Gap Probabilities for Edge Intervals in Finite Gaussian and Jacobi Unitary Matrix Ensembles

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Abstract

The probabilities for gaps in the eigenvalue spectrum of the finite dimension $N \times N$ random matrix Hermite and Jacobi unitary ensembles on some single and disconnected double intervals are found. These are cases where a reflection symmetry exists and the probability factors into two other related probabilities, defined on single intervals. Our investigation uses the system of partial differential equations arising from the Fredholm determinant expression for the gap probability and the differential-recurrence equations satisfied by Hermite and Jacobi orthogonal polynomials. In our study we find second and third order nonlinear ordinary differential equations defining the probabilities in the general N case. For N = 1 and N = 2 the probabilities and thus the solution of the equations are given explicitly. An asymptotic expansion for large gap size is obtained from the equation in the Hermite case, and also studied is the scaling at the edge of the Hermite spectrum as $N \to \infty$, and the Jacobi to Hermite limit; these last two studies make correspondence to other cases reported here or known previously. Moreover, the differential equation arising in the Hermite ensemble is solved in terms of an explicit rational function of a Painlevé-V transcendent and its derivative, and an analogous solution is provided in the two Jacobi cases but this time involving a Painlevé-VI transcendent.

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1 Introduction

It is a celebrated discovery of Jimbo, Miwa, Môri and Sato [14] that the probability of an eigenvalue free region in the bulk of the infinite GUE (Gaussian unitary ensemble of random Hermitian matrices) can be expressed exactly in terms of a Painlevé-V transcendent. This has initiated a number of studies [18, 1, 11] which, in addition to clarifying the general setting of the exact result, give formalisms that allow analogous results to be obtained in other cases. For example, the probability of an eigenvalue free region at the edge of the infinite GUE (appropriately scaled) has been expressed in terms of the Painlevé-II transcendent [17], while a Painlevé-III transcendent has been shown to determine the distribution of the smallest eigenvalue in the infinite LUE (Laguerre unitary ensemble of non-negative matrices of the form $A^{\dagger}A$ with A complex [18]). Moreover, for the finite classical ensembles with unitary symmetry, the probability of a single eigenvalue free region which includes an endpoint of the support of the weight has been expressed in terms of solutions of certain non-linear equations [18].

It is the objective of this work to provide three new evaluations of gap probabilities for particular finite classical random matrix ensembles with unitary symmetry. The ensembles considered are the finite GUE and the symmetric Jacobi unitary ensemble (JUE). We recall the eigenvalue p.d.f. for an ensemble with unitary symmetry is of the form

$$\prod_{l=1}^{N} w_2(\lambda_l) \prod_{1 \le j < k \le N} |\lambda_k - \lambda_j|^2,$$
(1.1)

where the weight function $w_2(\lambda)$ determines the specific unitary ensemble:

$$w_2(\lambda) = \begin{cases} e^{-\lambda^2}, & \text{GUE} \\ (1-\lambda)^{\alpha}(1+\lambda)^{\beta}, & \text{JUE} \end{cases}$$
(1.2)

(the symmetric Jacobi ensemble refers to the case $\alpha = \beta$ of the JUE). These ensembles can be realised in terms of matrices with independent Gaussian elements. For the GUE, each matrix X say must be Hermitian and have its diagonal elements x_{jj} (which must be real) and upper triangular elements $x_{jk} = u_{jk} + iv_{jk}$ chosen with p.d.f. N(0, $1/\sqrt{2}$) and N(0, 1/2) + iN(0, 1/2). For the JUE one first constructs [15] auxiliary rectangular $M_1 \times N$ and $M_2 \times N$ ($M_1, M_2 \ge N$) matrices a and b respectively, with complex elements independently distributed according to N(0, $1/\sqrt{2}$) + iN(0, $1/\sqrt{2}$). Then with $A = a^{\dagger}a$, $B = b^{\dagger}b$, it can be shown that the distribution of the eigenvalues of $A(A + B)^{-1}$ is an example of (1.1) with

$$w_2(\lambda) = \lambda^{M_1 - N} (1 - \lambda)^{M_2 - N}, \quad 0 < \lambda < 1.$$
 (1.3)

The change of variables $\lambda \mapsto (1-\lambda)/2$ shows that this is an example of the JUE.

With $E_2(0; I; w_2(\lambda); N)$ denoting the probability that there are no eigenvalues in the interval I of an ensemble with eigenvalue p.d.f. (1.1), the probabilities to be calculated are

$$E_2(0; (-\infty, -s) \cup (s, \infty); e^{-\lambda^2}; N)$$
(1.4)

and

$$E_2(0; (-1, -s) \cup (s, 1); (1 - \lambda^2)^{\alpha}; N), \qquad E_2(0; (-s, s); (1 - \lambda^2)^{\alpha}; N).$$
(1.5)

The quantity (1.4) gives the probability that there are no eigenvalues in the GUE with modulus greater than s, as does the first quantity in (1.5) for the JUE. The final quantity in

(1.5) gives the probability that there are no eigenvalues in the JUE with modulus less than s (the analogous probability for the GUE has previously been evaluated [18]).

These probabilities are special because, like the situation in which I is a single interval which includes an endpoint of the support of the weight function noted above, we will show that they can be evaluated exactly in terms of the solution of certain non-linear equations. Another motivating factor for our study is a recently discovered [9] general identity satisfied by the probability $E_2(0; I; w_2(\lambda); N)$ applicable whenever $w_2(\lambda)$ is even and I is symmetrical about the origin. The identity states

$$E_2(0; I; w_2(\lambda); N) = E_2(0; I^+; y^{-1/2} w_2(y^{1/2}); [(N+1)/2])$$

$$\times E_2(0; I^+; y^{+1/2} w_2(y^{1/2}); [N/2])$$
(1.6)

where I^+ denotes the portion of I on \mathbf{R}^+ in the variable $y = \lambda^2$ and $w_2(y^{1/2}) := 0$ for y < 0. The probabilities (1.4) and (1.5) are all of the form required by the LHS of this identity, and are thereby related to

$$E_2(0; (s^2, \infty); y^{\pm 1/2} e^{-y}; m)$$
(1.7)

and

$$E_2(0; (s^2, 1); y^{\pm 1/2}(1-y)^{\alpha}; m), \quad E_2(0; (0, s^2); y^{\pm 1/2}(1-y)^{\alpha}; m)$$
(1.8)

(m = [(N + 1)/2], [N/2]) respectively. The weight functions in (1.7) and (1.8) are again classical, but now the eigenvalues are excluded from a single interval which includes an endpoint of the support of the weight function. The corresponding probabilities are known in terms of Painlevé-V and VI transcendents from the works of [18] and [11]. Thus as a by-product of providing independent evaluations of the LHS of (1.6), we will be providing inter-relationships between solutions of nonlinear equations. This theme is further developed in [9].

In Section 2 the formalism of Tracy and Widom [18] giving coupled differential equations for the gap probability and some auxiliary quantities is revised. In Section 3 these coupled equations are reduced to a single ordinary differential equation specifying the probability (1.4), and this equation is used to compute the large gap size behaviour of the probability, as well as various scaling limits. A similar study is undertaken in Section 4 for the probabilities (1.5). In Section 5 the solution of the second order equations of Section 3 (Hermite case) and third order equations of Section 4 (Jacobi case) are given in terms of certain Painlevé-V and Painlevé-VI transcendents respectively.

2 The General Formalism

We are going to consider a set of the eigenvalue spectrum arising from the matrix ensembles consisting of an arbitrary number of disconnected intervals. In general the eigenvalues may be excluded from M disjoint intervals which together form I. Thus with the endpoints of these intervals denoted $\{a_j\}_{j=1}^{2M}$,

$$I = \bigcup_{m \ge 1}^{M} (a_{2m-1}, a_{2m}) .$$
 (2.1)

The probability of no eigenvalues being found in this interval is given by the general expression (see e.g. [10])

$$E(0;I) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_I dx_1 \dots \int_I dx_n \rho_n(x_1,\dots,x_n) , \qquad (2.2)$$

where ρ_n is the *n*-point distribution function of the eigenvalue p.d.f. For matrix ensembles with unitary symmetry the eigenvalue p.d.f. is proportional to (1.2), and the corresponding *n*point distribution function is given in terms of the orthonormal polynomials $\{p_j(x)\}_{j=0,1,2,...}$ associated with the weight function $w_2(x)$ according to the formula

$$\rho_n(x_1, \dots, x_n) = \det \left[K_N(x_i, x_j) \right]_{1 \le i, j \le n} , \qquad (2.3)$$

where

$$K_N(x,y) = [w_2(x)w_2(y)]^{1/2} \sum_{l=0}^{N-1} p_l(x)p_l(y) .$$
(2.4)

Substituting into (2.2), the Fredholm theory of integral operators then gives

$$E(0;I) = \det(\mathbb{I} - \mathbb{K}_N) , \qquad (2.5)$$

where \mathbb{K}_N is the integral operator with kernel $K_N(x, y)$ defined on the interval I. A crucial point is that (2.4) can be summed according to the Christoffel-Darboux formula and so written in the special form [12]

$$K_N(x,y) = \frac{\phi(x)\psi(y) - \phi(y)\psi(x)}{x - y} , \qquad (2.6)$$

where with a_N denoting the coefficient of x^N in $p_N(x)$

$$\phi(x) = \left(\frac{a_{N-1}}{a_N}w_2(x)\right)^{1/2} p_N(x) ,$$

$$\psi(x) = \left(\frac{a_{N-1}}{a_N}w_2(x)\right)^{1/2} p_{N-1}(x) .$$
(2.7)

We suppose furthermore that $\phi(x)$ and $\psi(x)$ satisfy the recurrence-differential relations

$$m(x)\phi'(x) = A(x)\phi(x) + B(x)\psi(x) ,$$

$$m(x)\psi'(x) = -C(x)\phi(x) - A(x)\psi(x) ,$$
(2.8)

where the coefficient functions m(x), A(x), B(x), C(x) are polynomials in x.

In the above setting Tracy and Widom [18] have derived a set of coupled differential equations for the determinant (2.5), as well as some auxiliary quantities. The latter we introduce in the following definitions.

Definition 1 Let $A \doteq A(x, y)$ denote that the integral operator A has kernel A(x, y). Then the kernels $\rho(x, y)$ and R(x, y) are specified by

$$(1-K)^{-1} \doteq \rho(x,y) ,$$

 $K(1-K)^{-1} \doteq R(x,y) ,$
(2.9)

The operator $K(1-K)^{-1}$ is called the resolvent and R(x, y) the resolvent kernel.

Definition 2 For $k \in \mathbb{Z}_{\geq 0}$ the functions Q_k and P_k are defined by

$$Q_k(x) = \int_I dy \ \rho(x, y) y^k \phi(y) ,$$

$$P_k(x) = \int_I dy \ \rho(x, y) y^k \psi(y) ,$$
(2.10)

and their values at the endpoints a_j of I are denoted q_{kj} , p_{kj} so that

$$q_{kj} = Q_k(a_j) \equiv \lim_{x \to a_j} Q_k(x)$$

$$p_{kj} = P_k(a_j) \equiv \lim_{x \to a_j} P_k(x)$$
(2.11)

Where there is no confusion, we denote q_{0j} , p_{0j} by q_j , p_j .

Definition 3 The inner products u, v, w are defined by

$$u = \langle \phi | Q \rangle = \int_{I} dy \ Q_{0}(y)\phi(y) ,$$

$$v = \langle \psi | Q \rangle = \int_{I} dy \ Q_{0}(y)\psi(y) = \langle \phi | P \rangle = \int_{I} dy \ P_{0}(y)\phi(y) , \qquad (2.12)$$

$$w = \langle \psi | P \rangle = \int_{I} dy \ P_{0}(y)\psi(y) .$$

Our goal is to characterise the probabilities (1.4) and (1.5) as the solution of certain nonlinear differential equations. Following [18] this is achieved by specifying partial differential equations for the quantities q_j, p_j, u, v, w and $R(a_j, a_k)$. These equations come in two types: a set of universal equations which are independent of the recurrence-differential equations (2.8), and a second set of equations which depend on the details of (2.8). Let us first present the former.

Proposition 1 For general functions $\phi(x), \psi(x)$ we have the relations

$$\frac{\partial}{\partial a_j} \log \det(1-K) = (-1)^{j-1} R(a_j, a_j) , \qquad (2.13)$$

and for $j \neq k$,

$$R(a_j, a_k) = \frac{q_j p_k - p_j q_k}{a_j - a_k} , \qquad (2.14)$$

and

$$\frac{\partial}{\partial a_k} R(a_j, a_j) = (-1)^k R(a_j, a_k) R(a_k, a_j) , \qquad (2.15)$$

with

$$\frac{\partial q_j}{\partial a_k} = (-1)^k R(a_j, a_k) q_k ,
\frac{\partial p_j}{\partial a_k} = (-1)^k R(a_j, a_k) p_k ,$$
(2.16)

and

$$\frac{\partial u}{\partial a_k} = (-1)^k q_k^2 ,$$

$$\frac{\partial v}{\partial a_k} = (-1)^k q_k p_k ,$$

$$\frac{\partial w}{\partial a_k} = (-1)^k p_k^2 .$$
(2.17)

The second set of equations, which depend on the details of (2.8), give the j = k cases of (2.14) and (2.16). Now for weights (1.2) the equations hold for m(x) a quadratic and A(x), B(x), C(x) linear functions, and thus of the general form

$$m(x) = \mu_0 + \mu_1 x + \mu_2 x^2 ,$$

$$A(x) = \alpha_0 + \alpha_1 x ,$$

$$B(x) = \beta_0 + \beta_1 x ,$$

$$C(x) = \gamma_0 + \gamma_1 x .$$

(2.18)

One then has the following equations [18].

Proposition 2 In the case that $\phi(x), \psi(x)$ satisfy the equations (2.8) with coefficient functions (2.18) we have

$$m_{i} \frac{\partial q_{i}}{\partial a_{i}} = [\alpha_{0} + \alpha_{1}a_{i} + \gamma_{1}u - \beta_{1}w - \mu_{2}v]q_{i} + [\beta_{0} + \beta_{1}a_{i} + 2\alpha_{1}u + 2\beta_{1}v + \mu_{2}u]p_{i} - \sum_{k \neq i}^{2M} (-1)^{k}R(a_{i}, a_{k})q_{k}m_{k} ,$$

$$m_{i} \frac{\partial p_{i}}{\partial a_{i}} = [-\gamma_{0} - \gamma_{1}a_{i} + 2\gamma_{1}v + 2\alpha_{1}w - \mu_{2}w]q_{i} + [-\alpha_{0} - \alpha_{1}a_{i} + \beta_{1}w - \gamma_{1}u + \mu_{2}v]p_{i} - \sum_{k \neq i}^{2M} (-1)^{k}R(a_{i}, a_{k})p_{k}m_{k} ,$$
(2.19)

and

$$m_{i}R(a_{i},a_{i}) = [\gamma_{0} + \gamma_{1}a_{i} - 2\gamma_{1}v - 2\alpha_{1}w + \mu_{2}w]q_{i}^{2} + [\beta_{0} + \beta_{1}a_{i} + 2\alpha_{1}u + 2\beta_{1}v + \mu_{2}u]p_{i}^{2} + [\alpha_{0} + \alpha_{1}a_{i} + \gamma_{1}u - \beta_{1}w - \mu_{2}v]2q_{i}p_{i} + \sum_{k \neq i}^{2M} (-1)^{k}m_{k}\frac{[q_{i}p_{k} - p_{i}q_{k}]^{2}}{a_{i} - a_{k}}, \qquad (2.20)$$

and furthermore,

$$\frac{\partial}{\partial a_i} [m_i R(a_i, a_i)] = 2\alpha_1 q_i p_i + \beta_1 p_i^2 + \gamma_1 q_i^2 - \sum_{k \neq i}^{2M} (-1)^k m_k R^2(a_i, a_k) , \qquad (2.21)$$

where $m_i = m(a_i)$.

3 The Gaussian Ensemble

In this section we take up the problem of computing (1.5). We recall (see e.g. [16]) that for the Gaussian weight e^{-x^2} the corresponding orthonormal polynomials $\{p_N(x)\}_{N=0,1,\ldots}$ are given in terms of the standard Hermite polynomials $H_N(x)$ by

$$p_N(x) = 2^{-N/2} \pi^{-1/4} (N!)^{-1/2} H_N(x) , \qquad (3.1)$$

and for the coefficient a_N of x^N in $p_N(x)$ we have

$$a_N = 2^{N/2} \pi^{-1/4} (N!)^{-1/2} . aga{3.2}$$

Substituting in (2.7,2.8) and making use of the differential relation for the Hermite polynomials gives [18]

$$\phi'(x) = -x\phi(x) + \sqrt{2N}\psi(x) ,$$

$$\psi'(x) = +x\psi(x) - \sqrt{2N}\phi(x) ,$$
(3.3)

and so we have

$$m(x) = 1, \quad \alpha_1 = -1, \quad \beta_0 = \gamma_0 = \sqrt{2N}, \quad \alpha_0 = \beta_1 = \gamma_1 = 0.$$
 (3.4)

3.1 Differential Equation

The interval from which the eigenvalues are excluded in the probability (1.4) consists of the union of the two intervals $(-\infty, -s)$ and (s, ∞) . Thus in (2.1) we have M = 2, and $a_1 = -\infty$, $a_2 = -s$, $a_3 = s$ and $a_4 = \infty$. Because a_1 and a_4 are infinite, only the quantities in Propositions 1 and 2 relating to a_2 and a_3 are of interest. Furthermore it turns out that the evenness symmetry of I and the odd/even symmetry of (3.1) implies the equations relating to a_2 are equivalent to those relating to a_3 . Thus $\phi(-x) = (-1)^N \phi(x)$, $\psi(-x) = (-1)^{N-1} \psi(x)$, and so $K_N(-x, -y) = K_N(x, y)$, which in turn implies $\rho(-x, -y) = \rho(x, y)$ and so $q_2 =$ $(-1)^N q_3$, $p_2 = (-1)^{N-1} p_3$, R(-x, -y) = R(x, y).

Proposition 3 The coupled differential equations for $q = q_3, p = p_3$ and R(s), R(-s, s) of the finite N GUE on the interval $I = (-\infty, -s) \cup (s, \infty)$ are

$$\frac{d}{ds}\ln E_2 = 2R(s) , \qquad (3.5)$$

$$q' = -sq + p[\beta_0 - 2u] + 2q^2 p/s , \qquad (3.6)$$

$$p' = +sp - q[\gamma_0 + 2w] - 2qp^2/s , \qquad (3.7)$$

$$u' = -2q^2 , (3.8)$$

$$w' = -2p^2$$
, (3.9)

$$R(-s,s) = (-1)^{N-1} qp/s , \qquad (3.10)$$

$$R(s) = q^{2}[\gamma_{0} + 2w] + p^{2}[\beta_{0} - 2u] - 2sqp + 2q^{2}p^{2}/s , \qquad (3.11)$$

$$R'(s) = -2qp - 2q^2 p^2 / s^2 , \qquad (3.12)$$

where primes indicate derivatives with respect to s.

Proof - The first equation follows from (2.13), (3.6) and (3.7) follow from (2.19), (3.8) and (3.9) follow from (2.17), (3.10) follows from (2.14), (3.11) from (2.20) and (3.12) from (2.21). In the derivation of (3.5) and (3.12) use is made of the general formula for the total derivative

$$\frac{d}{ds}f_I(s,s) = \left(\frac{\partial}{\partial a_3}f_I(a_3,a_3) - \frac{\partial}{\partial a_2}f_I(a_3,a_3)\right)\Big|_{a_3=-a_2=s} = 2\frac{\partial}{\partial s}f_I(s,s) , \qquad (3.13)$$

valid whenever $f_I(a_2, a_2) = f_I(a_3, a_3)$. \Box

There are a number of differences when one compares this set of coupled ODEs with that arising from a single interval (s, ∞) , as described in Ref.[18]. Firstly there is the appearance of factors of two, reflecting the contributions from the two free endpoints, but more significantly is the addition of nonlinear quadratic terms in (3.6, 3.7, 3.11, 3.12). Regarding the reduction of these equations, it is straightforward to show that (3.6) and (3.7) together imply

$$pq = uw - \frac{1}{2}\beta_0 w + \frac{1}{2}\gamma_0 u , \qquad (3.14)$$

or equivalently

$$\beta_0 \gamma_0 - 4pq = (\beta_0 - 2u)(\gamma_0 + 2w) , \qquad (3.15)$$

which is useful in the subsequent determination of a single equation for R(s). In fact it is possible to reduce the coupled system of PDEs in Proposition 3 down to a single ODE for R(s), or a single ODE for the quantity

$$\tilde{R}(s) = (-1)^N R(-s,s) . (3.16)$$

The boundary conditions satisfied by R(s) and $\tilde{R}(s)$ are found from the large s form of the kernel

$$R(s) \underset{s \to \infty}{\sim} K_N(s, s) , \qquad (3.17)$$

which implies

$$R(s) \sim_{s \to \infty} \frac{\pi^{-1/2} 2^{-N}}{(N-1)!} e^{-s^2} \left[H_N^2(s) - H_{N+1}(s) H_{N-1}(s) \right] ,$$

$$\tilde{R}(s) \sim_{s \to \infty} -\frac{\pi^{-1/2} 2^{-N}}{(N-1)!} s^{-1} e^{-s^2} H_N(s) H_{N-1}(s) .$$
(3.18)

Of course we could replace the Hermite polynomials by their leading order term, however of subsequent interest will be a scaling limit which requires the full expression as written.

We can reduce this coupled system of PDEs into a single second order ODE, and give two possible forms for this.

Proposition 4 The system of ODEs in Proposition 3 is equivalent to the following secondorder differential equation for $\tilde{R}(s)$

$$\left[s\tilde{R}'' + 2\tilde{R}' + 8Ns\tilde{R} + 24s^{2}\tilde{R}^{2}\right]^{2} = 4[s + 2\tilde{R}]^{2} \left[(\tilde{R} + s\tilde{R}')^{2} + 8Ns^{2}\tilde{R}^{2} + 16s^{3}\tilde{R}^{3}\right] , \qquad (3.19)$$

or to the second-order differential equation for R(s)

$$sR'' + 2R' = 2s[s-h] - 2h\sqrt{(R+sR')^2 - 4s^2[s-h]R - 2Ns^2[s-h]^2} .$$
(3.20)

$$h \equiv \sqrt{s^2 - 2R'} \ . \tag{3.21}$$

Proof - We adopt the notation a = sR, $b = s\tilde{R}$ for simplicity. Using (3.6) and (3.7) together with the definition (3.16) we find the relation

$$b' = q^2(\gamma_0 + 2w) - p^2(\beta_0 - 2u) , \qquad (3.22)$$

and utilising (3.11) and (3.12) the corresponding relation for a is

$$a' - 4sb = q^{2}(\gamma_{0} + 2w) + p^{2}(\beta_{0} - 2u) . \qquad (3.23)$$

By squaring and subtracting the right hand sides of these two equations and employing the integral (3.15) to eliminate the cross term we have

$$(a')^2 = 8ab + 8Nb^2 + (b')^2 . (3.24)$$

There is also another relation which follows directly from (3.12) and reads

$$sa' = a + 2s^2b - 2b^2 . aga{3.25}$$

If one proceeds to eliminate a using (3.24) and (3.25) then a second-order differential equation is obtained for \tilde{R} , namely (3.19). However if one eliminates b then a more useful second order equation for R emerges

$$\frac{1}{2} \left\{ s[s \mp \sqrt{s^2 - 2R'}] - \frac{1}{2}a'' \right\}^2 = (s^2 - 2R') \left\{ \frac{1}{2}(a')^2 - 2s[s \mp \sqrt{s^2 - 2R'}]a - Ns^2[s \mp \sqrt{s^2 - 2R'}]^2 \right\} .$$
 (3.26)

Firstly the square-root appearing above is well defined as

$$s^2 - 2R' = (s - 2\tilde{R})^2 . (3.27)$$

One now faces a decision regarding the choice of the branch to be taken for the square root of the right-hand side of (3.26), and if we take the negative branch then the correct choice is given in (3.20). This choice is the only acceptable one given the boundary conditions (3.18) which impose that all quantities a, R, \tilde{R} and derivatives vanish exponentially fast as $s \to \infty$. If one takes the upper, negative sign then s-h also vanishes in the same way and is consistent with the ODE (3.26) but if the other choice is made then s + h = O(s) in this limit and is incompatible with the ODE, both in the leading order s-dependence and the sign of the coefficient. \Box

3.2 Special Cases

Here we calculate (1.4) in the cases N = 1 and N = 2 directly from the expansion (2.2, 2.3, 2.6), which truncates after term n = N. The formulas (3.5) and (3.27) then allow R(s) and $\tilde{R}(s)$ to be computed in terms of the error function.

Proposition 5 Consider the probability (1.4) and the associated functions R(s) as specified by (3.5) and $\tilde{R}(s)$ as specified by (3.16). For N = 1, with the the standard definition of the error function $\operatorname{erf}(s)$, we have

$$E_{2}(0; I) = \operatorname{erf}(s) ,$$

$$R(s) = \frac{e^{-s^{2}}}{\pi^{1/2} \operatorname{erf}(s)} ,$$

$$\tilde{R}(s) = -\frac{e^{-s^{2}}}{\pi^{1/2} \operatorname{erf}(s)} ,$$
(3.28)

while for N = 2

$$E_{2}(0;I) = \operatorname{erf}(s) \left[\operatorname{erf}(s) - 2\pi^{-1/2} s e^{-s^{2}} \right] ,$$

$$R(s) = \frac{e^{-s^{2}}}{\pi^{1/2} \operatorname{erf}(s)} + \frac{s^{2} e^{-s^{2}}}{\frac{1}{2}\pi^{1/2} \operatorname{erf}(s) - s e^{-s^{2}}} ,$$

$$\tilde{R}(s) = \frac{e^{-s^{2}}}{\pi^{1/2} \operatorname{erf}(s)} - \frac{s^{2} e^{-s^{2}}}{\frac{1}{2}\pi^{1/2} \operatorname{erf}(s) - s e^{-s^{2}}} .$$
(3.29)

Using the kernel above, and (2.7) and (3.1), the results follow from direct evaluation of the integrals. \Box

We can check (by hand and using computer algebra) that these expressions satisfy the appropriate equations (3.20, 3.19) in Proposition 4, and furthermore satisfy the appropriate boundary conditions $s \to \infty$, equation (3.18).

For small s we find from (3.28, 3.29) that

$$R^{N=1} \sim \frac{1}{2s} - \frac{1}{3}s + \frac{4}{45}s^3 - \frac{8}{945}s^5 - \frac{16}{14175}s^7 + \frac{32}{93555}s^9 + \frac{1472}{638512875}s^{11} ,$$

$$R^{N=2} \sim \frac{2}{s} - \frac{14}{15}s + \frac{248}{1575}s^3 - \frac{128}{23625}s^5 - \frac{51104}{27286875}s^7 + \frac{1356032}{5320940625}s^9 + \frac{6898816}{558698765625}s^{11} .$$
(3.30)

We can also determine the small-s expansion for general N. The form of this expansion can be understood from the small s behaviour of (1.4). Now

$$E_{2}(0; (-\infty, -s) \cup (s, \infty); e^{-\lambda^{2}}; N) = \frac{1}{C} \prod_{l=1}^{N} \int_{-s}^{s} d\lambda_{l} e^{-\lambda_{l}^{2}} \prod_{j < k} |\lambda_{k} - \lambda_{j}|^{2}$$
$$= \frac{s^{N^{2}}}{C} \prod_{l=1}^{N} \int_{-1}^{1} d\lambda_{l} e^{-s^{2}\lambda_{l}^{2}} \prod_{j < k} |\lambda_{k} - \lambda_{j}|^{2}$$
$$\sim a_{0} s^{N^{2}} + a_{2} s^{N^{2}+2} + \dots , \qquad (3.31)$$

where the final equality follows by expanding the exponential. According to (3.5) and (3.27) this implies

$$R(s) \sim \frac{b_{-1}}{s} + b_1 s + b_3 s^3 + \dots$$
 (3.32)

Making an ansatz of this form we find that all the coefficients are uniquely determined by (3.20). Use of computer algebra gives the expansion in the following result.

Proposition 6 For fixed N, the asymptotic expansion of R(s) for small s is given by

$$R(s) \sim \frac{N^2}{2s} - \frac{N(2N^2 - 1)}{4N^2 - 1}s + \frac{2N^2(4N^4 - 9N^2 + 3)}{(4N^2 - 1)^2(4N^2 - 9)}s^3 + \frac{8N^3(4N^4 - 13N^2 + 6)}{(4N^2 - 1)^3(4N^2 - 9)(4N^2 - 25)}s^5 + \frac{8N^2(128N^{10} - 1312N^8 + 3304N^6 - 3430N^4 + 1355N^2 - 315)}{(4N^2 - 1)^4(4N^2 - 9)^2(4N^2 - 25)(4N^2 - 49)}s^7 .$$
 (3.33)

It is of interest to note that in fact the only Laurent series about s = 0 which satisfies (3.20) has the structure (3.33). To see this it is simplest to consider the equation (3.19). Making the Ansatz

$$s\tilde{R} = \sum_{n=k}^{\infty} r_n s^n , \qquad (3.34)$$

we find that with a lower exponent k and $k \ge 1$ the only solution for the coefficients is the null solution. However if $k \le -1$ then one finds $r_k = \ldots = r_{-1} = 0$, and furthermore the relations obtained by equating the coefficients of s^{-2}, s^{-1}, \ldots are the same as for the case of k = 0. Explicitly equating coefficients of s^{-2} in the ODE yields

$$r_1^2 + 8Nr_0^2 + 16r_0^3 = 0. (3.35)$$

Now $r_0 \neq 0$, therefore the odd-index terms must vanish because $s\tilde{R}(s)$ is even, as is seen from the ODE (3.19). Thus $r_1 = 0$ and an unique solution is found for $r_0 = -N/2$. The coefficient of s^{-1} automatically vanishes while equating coefficients of s^0 gives

$$r_2 = \frac{N^2}{4N^2 - 1}$$
 or $-N^2$. (3.36)

Choosing the former root, we find by equating coefficients of higher powers of s that each of r_3, r_4, \ldots is now uniquely determined.

3.3 Edge Scaling

To leading order the support of the GUE is within the interval $(-\sqrt{2N}, \sqrt{2N})$. The statistical properties of the eigenvalues in the vicinity of the edge $(\lambda \sim \sqrt{2N} \text{ say})$ can be studied by introducing the scale

$$\lambda \mapsto \sqrt{2N} + \frac{\lambda}{\sqrt{2}N^{1/6}} , \qquad (3.37)$$

which for $N \to \infty$ makes the separation between the largest and the second largest eigenvalue of order unity. In (1.4) the probability that the region $-\infty$ to the vicinity of the lowest eigenvalue, and the highest eigenvalue to ∞ , is free of eigenvalues can be studied in this limit by writing

$$s = \sqrt{2N} + \frac{t}{\sqrt{2N^{1/6}}} . \tag{3.38}$$

Because the two intervals in (1.4) become independent in this limit we would expect

$$\lim_{N \to \infty} E_2(0; (-\infty, -(\sqrt{2N} + \frac{t}{\sqrt{2N^{1/6}}})) \cup (\sqrt{2N} + \frac{t}{\sqrt{2N^{1/6}}}, \infty); e^{-\lambda^2}; N) = \left(E_2^{\text{soft}}(0; (t, \infty))\right)^2, \quad (3.39)$$

where $E_2^{\text{soft}}(0; (t, \infty))$ refers to the probability that the edge (t, ∞) is free of eigenvalue in the infinite GUE scaled according to (3.37). Now it follows from (3.5) that

$$E_2(0; (-\infty, -s) \cup (s, \infty); e^{-\lambda^2}; N) = \exp\left(-2\int_s^\infty du R(u)\right)$$
 (3.40)

On the other hand it is known that [17]

$$E_2^{\text{soft}}(0;(t,\infty)) = \exp\left(-\int_t^\infty du R^{\text{soft}}(u)\right) , \qquad (3.41)$$

where $R^{\text{soft}}(t)$ satisfies

$$\left(\ddot{R}^{\text{soft}}\right)^2 + 4\dot{R}^{\text{soft}} \left[\left(\dot{R}^{\text{soft}}\right)^2 - t\dot{R}^{\text{soft}} + R^{\text{soft}} \right] = 0 , \qquad (3.42)$$

which is the Jimbo-Miwa-Okamoto form of a particular Painlevé-II equation, subject to the boundary conditions

$$R^{\text{soft}}(t) \underset{t \to \infty}{\sim} -t \left[\text{Ai}(t) \right]^2 + \left[\text{Ai}'(t) \right]^2 .$$
(3.43)

Thus for (3.39) to be valid we must have

$$\lim_{N \to \infty} \frac{1}{\sqrt{2N^{1/6}}} R(\sqrt{2N} + \frac{t}{\sqrt{2N^{1/6}}}) = R^{\text{soft}}(t)$$
(3.44)

The equation (3.42) can be verified by putting

$$r(t) = \frac{1}{\sqrt{2N^{1/6}}} R(\sqrt{2N} + \frac{t}{\sqrt{2N^{1/6}}}) , \qquad (3.45)$$

and showing that for $N \to \infty r(t)$ satisfies the differential equation (3.42) and the boundary condition (3.43). Regarding the latter point, note that (3.17) implies

$$r(t) \underset{t \to \infty}{\sim} \frac{1}{\sqrt{2N^{1/6}}} K_N(\sqrt{2N} + \frac{t}{\sqrt{2N^{1/6}}}, \sqrt{2N} + \frac{t}{\sqrt{2N^{1/6}}}) , \qquad (3.46)$$

where $K_N(x, y)$ is specified by substituting (3.1) and (2.7) into (2.6). But we have the known limiting behaviour [10]

$$\lim_{N \to \infty} \frac{1}{\sqrt{2N^{1/6}}} K_N(\sqrt{2N} + \frac{X}{\sqrt{2N^{1/6}}}, \sqrt{2N} + \frac{Y}{\sqrt{2N^{1/6}}}) = K^{\text{soft}}(X, Y) = \frac{\text{Ai}(X)\text{Ai}'(Y) - \text{Ai}(Y)\text{Ai}'(X)}{X - Y} \quad (3.47)$$

which immediately implies (3.43). Thus it remains to show that for $N \to \infty$ (3.45) satisfies (3.42).

In fact it is more convenient to work with the ODE (3.19), and introduce the scaled quantity

$$\tilde{r}(t) = \frac{N^{1/6}}{\sqrt{2}} \tilde{R}(\sqrt{2N} + \frac{t}{\sqrt{2N^{1/6}}}) .$$
(3.48)

(this choice of scaling is consistent with the relation (3.12) between R(s) and $\tilde{R}(s)$). With this substitution one finds that for $N \to \infty$ (3.19) reduces to

$$2\tilde{r}\ddot{\tilde{r}} - (\dot{\tilde{r}})^2 + 16\tilde{r}^3 - 4t\tilde{r}^2 = 0 , \qquad (3.49)$$

while the relation (3.12) reduces to

$$\dot{r} = 2\tilde{r} \ . \tag{3.50}$$

Substituting (3.50) in (3.49) gives the third order ODE

$$2\dot{r}\ddot{r} - (\ddot{r})^2 + 8(\dot{r})^3 - 4t\dot{r}^2 = 0.$$
(3.51)

Differentiating (3.42) and using the original equation once more, it is easy to reduce it to (3.51), thus establishing the required result. Alternatively one prove the same correspondence by making the substitution

$$\tilde{r} = -\frac{1}{2}q^2$$
, (3.52)

in the ODE (3.49), and then show this leads to a Painlevé-II equation with $\alpha = 0$, namely

$$\ddot{q} = 2q^3 + tq$$
 . (3.53)

This equation has been derived for the kernel (3.47) in [17]).

4 The Jacobi Ensemble

In this part of our study we will treat the Jacobi ensemble, for general α, β in some parts, but mostly we will consider only the symmetrical case $\alpha = \beta$ with our objective to compute the quantities (1.5). For the Jacobi weight in (1.2) the orthonormal polynomials are given in terms of the standard Jacobi polynomials by

$$p_N(x) = \left[\frac{2N+\alpha+\beta+1}{2^{\alpha+\beta+1}} \frac{N!\Gamma(N+\alpha+\beta+1)}{\Gamma(N+\alpha+1)\Gamma(N+\beta+1)}\right]^{1/2} P_N^{(\alpha,\beta)}(x) , \qquad (4.1)$$

with the corresponding coefficient of x^N such that

$$\frac{a_{N-1}}{a_N} = 2 \left[\frac{N(N+\alpha)(N+\beta)(N+\alpha+\beta)}{(2N+\alpha+\beta)^2(2N+\alpha+\beta+1)(2N+\alpha+\beta-1)} \right]^{1/2} .$$
(4.2)

Making use of the differentiation formula

$$(2N+\alpha+\beta)(1-x^{2})\frac{d}{dx}P_{N}^{(\alpha,\beta)}(x) = N[\alpha-\beta-(2N+\alpha+\beta)x]P_{N}^{(\alpha,\beta)}(x) + 2(N+\alpha)(N+\beta)P_{N-1}^{(\alpha,\beta)}(x) , \quad (4.3)$$

and the three term recurrence for the Jacobi polynomials gives that the coupled first order equations hold with [18]

$$m(x) = 1 - x^{2} ,$$

$$A(x) = \frac{\beta^{2} - \alpha^{2}}{2(2N + \alpha + \beta)} - \frac{2N + \alpha + \beta}{2} x = \alpha_{0} + \alpha_{1} x ,$$

$$B(x) = \frac{2\sqrt{N(N + \alpha)(N + \beta)(N + \alpha + \beta)}}{2N + \alpha + \beta} \sqrt{\frac{2N + \alpha + \beta + 1}{2N + \alpha + \beta - 1}} = \beta_{0} ,$$

$$C(x) = \frac{2\sqrt{N(N + \alpha)(N + \beta)(N + \alpha + \beta)}}{2N + \alpha + \beta} \sqrt{\frac{2N + \alpha + \beta - 1}{2N + \alpha + \beta + 1}} = \gamma_{0} .$$

$$(4.4)$$

4.1 Differential Equations for End Intervals

We will begin by considering the first probability in (1.5), generalised so that the weight function is the general (nonsymmetric) Jacobi form in (1.2). We shall adopt the conventions $q_3, p_3 = q_+, p_+, q_2, p_2 = q_-, p_-$ and $R_+ = R(s, s), R_- = R(-s, -s), R_0 = R(-s, s)$.

Proposition 7 The coupled differential equations for the finite N JUE on the interval

 $(-1,-s) \cup (s,1)$ for general α,β are

$$[\ln E_2]' = R_- + R_+ , \qquad (4.5)$$

$$R_0 = \frac{q_+ p_- - q_- p_+}{2s} , \qquad (4.6)$$

$$(1-s^2)R_{-} = [\gamma_0 - w(2\alpha_1+1)]q_{-}^2 + [\beta_0 + u(2\alpha_1-1)]p_{-}^2 + [\alpha_0 - \alpha_1s + v]2q_{-}p_{-} + 2s(1-s^2)R_0^2 , \qquad (4.7)$$

$$(1-s^2)R_+ = [\gamma_0 - w(2\alpha_1+1)]q_+^2 + [\beta_0 + u(2\alpha_1-1)]p_+^2 + [\alpha_0 + \alpha_1s + v]2q_+p_+ + 2s(1-s^2)R_0^2 , \qquad (4.8)$$

$$u' = -q_{-}^2 - q_{+}^2 , (4.9)$$

$$v' = -q_{-}p_{-} - q_{+}p_{+} , \qquad (4.10)$$

$$w' = -p_{-}^2 - p_{+}^2 , \qquad (4.11)$$

$$(1-s^2)q'_{-} = -[\alpha_0 - \alpha_1 s + v]q_{-} - [\beta_0 + u(2\alpha_1 - 1)]p_{-} - 2(1-s^2)q_{+}R_0 , \qquad (4.12)$$

$$(1-s^2)p'_{-} = -[-\gamma_0 + w(2\alpha_1+1)]q_{-} - [-\alpha_0 + \alpha_1 s - v]p_{-} - 2(1-s^2)p_{+}R_0 , \qquad (4.13)$$

$$(1-s^2)q'_{+} = [\alpha_0 + \alpha_1 s + v]q_{+} + [\beta_0 + u(2\alpha_1 - 1)]p_{+} - 2(1-s^2)q_{-}R_0 , \qquad (4.14)$$

$$(1-s^2)p'_{+} = [-\gamma_0 + w(2\alpha_1+1)]q_{+} + [-\alpha_0 - \alpha_1 s - v]p_{+} - 2(1-s^2)p_{-}R_0 , \qquad (4.15)$$

$$\left[(1-s^2)R_{-} \right]' = -2\alpha_1 q_{-} p_{-} - 2(1-s^2)R_0^2 , \qquad (4.16)$$

$$\left[(1-s^2)R_+ \right]' = +2\alpha_1 q_+ p_+ - 2(1-s^2)R_0^2 .$$
(4.17)

The first equation follows from (2.13), (2.5) and the next equality in (4.6) follows from (2.14), (4.7) and (4.8) follow from (2.20), while (4.16) and (4.17) follow from (2.21). Furthermore (4.9)-(4.11) follow from (2.17) and the first equality in (4.6) and (4.12)-(4.15) follow from (2.19) and (4.6). \Box

We note from (4.16), (4.17) and (4.10) the integral

$$(1-s^2)(R_- - R_+) = 2\alpha_1 v . (4.18)$$

The boundary conditions satisfied by R(s,s) as $s \to 1^-$ or $s \to -1^+$ can be expressed as

$$R(s,s) \sim \frac{N!\Gamma(N+\alpha+\beta+1)}{\Gamma(N+\alpha)\Gamma(N+\beta)} \frac{(1-s)^{\alpha-1}(1+s)^{\beta-1}}{2^{\alpha+\beta}(2N+\alpha+\beta)}$$

$$\times \left[\left(\frac{\beta^2 - \alpha^2}{(2N+\alpha+\beta)(2N+\alpha+\beta+2)} - s \right) P_N^{(\alpha,\beta)}(s) P_{N-1}^{(\alpha,\beta)}(s) - \frac{2(N+1)(N+\alpha+\beta+1)}{2N+\alpha+\beta+2} P_{N+1}^{(\alpha,\beta)}(s) P_{N-1}^{(\alpha,\beta)}(s) + \frac{2N(N+\alpha+\beta)}{2N+\alpha+\beta} \left[P_N^{(\alpha,\beta)}(s) \right]^2 \right].$$

$$(4.19)$$

Our objective is to use equations such as (4.18) to reduce the equations of Proposition 7 down to a single equation for $R_+(s)$. For this purpose we restrict attention to the case

 $\alpha = \beta$, then we see from (4.4) that $\alpha_0 = 0$, while the fact that

$$P_N^{(\alpha,\alpha)}(-x) = (-1)^N P_N^{(\alpha,\alpha)}(x) , \qquad (4.20)$$

implies v = 0, $q_2 = (-1)^N q_3$, $p_2 = (-1)^{N-1} p_3$ and $R_+ = R_-$ (= R say), thus reducing the number of unknowns in Proposition 7.

Proposition 8 The coupled differential equations for the finite N JUE on the interval $(-1, -s) \cup (s, 1)$ for $\beta = \alpha$ are

$$[\ln E_2]' = 2R , \qquad (4.21)$$

$$R(-s,s) = (-1)^{N-1} \frac{qp}{s} \equiv (-1)^{N-1} R_0 , \qquad (4.22)$$

$$(1-s^2)R = [\gamma_0 - w(2\alpha_1+1)]q^2 + [\beta_0 + u(2\alpha_1-1)]p^2 + 2\alpha_1 sqp + 2s(1-s^2)R_0^2 , \quad (4.23)$$

$$u' = -2q^2 , (4.24)$$

$$w' = -2p^2$$
, (4.25)

$$(1-s^2)q' = +\alpha_1 sq + [\beta_0 + u(2\alpha_1 - 1)]p + \frac{2(1-s^2)}{s}q^2p , \qquad (4.26)$$

$$(1-s^2)p' = -\alpha_1 sp + [-\gamma_0 + w(2\alpha_1+1)]q - \frac{2(1-s^2)}{s}qp^2 , \qquad (4.27)$$

$$\left[(1-s^2)R\right]' = 2\alpha_1 qp - \frac{2(1-s^2)}{s^2} q^2 p^2 , \qquad (4.28)$$

where we have redefined R_0 from the previous usage $(R_0 \mapsto (-1)^{N-1}R_0)$ and made the notation $q_+ = q, p_+ = p$.

We now indicate how to reduce such a system to a single third order differential equation for R = R(s) = R(s, s).

Proposition 9 The coupled set of ODEs given in Proposition 8 reduce to the third order ODE for $\sigma(s) = (1-s^2)R(s)$,

$$(1-s^{2})^{2}\sigma''' - 2s(1-s^{2})\sigma'' - 2s^{-2}(1-s^{2})\sigma' - (1-s^{2})^{2}(\sigma'')^{2}/\sigma' + \alpha_{1}\frac{s}{2\sigma'}\frac{\alpha_{1}s + H}{H^{2}}\left[(1-s^{2})\sigma'' - 2s^{-1}\sigma'\right]^{2} + 4\alpha_{1}H\sigma + 2[\alpha_{1}s + H](\alpha_{1}s^{-1} - H\sigma') + 2\frac{\sigma^{2}}{s^{2}\sigma'}H[H - \alpha_{1}s] = 0 , \qquad (4.29)$$

where $H \equiv \sqrt{\alpha_1^2 s^2 - 2(1 - s^2)\sigma'}$.

Proof - Firstly we make some auxiliary definitions similar to the Gaussian case

$$a = qp$$
,
 $A = [\beta_0 + u(2\alpha_1 - 1)]p^2$, (4.30)
 $B = [-\gamma_0 + w(2\alpha_1 + 1)]q^2$.

Combining the relations (4.26) and (4.27) we find

$$(1-s^2)a' = A + B , (4.31)$$

and we differentiate this once more, using the relations (4.24), (4.25) (4.26) and (4.27), to arrive at

$$(1-s^2)^2 a'' = 2s(A+B) - 8\alpha_1(1-s^2)a^2 + [2\alpha_1s + 4s^{-1}(1-s^2)a](B-A) + 4AB/a .$$
(4.32)

Now the idea is to express A, B in terms of σ and its derivatives, so we employ (4.31) above and the relation

$$s\sigma' + \sigma - 4\alpha_1 sa = A - B , \qquad (4.33)$$

which follows from (4.23) and (4.28). In order to express a in terms of $\sigma'(s)$ we have to solve the quadratic relation (4.28) for a and this is how the square-root variable H arises. This quantity is well defined because

$$\alpha_1^2 s^2 - 2(1-s^2)\sigma' = \left[2\frac{1-s^2}{s}a - \alpha_1 s\right]^2 .$$
(4.34)

Having expressed A, B and a in terms of σ and it derivatives it is then a matter of substituting these into (4.32) and after considerable simplification we find the final result (4.29). \Box

4.2 Special Cases for End Intervals

In this part we present the calculations for the first two finite-N cases, that is N = 1 and N = 2, by direct means using the probability $E_2(0; I)$.

Proposition 10 The probability that there are no eigenvalues in the interval $I = (-1, -s) \cup (s, 1)$ of the JUE with N = 1, and general $\alpha \neq \beta$ is

$$E_2(0;I) = I_{(1+s)/2}(\alpha+1,\beta+1) - I_{(1-s)/2}(\alpha+1,\beta+1)$$
(4.35)

and for N = 2, is

$$E_{2}(0;I) = (\alpha+\beta+3) \left[I_{(1+s)/2}(\alpha+1,\beta+2) - I_{(1-s)/2}(\alpha+1,\beta+2) \right] \\ \times \left[I_{(1+s)/2}(\alpha+2,\beta+1) - I_{(1-s)/2}(\alpha+2,\beta+1) \right] \\ - (\alpha+\beta+2) \left[I_{(1+s)/2}(\alpha+2,\beta+2) - I_{(1-s)/2}(\alpha+2,\beta+2) \right] \\ \times \left[I_{(1+s)/2}(\alpha+1,\beta+1) - I_{(1-s)/2}(\alpha+1,\beta+1) \right]$$
(4.36)

where the normalised incomplete beta functions are defined by

$$I_x(a,b) = \frac{B_x(a,b)}{B(a,b)}$$
(4.37)

in terms of the incomplete and complete beta functions, $B_x(a,b)$ and B(a,b) respectively.

Proof - as in the case of Proposition 5, this follows from the expansion (2.2) with (2.3) which truncates after the term n = N used in conjunction with (2.6), (2.7), (4.1) and (4.2). Extensive use of the identities for the incomplete beta function

$$I_x(a,b) = 1 - I_{1-x}(b,a) ,$$

(a+b)I_x(a,b) = aI_x(a+1,b) + bI_x(a,b+1) ,

are made to reduce the number of their occurrences. \Box

In the symmetric case $\alpha = \beta$ considerable simplification ensues and we have the following results.

Proposition 11 In the case $\beta = \alpha$, the probability (4.35) and the associated quantities of Proposition 10 are given in terms of the Gauss hypergeometric function $_2F_1(a, b; c; z)$ by

$$E_{2}(0;I) = \frac{1}{4^{\alpha}B(\alpha+1,\alpha+1)} s_{2}F_{1}(-\alpha, \frac{1}{2}; \frac{3}{2}; s^{2}) ,$$

$$\sigma(s) = \frac{1}{2} \frac{(1-s^{2})^{\alpha+1}}{s_{2}F_{1}(-\alpha, \frac{1}{2}; \frac{3}{2}; s^{2})} ,$$

$$H(s) = (\alpha+1)s + \frac{(1-s^{2})^{\alpha+1}}{s_{2}F_{1}(-\alpha, \frac{1}{2}; \frac{3}{2}; s^{2})} ,$$

(4.38)

and similarly for the probability (4.36),

$$E_{2}(0;I) = \frac{2\alpha+3}{4^{2\alpha+1}B^{2}(\alpha+1,\alpha+2)} \frac{1}{3}s^{4} {}_{2}F_{1}(-\alpha,\frac{1}{2};\frac{3}{2};s^{2}) {}_{2}F_{1}(-\alpha,\frac{3}{2};\frac{5}{2};s^{2}) ,$$

$$\sigma(s) = \frac{(1-s^{2})^{\alpha+1}}{2s} \left\{ \frac{1}{{}_{2}F_{1}(-\alpha,\frac{1}{2};\frac{3}{2};s^{2})} + \frac{3}{{}_{2}F_{1}(-\alpha,\frac{3}{2};\frac{5}{2};s^{2})} \right\} ,$$

$$H(s) = (\alpha+2)s + \frac{(1-s^{2})^{\alpha+1}}{s} \left\{ -\frac{1}{{}_{2}F_{1}(-\alpha,\frac{1}{2};\frac{3}{2};s^{2})} + \frac{3}{{}_{2}F_{1}(-\alpha,\frac{3}{2};\frac{5}{2};s^{2})} \right\} .$$

$$(4.39)$$

Proof - These follow from the reduction of the incomplete beta functions in the symmetric case to Hypergeometric functions, such as

$$I_{(1+s)/2}(\alpha+1,\alpha+1) - I_{(1-s)/2}(\alpha+1,\alpha+1) = \frac{1}{4^{\alpha}B(\alpha+1,\alpha+1)} s_2 F_1(-\alpha, \frac{1}{2}; \frac{3}{2}; s^2) , \quad (4.40)$$

the differentiation formulae for these Hypergeometric functions, like

$$\frac{d}{ds}[s_2F_1(-\alpha, \frac{1}{2}; \frac{3}{2}; s^2)] = (1-s^2)^{\alpha}$$
(4.41)

and the use of their contiguous relations

$$(2\alpha+3)\frac{1}{3}s^{3}{}_{2}F_{1}(-\alpha,\frac{3}{2};\frac{5}{2};s^{2}) - s_{2}F_{1}(-\alpha,\frac{1}{2};\frac{3}{2};s^{2}) + s_{2}F_{1}(-\alpha-1,\frac{3}{2};\frac{3}{2};s^{2}) = 0.$$
(4.42)

One can show that these two specific cases are solutions to our third order differential equation (4.29), after noting the contiguous relation above (4.42) linking the two Hypergeometric functions in the case of N = 2.

Another special case for which the probability can be computed independently of the ODE (4.29) is that with $\alpha = \beta = 0$ and general N. We then have

$$E_2(0; (-1, -s) \cup (s, 1); \chi_{[0,1]}; N) = \frac{1}{C} \int_{-s}^{s} d\lambda_1 \cdots \int_{-s}^{s} d\lambda_N \prod_{1 \le j < k \le N} |\lambda_j - \lambda_k|^2 , \qquad (4.43)$$

where C is such that $E_2 = 1$ for s = 1, which can be evaluated by a change of variable.

Proposition 12 The probability (4.43) and the associated quantities of Proposition 10 have the evaluation

$$E_{2}(0;I) = s^{N^{2}},$$

$$\sigma(s) = N^{2} \frac{(1-s^{2})}{2s},$$

$$H(s) = \frac{N}{s}.$$
(4.44)

This exact form of $\sigma(s)$ can be verified to satisfy (4.29) with $\alpha = \beta = 0$. Furthermore we can take the limit $\alpha \to 0$ in (4.38) and (4.39) and reclaim (4.44) in the cases N = 1 and N = 2 respectively.

4.3 Jacobi to Hermite Limit for the End Intervals

From the definition

$$E_{2}(0; (-1, -s) \cup (s, 1); (1 - \lambda^{2})^{\alpha}; N) = \frac{1}{C} \int_{-s}^{s} d\lambda_{1} \cdots \int_{-s}^{s} d\lambda_{N} \prod_{l=1}^{N} (1 - \lambda_{l}^{2})^{\alpha} \prod_{1 \le j < k \le N} |\lambda_{k} - \lambda_{j}|^{2}, \quad (4.45)$$

where again the constant C ensures the normalisation. Replacing s by $t/\sqrt{\alpha}$, changing variables $\lambda_l \mapsto \lambda_l/\sqrt{\alpha}$ and taking $\alpha \to \infty$ shows

$$E_2(0; (-1, -t/\sqrt{\alpha}) \cup (t/\sqrt{\alpha}, 1); (1-\lambda^2)^{\alpha}; N) \underset{\alpha \to \infty}{\sim} E_2(0; (-\infty, -t) \cup (t, \infty); e^{-\lambda^2}; N) . \quad (4.46)$$

Recalling (3.5) and (4.21) this is equivalent to the statement that

$$\lim_{\alpha \to \infty} \frac{1}{\sqrt{\alpha}} R(\frac{t}{\sqrt{\alpha}}) = R^{\text{GUE}}(t) , \qquad (4.47)$$

where R(s) on the left-hand side is as in Propositions 7 and 8 while $R^{\text{GUE}}(t)$ is the R defined in Proposition 3. Thus it must be that with

$$\frac{1}{\sqrt{\alpha}}R(\frac{t}{\sqrt{\alpha}}) = r(t) , \qquad (4.48)$$

in (4.29), taking the limit $\alpha \to \infty$ must take the third order equation into an equation equivalent to the second order equation (3.20).

To verify this, we first note that the substitution (4.48), to leading order in α^2 reads

$$r''' + 2 - \frac{(r'')^2}{r'} - \frac{2}{t^2}r' + \frac{2}{t^2}\frac{r^2}{r'}h[t+h] + 2r'h[t-h] - \left(4r + \frac{2}{t}\right)h + \frac{(tr'' - 2r')^2}{2tr'h^2}[t-h] = 0 , \qquad (4.49)$$

and that differentiation of the second order Hermite ODE, (3.20), gives

$$tr''' + 3r'' = 4t + 4h^{2}(tr' + r + 2Nt) - 2h$$
$$-8t(r + Nt)h - (t - r'')(2t^{2} - tr'' - 2r')/h^{2}, \qquad (4.50)$$

with $h = \sqrt{t^2 - 2r'}$, as before. By effecting a suitable subtraction of these two equation in order to eliminate the term in r''', and using the second order ODE once more, one can show this difference is identically zero.

Employing this limit one can show that the special cases of N = 1 and N = 2 for the Jacobi weight (4.38) and (4.39) lead exactly to those for the Hermite weights (3.28) and (3.29), respectively.

4.4 Differential Equations for an Interior Interval

We will now consider our last case, the second probability in (1.5), in a parallel manner to that of the previous case. Treating first the nonsymmetrical form $\alpha \neq \beta$, we make the new conventions $q_2, p_2 = q_+, p_+, q_1, p_1 = q_-, p_-$ and $R_+ = R(s, s), R_- = R(-s, -s), R_0 = R(-s, s)$.

Proposition 13 The coupled differential equations for the finite N JUE on the interval

(-s,s) for general α,β are

$$[\ln E_2]' = -R_- - R_+ , \qquad (4.51)$$

$$R_0 = \frac{q_+ p_- - q_- p_+}{2s} , \qquad (4.52)$$

$$(1-s^2)R_{-} = [\gamma_0 - w(2\alpha_1+1)]q_{-}^2 + [\beta_0 + u(2\alpha_1-1)]p_{-}^2 + [\alpha_0 - \alpha_1s + v]2q_{-}p_{-} - 2s(1-s^2)R_0^2 , \qquad (4.53)$$

$$(1-s^2)R_+ = [\gamma_0 - w(2\alpha_1+1)]q_+^2 + [\beta_0 + u(2\alpha_1-1)]p_+^2 + [\alpha_0 + \alpha_1s + v]2q_+p_+ - 2s(1-s^2)R_0^2 , \qquad (4.54)$$

$$u' = +q_{-}^2 + q_{+}^2 , \qquad (4.55)$$

$$v' = +q_-p_- + q_+p_+ , \qquad (4.56)$$

$$(100)$$

$$w' = +p_{-}^{2} + p_{+}^{2} , \qquad (4.57)$$

$$(1-s^2)q'_{-} = -[\alpha_0 - \alpha_1 s + v]q_{-} - [\beta_0 + u(2\alpha_1 - 1)]p_{-} + 2(1-s^2)q_{+}R_0 , \qquad (4.58)$$

$$(1-s^2)p'_{-} = -[-\gamma_0 + w(2\alpha_1+1)]q_{-} - [-\alpha_0 + \alpha_1 s - v]p_{-} + 2(1-s^2)p_{+}R_0 , \qquad (4.59)$$

$$(1-s^2)q'_{+} = [\alpha_0 + \alpha_1 s + v]q_{+} + [\beta_0 + u(2\alpha_1 - 1)]p_{+} + 2(1-s^2)q_{-}R_0 , \qquad (4.60)$$

$$(1-s^2)p'_{+} = [-\gamma_0 + w(2\alpha_1+1)]q_{+} + [-\alpha_0 - \alpha_1 s - v]p_{+} + 2(1-s^2)p_{-}R_0 , \qquad (4.61)$$

$$\left[(1-s^2)R_{-} \right]' = -2\alpha_1 q_{-} p_{-} + 2(1-s^2)R_0^2 , \qquad (4.62)$$

$$\left[(1-s^2)R_+ \right]' = +2\alpha_1 q_+ p_+ + 2(1-s^2)R_0^2 .$$
(4.63)

Proof - these follow in an entirely parallel manner as for the derivation of the previous set (4.5)-(4.17). \Box

The only, apparently minor, difference between the interior interval set of equations, (4.51)-(4.63), and the endpoint interval set, (4.5)-(4.17), is a change in sign of a number of terms in the expressions for derivatives. However, in reality, the two cases are quite distinct.

Again we note that (4.62), (4.63) and (4.56) imply the integral

$$(1-s^2)(R_+ - R_-) = 2\alpha_1 v . (4.64)$$

The boundary conditions satisfied by R(s, s) now apply as $s \to 0$ and the limiting value takes the same form as (4.19).

We continue by considering the symmetrical case $\alpha = \beta$, and find again that $\alpha_0 = 0$, and the parity relation (4.20) implies v = 0 and $R_+ = R_-$, which is denoted by R.

Proposition 14 The coupled differential equations for the finite N JUE on the interval

(-s,s) for $\beta = \alpha$ are

$$[\ln E_2]' = -2R , \qquad (4.65)$$

$$R(-s,s) = (-1)^{N-1} \frac{qp}{s} = (-1)^{N-1} R_0 , \qquad (4.66)$$

$$(1-s^2)R = [\gamma_0 - w(2\alpha_1+1)]q^2 + [\beta_0 + u(2\alpha_1-1)]p^2 + 2\alpha_1 sqp - 2s(1-s^2)R_0^2 , \quad (4.67)$$

$$u' = +2q^2 , (4.68)$$

$$w' = +2p^2$$
, (4.69)

$$(1-s^2)q' = +\alpha_1 sq + [\beta_0 + u(2\alpha_1 - 1)]p - \frac{2(1-s^2)}{s}q^2p , \qquad (4.70)$$

$$(1-s^2)p' = -\alpha_1 sp + [-\gamma_0 + w(2\alpha_1+1)]q + \frac{2(1-s^2)}{s}qp^2 , \qquad (4.71)$$

$$\left[(1-s^2)R \right]' = 2\alpha_1 qp + \frac{2(1-s^2)}{s^2} q^2 p^2 , \qquad (4.72)$$

where we employ the notation $q_2 = q, p_2 = p$.

Again such a system can be reduced to a single third order differential equation for R(s).

Proposition 15 The coupled set of ODEs given in Proposition 13 are equivalent to the third order ODE for $\sigma(s) = (1-s^2)R(s)$,

$$(1-s^{2})^{2}\sigma''' - 2s(1-s^{2})\sigma'' - 2s^{-2}(1-s^{2})\sigma' - (1-s^{2})^{2}(\sigma'')^{2}/\sigma' + \alpha_{1}\frac{s}{2\sigma'}\frac{\alpha_{1}s + G}{G^{2}}\left[(1-s^{2})\sigma'' - 2s^{-1}\sigma'\right]^{2} + 4\alpha_{1}G\sigma - 2[\alpha_{1}s + G](\alpha_{1}s^{-1} + G\sigma') + 2\frac{\sigma^{2}}{s^{2}\sigma'}G[G - \alpha_{1}s] = 0 , \qquad (4.73)$$

where $G \equiv \sqrt{\alpha_1^2 s^2 + 2(1-s^2)\sigma'}$.

Proof - this proceeds in an identical manner to the proof of Proposition 9 but with a number of minor alterations. As in the previous reduction we have to express the following equation

$$(1-s^2)^2 a'' = 2s(A+B) + 8\alpha_1(1-s^2)a^2 + [2\alpha_1 s - 4s^{-1}(1-s^2)a](B-A) + 4AB/a , \qquad (4.74)$$

in terms of σ and its derivatives alone. We have an analogous expression for σ' which is quadratic in a and its solution contains the square-root variable G defined above. This is well defined due to the relation

$$\alpha_1^2 s^2 + 2(1-s^2)\sigma' = \left[2\frac{1-s^2}{s}a + \alpha_1 s\right]^2 .$$
(4.75)

The final result is the above third order ODE (4.73). \Box

4.5 Special Cases for the Interior Interval

In this part we present the analogous results for the first two finite-N cases N = 1 and N = 2 by direct calculation of the probability $E_2(0; I)$.

Proposition 16 The probability that no eigenvalues are found in the interval (-s, s) for the JUE with N = 1, and general $\alpha \neq \beta$ is

$$E_2(0;I) = 1 - I_{(1+s)/2}(\alpha+1,\beta+1) + I_{(1-s)/2}(\alpha+1,\beta+1)$$
(4.76)

and for N = 2, is

$$E_{2}(0;I) = (\alpha + \beta + 3) \left[1 + I_{(1-s)/2}(\alpha + 1, \beta + 2) - I_{(1+s)/2}(\alpha + 1, \beta + 2) \right] \\ \times \left[1 + I_{(1-s)/2}(\alpha + 2, \beta + 1) - I_{(1+s)/2}(\alpha + 2, \beta + 1) \right] \\ - (\alpha + \beta + 2) \left[1 + I_{(1-s)/2}(\alpha + 2, \beta + 2) - I_{(1+s)/2}(\alpha + 2, \beta + 2) \right] \\ \times \left[1 + I_{(1-s)/2}(\alpha + 1, \beta + 1) - I_{(1+s)/2}(\alpha + 1, \beta + 1) \right]$$
(4.77)

in terms of the normalised incomplete beta functions, $I_x(a, b)$.

Proof - these follow from direct evaluations of the terminating series for the probability $E_2(0; I)$ given in (2.2) as indicated in the proof of Proposition 10. \Box

When equality $\alpha = \beta$ holds then the simpler results follow -

Proposition 17 For equal parameters $\beta = \alpha$ the probability (4.76) and associated functions arising from Proposition 16 are given by

$$E_{2}(0;I) = \frac{s(1-s^{2})^{\alpha+1}}{2^{2\alpha+1}(\alpha+1)B(\alpha+1,\alpha+1)} {}_{2}F_{1}(\alpha+3/_{2},1;\alpha+2;1-s^{2}) ,$$

$$\sigma(s) = \frac{\alpha+1}{s {}_{2}F_{1}(\alpha+3/_{2},1;\alpha+2;1-s^{2})} ,$$

$$G(s) = (\alpha+1) \left\{ s - \frac{2}{s {}_{2}F_{1}(\alpha+3/_{2},1;\alpha+2;1-s^{2})} \right\} ,$$

$$(4.78)$$

and for the corresponding probability (4.77)

$$E_{2}(0;I) = \frac{(1-s^{2})^{2\alpha+2}s^{4}}{4^{2\alpha+2}(2\alpha+3)B^{2}(\alpha+2,\alpha+2)} {}_{2}F_{1}(\alpha+\frac{3}{2},1;\alpha+2;1-s^{2}) {}_{2}F_{1}(\alpha+\frac{5}{2},1;\alpha+2;1-s^{2}) ,$$

$$\sigma(s) = (\alpha+1) \left\{ \frac{1}{s {}_{2}F_{1}(\alpha+\frac{3}{2},1;\alpha+2;1-s^{2})} + \frac{1}{s {}_{2}F_{1}(\alpha+\frac{5}{2},1;\alpha+2;1-s^{2})} \right\} ,$$

$$G(s) = (\alpha+2)s + 2(\alpha+1) \left\{ \frac{1}{s {}_{2}F_{1}(\alpha+\frac{3}{2},1;\alpha+2;1-s^{2})} - \frac{1}{s {}_{2}F_{1}(\alpha+\frac{5}{2},1;\alpha+2;1-s^{2})} \right\} ,$$

$$(4.79)$$

in terms of the Gauss hypergeometric function $_2F_1(a,b;c;z)$.

Proof - we proceed in a parallel manner as in the proof of Proposition 11 and the relations given there, however also using the linear transformation formulae for the Hypergeometric functions. One such an example is

$$4^{\alpha}B(\alpha+1,\alpha+1) - s_{2}F_{1}(-\alpha,\frac{1}{2};\frac{3}{2};s^{2}) = \frac{1}{2}\frac{s(1-s^{2})^{\alpha+1}}{\alpha+1}{}_{2}F_{1}(\alpha+\frac{3}{2},1;\alpha+2;1-s^{2}) .$$
(4.80)

We also require the comparable contiguous relation

$$-(2\alpha+3)s^{2}{}_{2}F_{1}(\alpha+5/2,1;\alpha+2;1-s^{2}) + {}_{2}F_{1}(\alpha+3/2,1;\alpha+2;1-s^{2}) + 2(\alpha+1) = 0.$$
(4.81)

One can show that these two specific cases are solutions to our third order differential equation (4.73), after noting the above contiguous relation linking the two Hypergeometric functions for the case of N = 2.

The special case of $\alpha = \beta = 0$ and general N is no longer a simple case as was the situation for the endpoint interval.

4.6 Jacobi to Hermite Limit for the Interior Interval

As we have shown in Subsection 4.3 one would also expect the limit $\alpha \to \infty$ would recover the result for the Hermite ensemble on (-t, t), under the scaling in (4.48). This result is given in [18], although not in a form useful for our purposes. In that reference a second order ODE is given for $\tilde{R}(t)$ whereas we want the corresponding ODE for R(t). Using the same kind of elimination indicated in the proof of Proposition 4 on their Equations (5.31) and (5.32) we find the result

$$tr'' + 2r' = -2t[t-g] - 2g\sqrt{(a')^2 + 4t[t-g]a - 2Nt^2[t-g]^2}, \qquad (4.82)$$

where $g = \sqrt{t^2 + 2r'}$ and a(t) = tr(t). To verify that (4.73) has the correct limiting behaviour, we first find that the substitution (4.48), to leading order in α^2 yields

$$r''' - 2 - \frac{(r'')^2}{r'} - \frac{2}{t^2}r' + \frac{2}{t^2}\frac{r^2}{r'}g[t+g] + 2r'g[t-g] - \left(4r - \frac{2}{t}\right)g + \frac{(tr'' - 2r')^2}{2tr'g^2}[t-g] = 0$$
(4.83)

and that differentiation of the second order Hermite ODE, (4.82), gives

$$tr''' + 3r'' = -4t + 4g^{2}(tr' + r - 2Nt) + 2g$$
$$-8t(r - Nt)g + (t + r'')(2t^{2} + tr'' + 2r')/g^{2}$$
(4.84)

with g as before. By repeating the steps described in Subsection 4.3 we can show that both (4.82) and (4.84) can be combined to give the limiting case, (4.83).

5 Reductions to Painlevé transcendents

The analysis in the preceding sections yielded a number of nonlinear differential equations of the second and third order. In this section we present their solutions in terms of the fifth and sixth Painlevé transcendents. In all cases, we can arrive at one of the equations of the second order and second degree found by Chazy [4, 5] and subsequently rederived by a number of authors.

We begin with equation (3.19) for the variable $\hat{R}(s)$. This equation is equivalent under a gauge transformation to an equation first derived by Chazy by transformation of the Painlevé-V equation. It is the fourth member of the set of five equations denoted (II) in [4] and (C) in [5, p. 342]. It was also derived by Bureau [3, pp. 204–206] in his investigation of second-order second-degree equations.

The solution of (3.19) is

$$\tilde{R}(s) = \frac{\epsilon_1 w' - 2sw}{4w(w-1)} - \frac{N(w-1) + \epsilon_1}{4sw}, \qquad (5.1)$$

where $\epsilon_1 := \pm 1$ and w(s) (not to be confused with w used previously) satisfies the differential equation,

$$w'' = \left\{ \frac{1}{2w} + \frac{1}{w-1} \right\} (w')^2 - \frac{1}{s} w' + \frac{(w-1)^2}{2s^2} \left\{ N^2 w - \frac{(N-\epsilon_1)^2}{w} \right\} + 2\epsilon_1 w - \frac{2s^2 w (w+1)}{w-1}.$$
(5.2)

The prime denotes d/ds. Under the change of variable, $x = s^2$, the latter equation becomes the standard Painlevé-V equation,

$$\frac{d^2w}{dx^2} = \left\{\frac{1}{2w} + \frac{1}{w-1}\right\} \left(\frac{dw}{dx}\right)^2 - \frac{1}{x}\frac{dw}{dx} + \frac{(w-1)^2}{x^2} \left\{\alpha w + \frac{\beta}{w}\right\} + \frac{\gamma w}{x} + \frac{\delta w(w+1)}{w-1}, \quad (5.3)$$

with parameters,

$$\alpha = \frac{1}{8}N^2, \qquad \beta = -\frac{1}{8}(N - \epsilon_1)^2, \qquad \gamma = \frac{1}{2}\epsilon_1, \qquad \delta = -\frac{1}{2}.$$
 (5.4)

In terms of the same w(s), the solution of the companion equation (3.20) is

$$R(s) = \frac{1}{8sw(w-1)^2} \left\{ sw' + N(w-1)^2 + (2s^2 - 1)w + 1 \right\} \\ \times \left\{ sw' - N(w-1)^2 - (2s^2 + 1)w + 1 \right\},$$
(5.5)

and the auxiliary variable h(s) is given by

$$h(s) = \frac{2sw^2 - \epsilon_1 w'}{2w(w-1)} + \frac{N(w-1) + \epsilon_1}{2sw}.$$
(5.6)

The last two equations also furnish the solution of (3.26) with h(s) being identified with the square root with the upper sign. The identity (3.27) is satisfied identically by the forms of R(s) and $\tilde{R}(s)$ given here. They also solve equation (4.49) after renaming the variables in (4.49) according to $t \to s$, $r(t) \to R(s)$. In that case, the parameter N plays the role of the third integration constant because (4.49) can be obtained from (3.20) by differentiating out the parameter N.

On the other hand, equation (4.50), which was also obtained by differentiation of (3.20), is a true generalisation of (3.20). Its first integral is

$$sR'' + 2R' = 2s(s-h) - 2h\sqrt{(R+sR')^2 - 4s^2(s-h)R - 2Ns^2(s-h)^2 + K_1},$$
 (5.7)

where h denotes $\sqrt{s^2 - 2R'}$ and K_1 is the integration constant. Although the case of nonzero K_1 is not relevant to the discussion in Section 4, we nevertheless have another integrable equation. Its solution is

$$R = \frac{1}{8sw(w-1)^2} \left\{ sw' + N(w-1)^2 + (2s^2 - 1)w + 1 \right\}$$

$$\times \left\{ sw' - N(w-1)^2 - (2s^2 + 1)w + 1 \right\}$$

$$- \frac{K(w+1) \left\{ N(w-1)^2 - 2s^2w \right\}}{2sw(w-1)}$$

$$+ \frac{K^2(w-1)(3w+1)}{2sw}, \qquad (5.8)$$

$$h = \frac{2sw^2 - \epsilon_1 w'}{2w(w-1)} + \frac{(N-2K)(w-1) + \epsilon_1}{2sw},$$
(5.9)

where $\epsilon_1 := \pm 1$, $K_1 = 8K^2(N - 2K)$, and w(s) satisfies the differential equation,

$$w'' = \left\{ \frac{1}{2w} + \frac{1}{w-1} \right\} (w')^2 - \frac{1}{s} w' + \frac{(w-1)^2}{2s^2} \left\{ (N-2K)(N+6K)w - \frac{(N-2K-\epsilon_1)^2}{w} \right\} + 2(4K+\epsilon_1)w - \frac{2s^2w(w+1)}{w-1}.$$
(5.10)

As before, the change of variable $x = s^2$ transforms (5.10) to a standard Painlevé-V equation. When K = 0, these results reduce to the above results for (3.20). By changing the sign of R, these formulae, with zero or nonzero K_1 as appropriate, also solve equations (4.82)–(4.84).

We now turn our attention to equation (4.29), which presents a somewhat greater challenge. This differential equation is of the third order and second degree. We find that it is more manageable when written in terms of the auxiliary variable H(s) rather than $\sigma(s)$. After that, we observe a slight improvement by choosing a new variable y(s), in terms of which we have

$$H(s) = (1 - s^{2})y - \alpha_{1}s,$$

$$\sigma(s) = \frac{1}{4} \Big\{ \alpha_{1}s^{2}y' - (s^{2} - 1)y^{2} \Big\}^{-1} \Big\{ s^{2}(s^{2} - 1)^{2} \big(yy''' - y'y'' \big) \\ + 2s(s^{2} - 1) \big[(4s^{2} - 1)yy'' - (2s^{2} - 1)(y')^{2} \big] \\ - 2y \big[2s^{2}(s^{2} - 1)^{2}y^{2} + 3\alpha_{1}s^{3}(s^{2} - 1)y - 6s^{4} + 3s^{2} + 1 \big] y' \\ - 2sy^{2} \big[(s^{2} - 1)(3s^{2} - 1)y^{2} + 2\alpha_{1}s(3s^{2} - 2)y \\ + 2(\alpha_{1}^{2} - 1)s^{2} \big] \Big\}.$$

$$(5.11)$$

The differential equation satisfied by y(s) takes the form,

$$\{2\sigma - \alpha_1 s^2 y\}^2 = -s^2 (s^2 - 1)^2 \{yy'' - (y')^2\} - 2s^3 (s^2 - 1)yy' + y^2 \{s^2 (s^2 - 1)^2 y^2 + 2\alpha_1 s^3 (s^2 - 1)y + (\alpha_1^2 - 1)s^4 + 1\},$$
(5.13)

where it is understood that σ is to be replaced by the right-hand side of (5.12). This differential equation of the third order and second degree admits the first integral,

$$\left\{ s(s^{2}-1)^{2}y'' + 2(s^{2}-1)(2s^{2}-1)y' + 8s^{3}y^{3} + 12\alpha_{1}s^{2}y^{2} + s(2s^{2}+K_{1}-6)y \right\}^{2}$$
$$= 4\left\{ (s^{2}+1)y + \alpha_{1}s \right\}^{2} \left\{ (s^{2}-1)^{2}(sy'+y)^{2} + s^{2}y^{2}(4s^{2}y^{2} + 8\alpha_{1}sy + K_{1}-4) \right\},$$
(5.14)

where K_1 is the constant of integration.

Equation (5.14) is equivalent under a gauge transformation to the fifth member of the aforementioned set of Chazy equations. Chazy found this equation by transforming the Painlevé-VI equation, but he did not show the actual formula. Bureau also obtained this equation [3, pp. 200–202] but did not solve it. The same equation in a different gauge was obtained by Fokas and Yortsos [8], who gave the reduction to Painlevé-VI.

The variables y(s), H(s), and $\sigma(s)$ are given in terms of a function w(s) by the formulae,

$$y = \frac{\epsilon_1 s(s^2 - 1)w' - (w - 1)\{\epsilon_1(w + s^2) + 2\alpha_1 s^2\}}{2s(s^2 - 1)w},$$
(5.15)

$$H = \frac{\epsilon_1 s (1 - s^2) w' + \epsilon_1 (w - 1) (w + s^2) - 2\alpha_1 s^2}{2sw}, \qquad (5.16)$$

$$\sigma = -\frac{\left\{s(s^2 - 1)w' - (w - 1)(w + s^2)\right\}^2}{8sw(w - 1)(w - s^2)} - \frac{s(w - 1)\left\{(K_1 - 4)w - 4\alpha_1^2 s^2\right\}}{8w(w - s^2)},$$
(5.17)

where $\epsilon_1 := \pm 1$. The function w(s) satisfies the differential equation,

$$w'' = \frac{1}{2} \left\{ \frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-s^2} \right\} (w')^2 - \left\{ \frac{1}{s} + \frac{2s}{s^2 - 1} + \frac{2s}{w-s^2} \right\} w' + \frac{w(w-1)(w-s^2)}{2s^2(s^2 - 1)^2} \times \left\{ 1 - \frac{(1 + 2\epsilon_1\alpha_1)^2s^2}{w^2} + \frac{(K_1 - 4\alpha_1^2)s^2(s^2 - 1)}{(w-s^2)^2} \right\}.$$
 (5.18)

Under the change of variable, $x = s^2$, the latter equation becomes the Painlevé-VI equation,

$$\frac{d^2 w}{dx^2} = \frac{1}{2} \left\{ \frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-x} \right\} \left(\frac{dw}{dx} \right)^2
- \left\{ \frac{1}{x} + \frac{1}{x-1} + \frac{1}{w-x} \right\} \frac{dw}{dx}
+ \frac{w(w-1)(w-x)}{x^2(x-1)^2} \left\{ \alpha + \frac{\beta x}{w^2} + \frac{\gamma(x-1)}{(w-1)^2} + \frac{\delta x(x-1)}{(w-x)^2} \right\},$$
(5.19)

with parameters,

$$\alpha = \frac{1}{8}, \qquad \beta = -\frac{1}{8}(1 + 2\epsilon_1\alpha_1)^2, \qquad \gamma = 0, \qquad \delta = \frac{1}{8}(K_1 - 4\alpha_1^2). \tag{5.20}$$

These formulae also solve (4.73) with the sign of σ changed.

We draw the reader's attention to another set of five second-order second-degree Chazy equations which make regular appearances in random matrix theory. This set is denoted (III) in [4] and (B) in [5, p. 340] and has also been studied by Bureau [2, 3], Fokas and Ablowitz [7], Jimbo and Miwa [13, Appendix C], and Cosgrove and Scoufis [6]. This set can be embraced in a single equation,

$$(y'')^{2} = -\frac{4}{g^{2}(x)} \Big\{ c_{1}(xy'-y)^{3} + c_{2}y'(xy'-y)^{2} + c_{3}(y')^{2}(xy'-y) \\ + c_{4}(y')^{3} + c_{5}(xy'-y)^{2} + c_{6}y'(xy'-y) + c_{7}(y')^{2} \\ + c_{8}(xy'-y) + c_{9}y' + c_{10} \Big\},$$
(5.21)

where $g(x) := c_1 x^3 + c_2 x^2 + c_3 x + c_4$ and the prime denotes d/dx. This is equation SD-I, introduced in [6, pp. 57, 65–73]. It was called there the "master Painlevé equation" because it unified all of the six Painlevé transcendents into a single equation. Only four of the 10 parameters in SD-I are essential because the equation retains its shape under gauge transformations of the form,

$$\bar{x} = \frac{ax+b}{cx+d}, \qquad \bar{y} = \frac{hy+kx+m}{cx+d}.$$
 (5.22)

By using this gauge freedom appropriately, equation SD-I can be split into six normalised forms, whose solutions in terms of Painlevé transcendents are given in [6].

The full version of SD-I generates an abundant supply of third-order equations under differentiation. For example, replacing c_i by $c_i + Kc_{i+6}$ for i = 5, ..., 10 and differentiating out the parameter K produces a 15-parameter equation (after normalising c_{11} , say, to 1 or 0). In a similar fashion, one can generate a 25-parameter equation of the third order and second degree. These big equations are all integrable in terms of Painlevé-VI in the generic case and one of the other five Painlevé transcendents or elliptic functions in the remaining cases.

A special case of the aforementioned 15-parameter equation appears, for example, in Tracy and Widom [18]. This case was solved recently in terms of Painlevé-VI by Haine and Semengue [11]. A more recent example, also by Tracy and Widom [19], is the equation,

$$y''' = \frac{1}{2} \left\{ \frac{1}{y'} + \frac{1}{y'-1} \right\} (y'')^2 - \frac{1}{x} y'' - \frac{2(k+n)}{x} y'(y'-1) + \frac{x+n}{2x^2} (n-x+2y) - \frac{(n+y)^2}{2x^2y'} - \frac{(x-y)^2}{2x^2(y'-1)},$$
(5.23)

the prime denoting d/dx. This equation has the first integral,

$$x^{2}(y'')^{2} = -4(k+n)x(y')^{3} + \{4(k+n)y+x^{2}+2(2k+3n)x\}(y')^{2} - \{2(x+2k+3n)y+2nx+n^{2}\}y' + (y+n)^{2} - 4K_{1}y'(y'-1),$$
(5.24)

where K_1 is the constant of integration.

The latter equation is gauge-equivalent to the normalised form SD-I.b in [6]. Its solution is

$$y = \frac{1}{4(k+n)w} \left\{ \frac{xw'}{w-1} - w \right\}^{2} - \frac{(w-1)\{k(w-1) + n(w+1)\}}{4w} + \frac{1}{4(k+n)} \left\{ \frac{2nx(w-2)}{w-1} - \frac{x^{2}w}{(w-1)^{2}} + 2kx + \frac{4K_{1}}{w} \right\},$$
(5.25)

where w(x) satisfies the Painlevé-V equation (5.3) with parameters,

 $\alpha = \frac{1}{2}(k+n+\epsilon_1)^2, \qquad \beta = \frac{1}{2}(n^2-k^2+4K_1), \qquad \gamma = -n, \qquad \delta = -\frac{1}{2}, \qquad (5.26)$ with $\epsilon_1 := \pm 1$.

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