Some Remarks Concerning the Feynman "Integral over All Paths" Method

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Dedicated to Roman S. Ingarden on the occasion of his 80th birthday

Abstract

Suppose we have two nonequivalent but s-equivalent Lagrange functions, the question arises: are they both equally well fitted for the Feynman quantization procedure or do they lead to two different quantization schemes.

1. The goal of this note is to exhibit the following problem. It is well known that in the quantization prescription, based on the Feynman "integral over all paths" the *classical* Lagrange function is used in the exponent of the integrand of the Feynman integral. The physical content of a dynamical system is, however, mainly characterized by the equations of motion of this systems; the Lagrange function, if such one exists at all for these equations, plays a secondary rôle, as there can be many nonequivalent Lagrange functions linked to equations of motion (Euler Lagrange Equations), yielding the same set of solutions - so called s-equivalent equations.

The question arises: suppose we have two nonequivalent but s-equivalent Lagrange functions, are they both equally well fitted for the Feynman quantization procedure or do they lead to two different quantization schemes.

2. To begin with let us consider the case of one classical particle in a (1+1)-dimensional space-time and the largest set of s-equivalent Lagrange functions, corresponding to the equation of motion of this particle. We do not need to specify the form of this equation; to each equation written in the normal form, viz.

$$\ddot{x} = f(x, \dot{x}, t) \tag{1}$$

corresponds always a Lagrange function [1]. The inverse problem for the case of (1+1) dimensions was treated extensively by many scientists [2], [3].

It is known that the most general form of an autonomous Lagrange function, s-equivalent to a given autonomous Lagrange function $L(x\dot{x})$, the form of which we do not specify, is

$$L' = \dot{x} \int_c^{\dot{x}} G(x, u) du - \Sigma(H) \tag{2}$$

where

$$G(x, \dot{x}) \equiv \frac{d\Sigma(H)}{dH} \frac{\partial^2 L}{\partial \dot{x}^2} , \tag{3}$$

$$H \equiv \dot{x} \frac{\partial L}{\partial \dot{x}} - L , \tag{4}$$

and $\Sigma(z)$ is an arbitrary differentiable function of z. The constant c is so chosen that the integral on the r.h.s. of (2) does not diverge[1]. The Hamilton function reads

$$H' = \dot{x} \frac{\partial L'}{\partial \dot{x}} - L' = \Sigma(H) + const.. \tag{5}$$

The Lagrange function L' for different choices of Σ, assuming $\frac{d\Sigma(z)}{dz}$ is not a constant, are not equivalent to each other as well as to L; in other words they do not differ from each other by a function $\frac{d\Phi(t)}{dt}$.

To make things more specific let us now specify the original Lagrange function L as well as Σ and c, viz.

$$L = \frac{1}{2}\dot{x}^2 - V(x) , \tag{6}$$

$$\Sigma(H) = \frac{1}{2}H^2 , \tag{7}$$

$$c = 0 . \tag{8}$$

[1] we could even assume $c = c(x)$.

Then

$$H = \frac{1}{2}\dot{x}^2 \; + V(x)\,, \tag{9}$$

$$L' = \frac{1}{24}\,\dot{x}^4 + \frac{1}{2}\dot{x}^2\,V - \frac{1}{2}V^2\,, \tag{10}$$

$$H' = \frac{1}{2}H^2 = \frac{1}{8}\dot{x}^4 + \frac{1}{2}\dot{x}^2\,V + \frac{1}{2}V^2\,, \tag{11}$$

and[2]

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[2]With the notation $\dot{x} = z$ we have

$$\frac{dz}{dp'} = \frac{1}{H} = (2H')^{-\frac{1}{2}}\,, \tag{A}$$

$$\frac{d^2z}{dp'^2} = -\frac{z}{H^3}\,.$$

For z becoming large $\frac{dz}{dp'}$ vanishes like z^{-2} and $\frac{d^3z}{dp'^2}$ like z^{-5}.

Using the canonical Hamilton equation

$$z = \frac{\partial H'}{\partial p'}$$

and taking into account (A) we get the equation for H', viz.

$$\frac{\partial^2 H'(x,p')}{\partial p'^2} - \frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{H'(x,p')}} = 0\,. \tag{B}$$

The particular solution of (B) independent of x reads

$$\widehat{H}\,' = \left(\frac{81}{32}\right)^{\frac{1}{3}}\,p'^{\frac{4}{3}}$$

which corresponds to large p' and $\dot{x}$ and discarding $V(x)$. The application of the first order perturbative procedure for small V and $\frac{dV}{dx}$ as well as the use of canonical Hamilton equations yields

$$\dot{x} = (6p')^{\frac{1}{3}} - 2\,(6p')^{-\frac{1}{3}}\;V$$

and

$$H\,' = \left(\frac{81}{32}\right)^{\frac{1}{3}}\,p'^{\frac{4}{3}} - \left[\left(\frac{9}{2}\right)^{\frac{1}{2}}\;V + a\right]p'^{\frac{2}{3}}$$

where a is a small number.

$$p' \equiv \frac{\partial L'}{\partial \dot{x}} = \frac{1}{6}\,\dot{x}^3 + \dot{x}\,V(x)\,. \tag{12}$$

Relation (12) is an algebraic equation of third degree with respect to

$$\dot{x}(p', x) = -\dot{x}(-p', x)\,.$$

For

$$p' = 0 \qquad \text{and} \qquad V(x) > 0 \tag{13}$$

we have three roots of (12)

$$\dot{x}_1 = 0\,, \qquad \dot{x}_{2,3} = \pm i\sqrt{6\,V(x)}\,. \tag{14}$$

For obvious reasons we choose the real solution. In case V is not always positive but it is bounded from below we may change V in (6) by adding to it a properly chosen constant so that V is then always positive.

The solution of (12) reads

$$\dot{x} = \frac{p'}{V} - \frac{1}{6}\frac{1}{V}\left(\frac{p'}{V}\right)^3 + \frac{1}{12}\,\frac{1}{V^2}\left(\frac{p'}{V}\right)^5 - \frac{1}{18}\,\frac{1}{V^3}\left(\frac{p'}{V}\right)^7 + o\left(\left(\frac{p'}{V}\right)^9\right)\,. \tag{15}$$

Notice that the few first terms of (15) coincide with

$$\frac{p'}{V}\left[1 + \frac{1}{6}\ln\left(1 - \frac{1}{V}\left(\frac{p'}{V}\right)^2\right)\right]\,. \tag{16}$$

For large $\dot{x}$ and p'

$$\dot{x} = (6\,p')^{\frac{1}{3}}\,. \tag{17}$$

We have

$$H' = \frac{1}{2}\,V^2 + \frac{1}{2}\,\frac{p'^{\,2}}{V} - \frac{1}{24}\,\frac{p'^{\,4}}{V^4} + o\left(\left(\frac{p'}{V}\right)^8\right)\,. \tag{18}$$

3. Let us now investigate the quantal case of one particle presented in the language of Feynman's approach.

It is well known [4], [3] that in case the Hamiltonian function consists of two terms from which one depends only on p and the other one only on x, the formula of Feynman‘s “integral over all paths” with the classical Lagrange function in the exponent of the integral can be recovered from standard quantum mechanical approach.

To remind the Reader on this procedure let us consider the Hamiltonian function (9),[3], viz.

$$H = \frac{1}{2}p^2 + V(x)\,. \tag{19}$$

Starting from the first principles of Quantum Mechanics we have for the transition amplitude

$$\phi(x', t_2|x, t_1) = \langle x'|\exp\{-i\widehat{H}(t_2 - t_1)\}|x\rangle\,, \tag{20}$$

where $\langle\cdot|$ and $|\cdot\rangle$ denote the bra - and ket - states resp. and $\widehat{H}$ is the Hamilton operator

$$\widehat{H} \equiv \frac{1}{2}\widehat{p}^{\,2} + V(x)\,, \qquad \widehat{p} = -i\,\frac{\partial}{\partial x}\,. \tag{21}$$

We may write (20) as follows

$$\begin{aligned}\langle x'|\exp\{-i\widehat{H}\,t\}|x\rangle &= \lim_{\substack{\Delta t\to 0\\ \Delta t n = t}} \int dx_{n-1}\ldots\int dx_1 \langle x'|e^{-i\widehat{H}\,\Delta t}|x_{n-1}\rangle\langle x_{n-1}|\ldots\\ &\quad \ldots|x_1\rangle\langle x_1|e^{-i\widehat{H}\,\Delta t}|x\rangle\,. \end{aligned}\tag{22}$$

If we use the formula

$$e^{(a+b)t} = \lim_{n\to\infty}\left(e^{a\frac{t}{n}}e^{b\frac{t}{n}}\right)^n \tag{23}$$

then

$$\begin{aligned}\langle x'|\exp\{-i\widehat{H}\,t\}|x\rangle &= \lim_{\substack{\Delta t\to 0\\ \Delta t n = t}} \int dx_{n-1}\ldots\int dx_1 \langle x'|e^{-i\frac{\widehat{p}^2}{2}\Delta t}|x_{n-1}\rangle\langle x_{n-1}|\ldots\\ &\quad \ldots|x_1\rangle\langle x_1|e^{-i\frac{\widehat{p}^2}{2}\Delta t}|x\rangle e^{-iV(x)t}\,. \end{aligned}\tag{24}$$

Further we have

$$\begin{aligned}\langle x'|e^{-i\frac{\widehat{p}^2}{2}\Delta t}|x\rangle &= \int dp\langle x'|e^{-i\frac{\widehat{p}^2}{2}\Delta t}|p\rangle\langle p|x\rangle\\ &= \frac{1}{2\pi}\int dp e^{-i\frac{p^2}{2}\Delta t}e^{-ip(x'-x)}\,. \end{aligned}\tag{25}$$

[3]We put the mass of the particle equal to one ($m = 1$).

as

$$\langle p|x\rangle = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} e^{ipx} . \qquad (26)$$

Notice that

$$\begin{aligned} -\frac{i\Delta t}{2}p^2 - i\,p(x'-x) &= -\frac{i\Delta t}{2}\left(p^2 + \frac{2}{\Delta t}p(x'-x) + \frac{1}{(\Delta t)^2}(x'-x)^2\right) \\ &\quad + \frac{i}{2}\frac{(x'-x)^2}{\Delta t} . \end{aligned} \qquad (27)$$

Consequently

$$\begin{aligned} \langle x'|e^{-i\frac{\widehat{p}^2}{2}\Delta t}|x\rangle &= \frac{1}{2\pi}\int dp\, \exp\left\{-\frac{i\Delta t}{2}\left(p + \frac{x'-x}{\Delta t}\right)^2\right\} \exp\left\{\frac{i}{2}\frac{(x-x')^2}{\Delta t}\right\} \\ &= (2\pi i\Delta t)^{-\frac{1}{2}} \exp\left\{\frac{i}{2}\left(\frac{x'-x}{\Delta t}\right)^2 \Delta t\right\} \end{aligned} \qquad (28)$$

where we used the saddle point method to evaluate

$$\frac{1}{2\pi}\int dp\, \exp\left\{-\frac{i\Delta t}{2}\left(p + \frac{x'-x}{\Delta t}\right)^2\right\} = (2\pi i\Delta t)^{-\frac{1}{2}} . \qquad (29)$$

Taking into account (24) and (28) we get eventually

$$\begin{aligned} \phi(\underline{x}', t_2|\underline{x}, t_1) &= \lim_{\substack{n\to\infty \\ n\Delta t = t_2 - t_1}} \prod_{j=1}^{n-1}\int dx_j \prod_{k=1}^{n}(2\pi i\Delta t)^{\frac{1}{2}} \\ &\quad \cdot \exp\left\{i\left[\left(\frac{x_k - x_{k-1}}{\Delta t}\right)^2 - V(x)\right]\Delta t\right\} \end{aligned} \qquad (30)$$

where $x_n \equiv x'$, $x_0 \equiv x$. Thus in the exponent in (30) we have, indeed,

$$i\int_{t_1}^{t_2} L(x(t), \dot{x}(t))dt , \qquad (31)$$

is conjectured at the start.

The procedure presented above can not be applied in case of H' and L' given by (18) and (10) resp. as

$$\widehat{H}\,' = \frac{1}{2}V^2 + \frac{1}{6}\left(\frac{1}{V}\widehat{p}\,^2 + \widehat{p}\,\frac{1}{V}\widehat{p} + \widehat{p}\,^2\frac{1}{V}\right) + \dots\,, \tag{32}$$

is a power series in expressions of type $\frac{1}{V^m}\widehat{p}\,^l$, $\widehat{p}\,^l\frac{1}{V^m}$, $l, m = 1, 2, \dots$ and $\widehat{p}$ and x can not be separated. So a new quantization prescription is needed.

It is also not at all clear whether L', given by (10), inserted into the exponent of the integral instead of L in (31) yields the same physical results as using L of (6). It seems rather that it leads to different value of the transition amplitude and to a different kind of quantization.

The question to be answered is: what are the limitations in using the Feynman rule for the "integral over all paths". Unfortunately, I do not feel to be able to give an answer to it. Thus the problem remains open, at least for me.

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