}(m)$ on both sides of the eigenvalue relation (3.20) and using the relations (3.19), we get

$$H(m-1)(A_{1}^{\dagger}(m)\Psi_{n,m}(r,\theta,\psi)) = (E_{n}+\omega)(A_{1}^{\dagger}(m)\Psi_{n,m}(r,\theta,\psi)),$$

therefore, $A_1^{\dagger}(m)\Psi_{n,m}(r,\theta,\psi)$) corresponds to the eigenfunction of H(m-1) with corresponding eigenvalue E_{n+1} , that is

$$A_{1}^{\dagger}(m)\Psi_{n,m}(r,\theta,\psi) = \sqrt{\frac{n-m}{2}} + 1\Psi_{n+1,m-1}(r,\theta,\psi),$$

where $\Psi_{n,m}(r,\theta,\psi) := \Psi_{(n,m,n_3,n_4)}(r,\theta,\psi)$ and $E_n := E_{(n,n_3,n_4)}$. Similarly, operating $A_2^{\dagger}(m)$ on both sides of (3.20) and using (3.19) we get:

$$H(m+1)(A_{2}^{\dagger}(m)\Psi_{n,m}(r,\theta,\psi)) = (E_{n}+\omega)(A_{2}^{\dagger}(m)\Psi_{n,m}(r,\theta,\psi)),$$

which leads to

$$A_{2}^{\dagger}(m)\Psi_{n,m}(r,\theta,\psi) = \sqrt{\frac{n+m}{2} + 1}\Psi_{n+1,m+1}(r,\theta,\psi).$$

Also by acting the operators $A_1(m)$ and $A_2(m)$ on the eigenvalue relation (3.20) and using the relations (3.19) we obtain

$$H(m+1)(A_1(m)\Psi_{n,m}(r,\theta,\psi)) = (E_n - \omega)(A_1(m)\Psi_{n,m}(r,\theta,\psi)),$$
$$H(m-1)(A_2(m)\Psi_{n,m}(r,\theta,\psi)) = (E_n - \omega(A_2(m)\Psi_{n,m}(r,\theta,\psi)),$$

which imply the following relations

$$A_{1}(m)\Psi_{n,m}(r,\theta,\psi) = \sqrt{\frac{n-m}{2}}\Psi_{n-1,m+1}(r,\theta,\psi),$$
$$A_{2}(m)\Psi_{n,m}(r,\theta,\psi) = \sqrt{\frac{n+m}{2}}\Psi_{n-1,m-1}(r,\theta,\psi).$$

From the above relations we conclude that the pair of operators $(A_2(m), A_1^{\dagger}(m))$ or $(A_2^{\dagger}(m), A_1(m))$ acting at eigenfunction $\Psi_{n,m}(r, \theta, \psi)$ of Hamiltonian H(m), give eigenfunction of Hamiltonian $H(m \pm 2)$ with same the energy as follows:

$$A_{2}(m-1)A_{1}^{\dagger}(m)\Psi_{n,m}(r,\theta,\psi) = \frac{1}{2}\sqrt{(n+m)(n-m+2)}\Psi_{n,m-2}(r,\theta,\psi),$$
$$A_{2}^{\dagger}(m+1)A_{1}(m)\Psi_{n,m}(r,\theta,\psi) = \frac{1}{2}\sqrt{(n-m)(n+m+2)}\Psi_{n,m-2}(r,\theta,\psi).$$

Now introducing the operators $A_{-}(m) := A_{2}(m-1)A_{1}^{\dagger}(m)$ and $A_{+}(m) := A_{2}^{\dagger}(m-1)A_{1}(m-2)$, we have:

$$A_{-}(m)A_{+}(m)\Psi_{n,m-2}(r,\theta,\psi) = E(n,m)\Psi_{n,m-2}(r,\theta,\psi),$$
$$A_{+}(m)A_{-}(m)\Psi_{n,m}(r,\theta,\psi) = E(n,m)\Psi_{n,m}(r,\theta,\psi),$$

where

$$E(n,m) = \frac{1}{4}(n+m)(n-m+2).$$

The above relations show the existence of shape invariance symmetry between the Hamiltonian H(m) and H(m-2) with same given eigenvalue E_n . Hence we can obtain the

$$\Psi_{n,m}(r,\theta,\psi) = c^{-1}A_{-}(m+2)A_{-}(m+4)...A_{-}(n-2)A_{-}(n)\Psi_{n,n}(r,\theta,\psi),$$

where

$$c = \frac{1}{2^{\frac{n-m}{2}}}\sqrt{(n-m)!!2n(2n-2)...(n+m+4)(n+m+2)},$$

and

$$(n-m)!! = (n-m)(n-m-2)...4 \times 2,$$

$$\Psi_{n,n}(r,\theta,\psi) \equiv \Psi_{(n,n,n_3,n_4)}(r,\theta,\psi) = (a_3^{\dagger})^{n_3} (a_4^{\dagger})^{n_4} A_2^{\dagger}(n-1) A_2^{\dagger}(n-2) \dots A_2^{\dagger}(1) A_2^{\dagger}(0) e^{-\frac{1}{2}\omega r^2} A_2^{\dagger}(n-1) A_2^{\dagger}(n-2) \dots A_2^{\dagger}(n-2$$

Of course we can obtain the eigenfunction $\Psi_{(n,m,n_3,n_4)}(r,\theta,\psi)$ by reduction of coordinate ϕ via Fourier transformation of (3.11), which has the following form:

$$\Psi_{(n,m,n_3,n_4)}(r,\theta,\psi) = N2^{\frac{n}{2}}e^{-(1/2)r^2}(r\sin(\psi)\sin(\theta))^n$$
$$\times \mathcal{H}_{n_3}(\sin(\psi)\sin(\theta))\mathcal{H}_{n_4}(r\cos(\psi))\Sigma_{i=0}^{\frac{n-m}{2}}(-1)^i i! \begin{pmatrix} \frac{n-m}{2} \\ i \end{pmatrix} \begin{pmatrix} \frac{n+m}{2} \\ i \end{pmatrix} (r\sin(\psi)\sin(\theta))^{2i}.$$

IV CONCLUSION

Here in this work having Fourier transformed 3 and 4-dimensional Hamiltonians associated with su(2) and Heisenberg Lie algebra we have been able to obtain 2 and 3-dimensional Hamiltonian whit shape invariance symmetry. It would be interesting to obtain many-body Hamiltonian in one dimension or higher, which possesses shape invariance symmetry by appropriate Fourier transformation over some coordinates of Hamiltonian associated with higher ranks semisimple and non semisimple Lie algebra. This is under investigation.

ACKNOWLEDGEMENT

We wish to thank Dr. S.K.A. Seyed Yagoobi for carefully reading the article and for his constructive comments.

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