Ground state and functional integral representations of the CCR algebra with free evolution

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Abstract

The problem of existence of ground state representations of the CCR algebra with free evolution are discussed and all the solutions are classified in terms of non regular or indefinite invariant functionals. In both cases one meets unusual mathematical structures which appear as prototypes of phenomena typical of gauge quantum field theory, in particular of the temporal gauge. The functional integral representation in the positive non regular case is discussed in terms of a generalized stochastic process satisfying the Markov property. In the indefinite case the unique time translation and scale invariant Gaussian state is suprisingly faithful and its GNS representation is characterized in terms of a Tomita-Takesaki operator. In the corresponding Euclidean formulation, one has a generalization of the Osterwalder-Schrader reconstruction and the indefinite Nelson space, defined by the Schwinger functions, has a unique Krein structure, allowing for the construction of Nelson projections, which satisfy the Markov property. Even if Nelson positivity is lost, a functional integral representation of the Schwinger functions exists in terms of a Wiener random variable and a Gaussian complex variable.

1 Introduction

Most of the wisdom on quantum field theory and more generally on quantum systems with infinite degrees of freedom relies on the existence of a vacuum or ground state, so that their formulation and control is obtained in terms of the ground state correlation functions. In particular the analytic continuation to imaginary time and the functional integral approach exploits this basic property [\[1,](#page-22-0) [2\]](#page-22-1). For a large class of models involving infrared singular canonical variables or fields the existence of a ground state is linked to non regular representations of such variables [\[3,](#page-22-2) [4,](#page-22-3) [5,](#page-22-4) [6\]](#page-22-5) if positivity is required, whereas regular, i.e. weakly continuous, representations are available if one allows indefinite metric [\[7,](#page-22-6) [8,](#page-22-7) [9\]](#page-22-8).

The mathematical problems emerging in these cases, including the euclidean functional integral formulation, already show up if one ask for the simple case of the ground state representations of the Heisenberg algebra with free time evolution. In particular the model reproduces the basic properties of the temporal gauge in quantum electrodynamics [\[10\]](#page-22-9), where the longitudinal variables $div\mathbf{A}$, $div\mathbf{E}$ correspond to a collection of free quantum mechanical canonical variables [\[11\]](#page-22-10).

Thus the analysis of the ground state representations of the model is an essential step for the solution of mathematical problems and for the construction of functional integral representations of the temporal gauge; in particular the model also displays the occurrence of unusual features in the temporal gauge, including the conflict between time translation invariance, energy spectral condition, positivity and the regularity of the vacuum state [\[11\]](#page-22-10). One also obtains non trivial information on the functional integral formulation of models with infrared singular variables, as in the temporal gauge. For these reasons, the analysis of the model has general implications on realistic quantum field theories and should not be dismissed on the basis of its trivial dynamics.

In Sect.2 we discuss the problem of ground state representations of the Heisenberg algebra, which correspond to the standard field quantization of the temporal gauge in terms of vacuum expectation of the canonical fields. Such representations are shown to violate positivity and share the basic features of the Tomita-Takesaki theory [\[12,](#page-22-11) [13\]](#page-22-12). The unique time translation and scale invariant state is in fact faithful on the Heisenberg algebra and a Tomita-Takesaki operator S is defined by $SA\Psi_0 = A^*\Psi_0$. The corresponding GNS representation, with cyclic ground state vector Ψ_0 , is given by the tensor product of a Fock and anti-Fock representations [\[14,](#page-22-13) [15\]](#page-22-14) of the canonical variables $Q_{\pm} \equiv (q \pm p')/\sqrt{2}$, $P_{\pm} \equiv (\pm p + q')/\sqrt{2}$, with $q' \equiv JqJ^*$, $p' \equiv JpJ^*$, where J is an antiunitary operator obtained from S as in the Tomita-Takesaki theory. This implies that the non positive vacuum of the (standard) temporal gauge has properties similar to those of a KMS state.

The ground state positive representations of the Weyl algebra are non regular and satisfy the energy spectral condition. The euclidean version satisfies Osterwalder-Schrader positivity and the corresponding functional integral representation satisfies Markov property [\[2,](#page-22-1) [16\]](#page-23-0). Actually, the euclidean correlation functions can be expressed as expectations of the euclidean fields $U(\alpha, \tau) = e^{i\alpha x(\tau)}$ with a functional measure which is the product of the conditional Wiener measure $dW_{0,x}(y(\tau))$ on trajectories $y(\tau)$ starting at x at time $\tau = 0$, times the ergodic mean $d\nu(x)$, which defines a measure on the Gelfand spectrum of the Bohr algebra [\[5\]](#page-22-4) generated by the $U(\alpha, \tau)$. The Osterwalder-Schrader reconstruction theorem provides the (non regular) quantum mechanical representation, with a ground state, which is cyclic with respect to the euclidean algebra at time zero.

In Sect. 3 we analyze the euclidean correlation functions of the indefinite representation of the Heisenberg algebra. We show that the failure of the Ostervalder-Schrader positivity does not prevent an Osterwalder-Schrader reconstruction, yielding the (indefinite) quantum mechanical representation. Nelson positivity fails and we have an indefinite inner product space with a Krein structure very similar to that of the massless scalar field in two space time dimensions [\[7\]](#page-22-6). A generalization of Nelson strategy [\[16\]](#page-23-0) can be performed and projection operators E_{\pm} , E_0 can be defined, yielding the same operator formulation of the Markov property as in the positive case.

A stochastic interpretation of the euclidean variables meets the problem of absence of Nelson positivity and of the non existence of invariant measures for the brownian motion. It is shown that a functional integral representation of the euclidean (indefinite) correlation functions $\langle x(\tau_1)...x(\tau_n)\rangle$ exists in terms of a Brownian variable $\xi(\tau)$ and of complex variables z, \bar{z}

$$
\langle x(\tau_1)...x(\tau_n) \rangle =
$$

$$
\int dW_{0,0}(\xi(\tau)) dz d\bar{z} e^{-2|z|^2} (\xi(\tau_1) + z - |\tau_1|\bar{z})...(\xi(\tau_n) + z - |\tau_n|\bar{z}).
$$

The above analysis provides a framework which can be applied e.g. to the massless scalar model in two space time dimensions to obtain a functional integral representation of its euclidean correlation functions.

2 Ground state representations of the Heisenberg algebra with free evolution

Our model is defined by the Heisenberg *-algebra \mathcal{A}_H generated by q, p (for simplicity we consider the one dimensional case), invariant under the *-operation (hermiticity) and satisfying the canonical commutation relations (CCR)

$$
[q, p] = i. \tag{2.1}
$$

invariant under the *-operation $q = q^*$, $p = p^*$ The time evolution is defined by the following one parameter group of $*$ -automorphisms α_t , $t \in \mathbf{R}$

$$
\alpha_t(q) = q + pt \equiv q(t) , \qquad \alpha_t(p) = p . \qquad (2.2)
$$

Motivated by Wightman formulation of quantum field theory, it is natural to ask whether there are time translationally invariant hermitean linear functionals ω on \mathcal{A}_H , $\omega(\alpha_t(A)) = \omega(A)$, and to investigate the generalized GNS representations defined by them.

By eq. (2.2) the correlation functions $\omega(A \alpha_t(B) C)$, $A, B, C \in \mathcal{A}_H$ are polynomials in t and therefore time translation invariance of ω implies that their Fourier transform has support at the origin and therefore satisfy the energy spectral condition, which characterizes ground states.

Proposition 2.1 *A hermitean linear functional* ω *on the Heisenberg algebra invariant under time translations, i.e. such that* $\forall A \in A_H$, $\omega(\alpha_t(A)) = \omega(A)$, *is not positive, defines a GNS representation in an indefinite inner product space* $D = \pi_{\omega}(\mathcal{A}_H)\Psi_{\omega}$ *and the two point functions have the following form*

$$
\omega(q^2) = c
$$
, $\omega(p^2) = 0$, , $\omega(qp) = -\omega(pq) = i/2$, (2.3)

where c *is a constant.*

If ω *is Gaussian, i.e. its truncated correlation functions vanish, the GNS representation* (D, π_{ω}) *defined by it has a non trivial commutant* $\pi_{\omega}(A_H)'$ *.*

Moreover, if ω *is Gaussian and* $c = 0$ *, then* ω *is invariant under scale transformations*

$$
q \to \lambda q, \quad p \to \lambda^{-1} p, \quad \lambda \in \mathbf{R},
$$

and such transformations are implemented by isometric operators in D*.*

Proof Since

$$
\omega(p^2) = \omega(d\alpha_t(q)/dt \, p) = d/dt \, \omega(\alpha_t(qp)) = 0,
$$

if positivity holds, $|\omega([q, p])| \leq \omega(q^2)^{1/2} \omega(p^2)^{1/2} = 0$, in contrast with the CCR.

Equations (2.3) follow from the condition of time translation invariance of the two point function $\omega(q(t) q(s))$ and the CCR.

By a standard GNS construction ω provides a representation π_{ω} of the Heisenberg algebra on a vector space $D = \pi_{\omega}(\mathcal{A}_H)\Psi_{\omega}$, with an indefinite non-degenerate inner product given by

$$
\langle A\Psi_{\omega}, B\Psi_{\omega} \rangle = \omega(A^*B).
$$

If ω is Gaussian, the matrix elements of the operator $U(t)$ which implements the time translations on D can be explicitly computed and the (weak) derivative $dU(t)/dt$ exists and defines a hermitean operator H on D. Without loss of generality, we can redefine H so that $H\Psi_{\omega} = 0$; henceforth Ψ_{ω} will be denoted by Ψ_0 . Now, the CCR imply

$$
H = p^2/2 + h , \quad h \in \pi_\omega(\mathcal{A}_H)'
$$
 (2.4)

and, if $\pi_{\omega}(\mathcal{A}_H)' = {\lambda \mathbf{1}}$, $\lambda \in \mathbf{C}$, the equations $(\Psi_0, H\Psi_0) = 0$, $\omega(p^2) = 0$ imply $h = 0$. This leads to a contradiction

$$
0 = \omega(q^2 H) = \omega(q p)^2 = -1/4.
$$

The last statement follows from scale invariance of the correlation functions. ш

The existence of a non-trivial commutant is reminiscent of the structure which characterizes the theory of KMS (non zero temperature) states and more generally the Tomita-Takesaki theory. The occurrence of such a structure in a ground state representation is a consequence of the fact that the functionals of Proposition 2.1 are faithful.

Proposition 2.2 *The Gaussian functionals* ω *of Proposition 2.1 are faithful, i.e.,*

$$
\omega(BA) = 0, \quad \forall B \in \mathcal{A}_H \tag{2.5}
$$

implies $A = 0$ *.*

Furthermore the commutant $\pi_{\omega}(\mathcal{A}_H)'$ *is given by*

$$
\pi_{\omega}(\mathcal{A}_H)' = S \pi_{\omega}(\mathcal{A}_H) S, \qquad (2.6)
$$

where S *is a Tomita-Takesaki antilinear operator defined by*

$$
SA\Psi_0 \equiv A^* \Psi_0, \quad \forall A \in \pi_\omega(\mathcal{A}_H), \tag{2.7}
$$

with Ψ_0 *the vector which represents the state* ω *.*

Proof Any $A \in \mathcal{A}_H$ can be written in an unique way as an ordered polynomial $A = \sum_{k=0}^{N} q^k P_k(p)$, with P_k a polynomial. Then, by time translation invariance of ω , eq.(2.5) implies $\omega(B \alpha_t(A)) = 0$, $\forall t$. Thus the term of highest power of t, $\omega(B p^N P_N(p))$ vanishes $\forall B \in \mathcal{A}_H$; in particular $\omega(q^k p^N P_N(p)) = 0$, $\forall k$ which implies $P_N(p) = 0$, by the gaussian property of ω and eqs.(2.3).

Since ω is faithful, eq.(2.7) defines the analogue of the Tomita-Takesaki antilinear operator S on D. Clearly, $S\Psi_0 = \Psi_0$ and $S^2 = 1$. As in the standard Tomita-Takesaki theory one has

$$
SAS \in \pi_{\omega}(\mathcal{A}_H)' , \quad \forall A \in \pi_{\omega}(\mathcal{A}_H).
$$

Furthermore, since Ψ_0 is cyclic for $\pi_\omega(\mathcal{A}_H)$ and therefore separating for $\pi_{\omega}(\mathcal{A}_H)'$, $\forall B \in \pi_{\omega}(\mathcal{A}_H)'$, $\exists A \in \pi_{\omega}(\mathcal{A}_H)$ such that $B\Psi_0 = A^* \Psi_0$ and therefore $B = SAS$; in conclusion

$$
S\pi_{\omega}(\mathcal{A}_H)S = \pi_{\omega}(\mathcal{A}_H)'.
$$
\n(2.8)

 \blacksquare

For $c = 0$ the Tomita-takesaki structure is particularly simple and close to the positive case; the case $c \neq 0$ requires a more general analysis of Gaussian indefinite functionals, which will be done elsewhere.

Proposition 2.3 If ω *is the Gaussian hermitean linear functional on the Heisenberg algebra* \mathcal{A}_H *, defined by eqs.(2.3) with* $c = 0$ *, then* $\pi_\omega(\mathcal{A}_H)'$ *is the* (concrete) algebra A'_H generated by pseudo-canonical variables q', p'

$$
q' \equiv J q J^*, \quad p' \equiv J p J^*, \quad [q', p'] = -i,
$$
\n(2.9)

where J *is an anti-unitary operator with respect to the indefinite inner product of* D*:*

$$
J J^* = J^* J = 1.
$$
\n(2.10)

J is related to S and to the (non positive) modular operator $\Delta \equiv S^* S$ by

$$
S = J \triangle^{1/2}.
$$
\n
$$
(2.11)
$$

 $\pi_{\omega}(\mathcal{A}_H)$ *is the tensor product of a Fock and an anti-Fock representation, respectively, for the pairs of canonical variables*

$$
Q_{\pm} \equiv (q \pm p')/\sqrt{2} \; , \quad P_{\pm} \equiv (\pm p + q')/\sqrt{2} \tag{2.12}
$$

which satisfy

$$
[Q_{\pm}, P_{\pm}] = \pm i
$$
, $[Q_{\mp}, P_{\pm}] = 0.$ (2.13)

In fact, by introducing the operators

$$
a \equiv (Q_+ + iP_+)/\sqrt{2}
$$
, $b^* \equiv (Q_- + iP_-)/\sqrt{2}$, (2.14)

which satisfy

$$
[a, a^*] = 1 = [b, b^*],
$$
, $[a, b] = [a, b^*] = 0$, (2.15)

one has that Ψ_0 *satisfies the Fock and anti-Fock conditions*

$$
a\Psi_0 = 0, \qquad b^* \Psi_0 = 0. \tag{2.16}
$$

Proof The adjoint of S satisfies

$$
S^*\Psi_0 = \Psi_0 \ , \quad S^{*2} = 1 \ , \quad S^*\pi_\omega(\mathcal{A}_H)'S^* = \pi_\omega(\mathcal{A}_H). \tag{2.17}
$$

In fact, one has

$$
\langle A\Psi_0, S^*\Psi_0 \rangle = \langle S A \Psi_0, \Psi_0 \rangle = \langle A \Psi_0, \Psi_0 \rangle;
$$

the last eq. (2.17) follows from eq. (2.8) .

Because of the indefinite inner product, the analog of the modular operator $\Delta \equiv S^*S$ is not positive and in fact it is given by

$$
S^* Sp^k q^j \Psi_0 = S^* q^j p^k S^* \Psi_0 = (-1)^{j+k} p^k q^j \Psi_0.
$$

Even if Δ is not positive one may introduce a hermitean square root of Δ

$$
\Delta^{1/2} p^k q^j \Psi_0 \equiv i^k (-i)^j p^k q^j \Psi_0. \tag{2.18}
$$

In fact

$$
\langle \Delta^{1/2} p^l q^m \Psi_0, p^k q^j \Psi_0 \rangle = (-i)^l i^m \omega (q^m p^l p^k q^j) =
$$

$$
i^k (-i)^j \omega (q^m p^l p^k q^j) = \langle p^l q^m \Psi_0, \Delta^{1/2} p^k q^j \Psi_0 \rangle,
$$

since the above correlation functions vanish unless $k+l = m+j$. A hermitean $\triangle^{-1/2}$ is defined by changing *i* into $-i$ in the definition of $\triangle^{1/2}$.

In analogy with the non zero temperature case, one may then introduce the analog of the modular conjugation J defined by $J = S \triangle^{-1/2}$. Equation (2.10) follows and as in the standard case $J\Delta^{1/2}J = \Delta^{-1/2}$.

We may then introduce the following hermitean operators

$$
q' \equiv J q J^* = iS q S , \qquad p' \equiv J p J^* = -iS p S. \tag{2.19}
$$

They satisfy the following (pseudo-)canonical commutation relations

$$
[q', p'] = J [q, p] J^* = -i.
$$

Furthermore, by the definition of the antilinear operator S , we have

$$
(q + iq')\Psi_0 = (q - SqS)\Psi_0 = 0 , \qquad (p - ip')\Psi_0 = (p - SpS)\Psi_0 = 0.
$$

The above equations imply eqs.(2.16). They also imply that the operator

$$
H \equiv (p^2 + p'^2)/2 = (p + ip')(p - ip')/2 = (a^* - b)(a - b^*)/2 \tag{2.20}
$$

annihilates Ψ_0 and can be taken as the Hamiltonian, since it has the right commutation relations with q, p. By the same kind of argument, Ψ_0 is annihilated also by $q^2 + (q')^2$.

In conclusion, even if the energy spectral condition is satisfied by the correlation functions of ω , the corresponding GNS representation of $\pi_{\omega}(\mathcal{A}_H)$ exhibits features similar to those of a KMS state. The modular group can be written as

$$
(\Delta^{1/2})^{it} = e^{-i\pi A t}, \ \ A = -\frac{1}{2} (pq + qp - p'q' - q'p').
$$

The representation of \mathcal{A}_H in an indefinite inner product space characterized in Proposition 2.3 can be given a Krein structure.

Proposition 2.4 *The indefinite inner product representation space* D = $\pi_{\omega}(\mathcal{A}_H)$) Ψ_0 *can be given a Krein structure by introducing a metric operator* η *with* $\eta \Psi_0 = \Psi_0$ *and*

$$
\eta a \eta = a \;, \qquad \eta b \eta = -b \;, \tag{2.21}
$$

equivalently,

$$
\eta q \eta = p' \;, \qquad \eta p \eta = q' \;.
$$
 (2.22)

Proof It follows easily from the positivity of the two point function $(b \Psi_0, \eta b \Psi_0)$.

The so obtained structure provides an example of the Tomita-Takesaki theory in indefinite inner product spaces; surprisingly, the commutant is described by pseudo-canonical variables, as it happens for the time component of the electromagnetic potential in the Gupta-Bleuler formulation of the free electromagnetic field. The above representation of b, b^* is the same as the anti-Fock representation of the CCR discussed in [\[14,](#page-22-13) [15\]](#page-22-14)

The above representation of the Heisenberg algebra in a Krein space allows for the construction of the Weyl algebra (equivalently of the Heisenberg group) as the algebra generated by the (pseudo) unitary operators $U(\alpha) \equiv \exp i\alpha q$, $V(\beta) \equiv \exp i\beta p$. Therefore, in this way one gets a ground state (regular) representation of the Weyl algebra for the free particle; however, the correlation functions of ther Weyl operators do not satisfy the energy spectral condition (see Appendix A).

The lack of positivity of the time translationally invariant state can be avoided by looking for representations of the Weyl algebra which do not yield representations of the Heisenberg algebra. The analog of eq.(2.2) is now

$$
\alpha_t(W(\alpha, \beta)) = W(\alpha, \beta + \alpha t). \tag{2.23}
$$

and the (unique) representation defined by a translationally invariant state Ω is identified by the following expectations

$$
\Omega(W(\alpha, \beta)) = 0, \quad if \alpha \neq 0; \quad \Omega(W(0, \beta)) = 1.
$$
 (2.24)

In this case the Fourier transforms of the correlation functions $\Omega(A\alpha_t(B))$ are measures with support contained in \mathbb{R}^+ , see Appendix B. The corresponding euclidean formulation satisfies Nelson positivity and therefore it admits a unique functional integral representation (see Appendix C).

3 Euclidean formulation and stochastic processes. Indefinite case

a. Schwinger functions and Osterwalder-Schrader reconstruction

The n-point Schwinger functions obtained by Laplace transforms of the Wightman functions of q factorize as usual for Gaussian states and define a functional Φ_S on the euclidean Borchers' algebra, fully determined by the two point function

$$
\Phi_S(\overline{f} \times g) \equiv \langle f, g \rangle \equiv \int d\tau_1 d\tau_2 \overline{f}(\tau_1) \mathcal{S}(\tau_1, \tau_2) g(\tau_2), f, g \in \mathcal{S}(\mathbf{R}) \equiv \mathcal{S},
$$

$$
\mathcal{S}(\tau_1, \tau_2) = S(\tau) = c - |\tau|/2.
$$
 (3.1)

Euclidean fields $x(f)$, $f \in S$, f real, can be introduced as usual, with two point function

$$
\langle x(f)x(g)\rangle = \langle f, g\rangle
$$

and n-point functions given by the functional $(f \text{ real})$

$$
\langle e^{ix(f)} \rangle \equiv e^{-\langle f, f \rangle/2} = e^{-\int f(\tau)f(\sigma) S(\tau - \sigma) d\tau \, ds/2}.
$$
 (3.2)

The Osterwalder-Schrader positivity is not satisfied, since $S(-\tau_1, \tau_2) = c (\tau_1 + \tau_2)/2$, $\tau_1, \tau_2 \ge 0$, is not a positive kernel and therefore

$$
\Phi_S(\overline{\theta f} \times f), \quad (\theta f)(\tau) \equiv f(-\tau),
$$

is not positive.

However, the Osterwalder-Schrader (O-S) reconstruction of the real time indefinite vector space, in terms of the Schwinger functions can be done by the standard extension of the recostruction without positivity [\[17\]](#page-23-1). The O-S scalar product is defined by factorization starting from

$$
\langle f, g \rangle_{OS} \equiv \langle \theta f, g \rangle = i/2 \left(\overline{\tilde{f}'(0)} \, \tilde{g}(0) - \overline{\tilde{f}(0)} \, \tilde{g}'(0) \right) + c \overline{\tilde{f}(0)} \, \tilde{g}(0)
$$

for $f, g \in \mathcal{S}(\mathbf{R}^+) \equiv \mathcal{S}^+$.

The null space of the O-S scalar product,

$$
N_{OS} \equiv \{ f \in \mathcal{S}(\mathbf{R}^+): \langle f, g \rangle_{OS} = 0 \ \forall g \in \mathcal{S}(\mathbf{R}^+)\},
$$

has codimension two, so that the space $\mathcal{S}(\mathbf{R}^+)/N_{OS}$ is two–dimensional.

b. Nelson space

In contrast with the standard case, $\langle \cdot, \cdot \rangle$ is not positive and therefore does not define a Gaussian measure for the functional integral representation of the correlation functions (3.2).

Proposition 3.1 *The inner product* < ... > *is positive on* $S_0(\mathbf{R}) \equiv \{f \in$ $\mathcal{S}(\mathbf{R}), \int f d\tau = 0$, and there is a function χ such that $\langle \chi, \chi \rangle = -1$. *Thus* S(R) *is an indefinite inner product space with one negative dimension (pre-Pontriagin space).*

Proof The proof follows easily since the Fourier transform of S is

$$
\tilde{S}(\omega) = 2\pi c \,\delta(\omega) - (d/d\omega) P(1/\omega),
$$

where P denotes the principal value.

For simplicity, in the following we consider the case $c = 0$. The above indefinite inner product space has a functional structure very similar to the indefinite inner product space defined by the (real time) two point function of the massless scalar field in two space time dimension [\[7\]](#page-22-6). As in that case, there is only one extension [\[15\]](#page-22-14) of $\mathcal{S}(\mathbf{R})$ to a weakly complete inner product space $\overline{S}(R)$, (also briefly denoted by \overline{S}), since the negative space has finite dimensions [\[7,](#page-22-6) [15\]](#page-22-14).

Proposition 3.2 *The space* $\overline{S}(R)$ *has the following decomposition, where* $<\oplus>$ *denotes orthogonal sum with respect to* $<\dots$,

$$
\overline{\mathcal{S}}(\mathbf{R}) = \overline{\mathcal{S}}_{00} < \oplus > \mathcal{V},\tag{3.3}
$$

$$
\mathcal{S}_{00} \equiv \left(\frac{d}{d\tau}\right)^2 \mathcal{S}^+ < \oplus > \left(\frac{d}{d\tau}\right)^2 \mathcal{S}^-,\ \ \mathcal{S}^\pm \equiv \mathcal{S}(\mathbf{R}^\pm),\ \mathcal{V} \equiv \{a\delta_0 + bw,\ a,b \in \mathbf{C}\},\
$$

$$
\langle \delta_0, \delta_0 \rangle = 0 = \langle w, w \rangle, \langle \delta_0, w \rangle = -1/2, \tag{3.4}
$$

$$
<\delta_0, f> = -\int d\sigma f(\sigma)|\sigma|, < w, f> = -\tilde{f}(0)/2,
$$
 (3.5)

the product $\langle \cdot, \cdot \rangle$ *is positive on* S_{00}

$$
\langle f, g \rangle = (F', G')_{L^2}, f = F'', g = G'', F, G \in S^{\pm}, \tag{3.6}
$$

and \overline{S}_{00} denotes the closure of S_{00} with respect to it.

 $\mathcal{S}(\mathbf{R})$ can be turned into a Krein space K_{α} , such that the above decompo*sition is also orthogonal with respect to the Krein scalar product* $(\alpha \in \mathbb{R}^+)$

$$
[f, g]_{\alpha} = \equiv +2 < \alpha \delta_0 + \alpha^{-1} w, g> (3.7)
$$

(3.7)
where the metric $p, x^2 = 1$ is defined by $\alpha \delta_0 + \alpha^{-1} w$ being its negative

where the metric η_{α} , $\eta_{\alpha}^2 = 1$ *is defined by* $\alpha \delta_0 + \alpha^{-1} w$ *being its negative eigenvector.*

Proof The sequences d_n , with $d_n(\tau)$ smooth positive approximations of the Dirac delta function $\delta(\tau)$, converge weakly in the topology of the inner product (3.1); also the sequences $f_n(\tau) = f(\tau - n)/n$, $f \in \mathcal{S}(\mathbf{R})$, $\tilde{f}(0) = 1$, are weakly convergent. In fact one has

$$
\lim_{n \to \infty} \langle \delta_n, g \rangle = -\frac{1}{2} \int g(s) |s - \sigma| ds, \quad \lim_{n \to \pm \infty} \langle f_n, g \rangle = \mp \tilde{g}(0)/2. \tag{3.8}
$$

Actually, all the above sequences are uniformly weakly compatible sequences in the sense of [\[15\]](#page-22-14), i.e. $\lim_{n,m} < u_n$, $u_m >$ exists and $\lim_{n} < u_n$, $g >$ exists $\forall g \in \mathcal{S}(\mathbf{R})$. Therefore they define a weak extension \mathcal{S}_{ext} of $\mathcal{S}(\mathbf{R})$ through the addition of the elements

$$
\delta_0 = w - \lim_{n \to \infty} d_n, \quad w = w - \lim_{n \to \infty} f_n.
$$
\n(3.9)

Since

$$
\lim_{(n,m)\to\infty} =0, \ \lim_{(n,m)\to\infty} =-1/2, \ \lim_{(n,m)\to\infty} =0
$$

we have

$$
\langle w, g \rangle = -\tilde{g}(0)/2, \langle w, w \rangle = 0, \langle \delta_0, w \rangle = -1/2, \langle \delta_0, \delta_0 \rangle = 0.
$$
\n(3.10)

The product (3.7) is well defined in \mathcal{S}_{ext} and makes it a pre-Krein space; $\mathcal{S}(\mathbf{R})$ is dense in the Krein completion of \mathcal{S}_{ext} , which therefore coincides with the unique weakly complete extension $S(\mathbf{R})$ of $\mathcal{S}(\mathbf{R})$.

 $\overline{S}(\mathbf{R})$ contains also the weak limits δ_{σ} of the sequences $d_n(\tau-\sigma)$, $\forall \sigma \in \mathbf{R}$, which satisfy

$$
\langle \delta_{\sigma}, \delta_{\rho} \rangle = -\frac{1}{2} |\sigma - \rho|.
$$
 (3.11)

In order to prove eq.(3.3), we have to show that $f \in \overline{S}$, $\langle f, V \rangle = 0$, and $\langle f, \mathcal{S}_{00}\rangle = 0$ implies $f = 0$. In fact, if $f_n \to f$ strongly in $\overline{S}(\mathbf{R})$, then

 $< f_n, \delta_0 > \to 0, \, < f_n, w > \to 0.$ One can then construct a sequence $h_n \to f$ with $\langle h_n, w \rangle = 0 \implies h_n, \delta \rangle$; by eq.(3.10) this implies $h_n \in \mathcal{S}_0(\mathbf{R})$, i.e. $h_n = d g_n / d\tau$. Then, one has

$$
[h_n - h_m, h_n - h_m]_{\alpha} = \langle h_n - h_m, h_n - h_m \rangle = ||g_n - g_m||_{L_2}^2,
$$

so that g_n converge in L_2 . Now, $\lt f$, $S_{00} \gt 0$ implies that $\forall k^{\pm} \in S^{\pm}$

$$
\langle h_n, d^2 k^{\pm}/d\tau^2 \rangle = (g_n, dk^{\pm}/d\tau)_{L_2} \to 0,
$$

so that $g_n \to 0$, in L_2 and $h_n \to 0$ in \overline{S} .

The other statements of the Proposition follow easily. \blacksquare

In the parametrization of the Krein scalar products, α plays the rôle of a scale parameter; thus, even if the correlation functions are scale invariant, the Krein products are not.

As usual, one can construct the Fock space $\Gamma(K)$ over K. Euclidean fields $x(f)$, f real, act on $\Gamma(K)$ and by standard arguments they can be extended from S to K; the existence of δ_0 and w are equivalent to the existence of the fields $x(\tau)$ at fixed times, $x \equiv x(0)$, and of the (positive time) ergodic mean of the velocity, namely $v \equiv \lim_{\tau \to +\infty} x(\tau)/\tau$.

The OS scalar product extends by continuity to the Krein closure of \mathcal{S}^+ and K_{OS} can be identified with the subspace V of the Krein closure $\mathcal{S}(\mathbb{R}^+)$ generated by δ_0 and w. In fact, by eqs. (3.4), (3.5), $\forall f, g \in \mathcal{S}^+$

$$
\langle f, g \rangle_{OS} = \langle E_0 f, E_0 g \rangle, \ E_0 f \equiv \tilde{f}(0) \, \delta_0 + i \tilde{f}'(0) \, w.
$$

The above decomposition allows to control the failure of Nelson positivity and from this point of view the model can be seen as an example of a general structure which substantially generalizes Nelson strategy to cases in which positivity fails and in particular allows for a generalization of the formulation of the Markov property in terms of projections (see Appendix D).

4 Functional integral representation

The decomposition (3.3) provides the tool for writing a functional integral representation of the functional (3.2). In fact, by eq.(3.6), $\langle e^{ix(f)} \rangle$, $f \in$ S_{00} , $f = F''$, can be represented in terms of a functional integral with Wiener measure

$$
\langle e^{ix(f)} \rangle = \int dW_0(\xi(\tau)) e^{i\xi(f)}, \tag{4.1}
$$

since for $f \in S_{00}$ one has $\int d\tau |\tau| f(\tau) = 0$ and therefore

$$
\langle f, f \rangle = \int dW_0(\xi(\tau)) \xi(f)^2.
$$

We are then left with the case $f \in \mathcal{V}$.

Quite generally, a Gaussian functional defined by a (not necessarily positive) quadratic form on a finite dimensional space V has a canonical representation in terms of a positive measure and complex random variables.

This is easily seen in our case, where the two dimensional space $\mathcal V$ is generated by δ_0 and w. Then, one has

$$
\langle \alpha \delta_0 + \beta w, \alpha \delta_0 + \beta w \rangle = -\alpha \beta = 2\pi^{-1} \int dz \, d\bar{z} \, e^{-2|z|^2} (\alpha z - \beta \bar{z}) (\alpha z - \beta \bar{z})
$$
\n(4.2)

Thus, for any polynomial P one has

$$
<\mathcal{P}(x, v) > = \pi^{-1} \int dz \, d\bar{z} \, e^{-2|z|^2} \mathcal{P}(z, \bar{z}).
$$
 (4.3)

In conclusion, the decomposition according to eq.(3.3)

$$
\delta_{\tau}(\sigma) = \delta_0 + |\tau|w + h_{\tau}(\sigma), \ \ h_{\tau}(\sigma) \in \overline{\mathcal{S}}_{00}
$$

gives the following decomposition of the field variables

$$
x(\tau) = x(\delta_{\tau}) = x + |\tau|v + x(h_{\tau}), \tag{4.4}
$$

with

$$
\langle x(h_{\tau}) x(h_{\sigma}) \rangle = +1/2(-|\tau - \sigma| + |\tau| + |\sigma|).
$$

Therefore, putting $z = z_1 + iz_2$,

$$
\langle x(\tau_1)...x(\tau_n) \rangle =
$$

$$
2\pi^{-1} \int dW_{0,0}(\xi(\tau)) dz_1 dz_2 e^{-2|z|^2} (\xi(\tau_1) + z - |\tau_1|\bar{z})...(\xi(\tau_n) + z - |\tau_n|\bar{z}).
$$

(4.5)

The occurrence of complex random variables should not be regarded as an oddity, since the operators x, v are hermitean with respect to the indefinite product but they are not self adjoint operators in the Hilbert-Krein K_{α} ,

since they do not commute with the metric η_{α} . Their realization as multiplication operators should therefore account for their non real spectrum and lead to complex variables. This phenomenon looks of general nature and therefore should generically appear in local formulations of gauge theories, where positivity does not hold (for an example see Ref. [\[11\]](#page-22-10).

The above functional integral representation gives direct information on the Hilbert-Krein structure which can be associated to the correlation functions (4.5). In fact, the Krein scalar product (3.7) can be associated to the Krein two point function of a Gaussian euclidean field $x_{\alpha}(f)$

$$
[f, g]_{\alpha} = [x(f)\Psi_0, x(g)\Psi_0]_{\alpha} = \langle \Psi_0, x_{\alpha}(f) x_{\alpha}(g) \Psi_0 \rangle,
$$

$$
x_{\alpha}(f) \equiv 1/2(x(f) + \eta_{\alpha} x(f) \eta_{\alpha}) - i/2(x(f) - \eta_{\alpha} x(f) \eta_{\alpha}).
$$

The positive Gaussian functional

$$
[e^{ix(f)}]_{\alpha} \equiv e^{-[f,f]_{\alpha}/2},\tag{4.6}
$$

defines a Gaussian measure on \mathcal{S}'

$$
\int d\mu_{\alpha} e^{ix(f)} \equiv e^{-[f,f]_{\alpha}/2} \tag{4.7}
$$

and it is not difficult to see that the explicit form of the corresponding functional integral is obtained by replacing the complex variables z, \bar{z} in eq.(4.5) by real variables $\alpha^{-1}(z_1 + z_2)$, $-\alpha(z_1 - z_2)$, corresponding to x, v.

An explicit analysis of the gaussian measure defined by the Krein scalar product (generating kernel, Markov properties etc.) immediately follows from such a representation.

The Markov property for $d\mu_{\alpha}$ holds in the variables $\tilde{\xi}(\tau) \equiv x + \xi(\tau)$, $v(\tau)$, with $v(\tau)$ constant in τ ; in fact, $d\mu_{\alpha}$ defines a measure on trajectories $\xi(\tau)$, $v(\tau)$, concentrated on $v(\tau) = v$ and satisfying the Markov property in the two variables $x(\tau)$, $v(\tau)$.

Clearly, the process defined by $d\mu_{\alpha}$ corresponds to Brownian motion, with gaussian distribution of variance $\alpha^{-2}/2$ for the position at $\tau = 0$, with an additional gaussian variable, with variance $\alpha^2/2$, describing non-zero mean velocities, with opposite and constant values for $\tau > 0$ and $\tau < 0$. The limit $\alpha \rightarrow 0$ exists in the sense of positive functionals on the Bohr algebra and coincides with the measure on its spectrum discussed in Appendix C.

A Appendix A. Indefinite representations of the Weyl algebra

Proposition A.1 *The formal series corresponding to the exponentials* $U(\alpha)$ *,* $V(\beta), \alpha, \beta \in \mathbf{R}$ *converge strongly in the Krein topology defined in Proposition 2.4 and define (pseudo) unitary operators in the corresponding Krein space. The so obtained Weyl operators* $U(\alpha)$, $V(\beta)$ *generate a Weyl algebra* \mathcal{A}_W *and in this way one gets a regular representation in a Krein space, defined by a state* ω *invariant under the free time evolution and characterized by the following expectations*

$$
\omega(e^{i\alpha q} e^{i\beta p}) = e^{-i\alpha \beta/2} \qquad \forall \alpha, \beta \in \mathbf{R} \tag{A.1}
$$

Proof By expressing q and p in terms of the destruction and creation operators a, a^*, b, b^* defined in Proposition 2.3, one gets (by Fock space methods) the analog of the standard Fock estimate

$$
||q^n \Psi_0||_K \le 2^n \sqrt{n!},
$$

where $|| \ ||_K$ denotes the Krein-Hilbert norm, and a similar estimate for $p^n \Psi_0$. As in the standard case, this yields the strong convergence of the series and the existence of the (pseudo) unitary operators $U(\alpha)$, $V(\beta)$. The above expectations follow from the correlation functions

$$
\omega(q^n p^m) = \delta_{n,m} (i/2)^n n! \tag{A.2}
$$

It is worthwhile to mention that the operators $U(\alpha)$, $V(\beta)$ do not commute with the metric operator η , and therefore they are not unitary with respect to the positive (Hilbert) scalar product (\cdot, η) defined by η . Since ω is time translation invariant, the time evolution automorphisms $\alpha_t, t \in \mathbb{R}$ are implemented by a one parameter group $\mathcal{U}(t)$ of (pseudo) unitary operators, and information about their spectral properties is given by the Fourier transforms of the correlation functions $\omega(A \alpha_t(B))$.

Proposition A.2 *In the representation of the Weyl algebra in the Krein space discussed above, the Fourier transform of the correlation functions* $\omega(A \alpha_t(B))$ are tempered distributions with the following properties: *i)* for $A, B \in \mathcal{A}_H$ *they have support at the origin, ii)* for $A, B \in A_W$ *they are measures with support in the real line.*

Proof In fact, by the definition of the time evolution and the invariance of ω one has

$$
\omega(p^k q^j \alpha_t(p^l q^m)) = P_m(t),
$$

with $P_m(t)$ a polynomial of degree m. On the other hand

$$
\omega(U(\alpha) \alpha_t(U(\beta))) = e^{-i\alpha\beta t/2}.
$$

In the latter case, the support of the Fourier transform is contained in the positive real axis for the diagonal expectations, $\alpha = -\beta$, but not in general.

B Appendix B. Ground state positive representations of the Weyl algebra

Proposition B.1 *A time translationally invariant positive state* Ω *on the Weyl algebra* A^W *satisfying the positivity of the energy spectrum is identified by having the following expectations of* $W(\alpha, \beta) \equiv U(\alpha)V(\beta) \exp(i\alpha\beta/2)$

$$
\Omega(W(\alpha, \beta)) = 0, \quad if \alpha \neq 0; \quad \Omega(W(0, \beta)) = 1.
$$
 (B.1)

Proof Time translation invariance implies that the above expectation is independent of β if $\alpha \neq 0$. On the other hand,

$$
\Omega(W(\alpha,0)\,\alpha_t(W(\gamma,0)) = \Omega(W(\alpha+\gamma,\gamma t))\,e^{-i\alpha\gamma t/2},\tag{B.2}
$$

so that, since for $\alpha = \gamma \neq 0$,

$$
\Omega(W(\alpha + \gamma, \gamma t)) = \Omega(W(\alpha + \gamma, 0)),
$$

positivity of the energy requires that it vanishes. For $\alpha = -\gamma$, the Fourier transform of eq.(B.2) has support in \mathbb{R}^+ , $\forall \gamma \in \mathbb{R}$, iff the Fourier transform of $\Omega(W(0, \gamma t))$ has positive support, which holds for all γ iff the support is at the origin. Positivity of the state Ω then implies that $\Omega(W(0, \alpha t)) = 1$. Thus, eqs.(B.1) hold. Conversely, eqs.(B.1) define a positive state (as a limit of ground states of harmonic oscillators [\[3\]](#page-22-2)). Moreover, eqs.(2.2) imply

$$
\Omega(W(\alpha,\beta)\,\alpha_t(W(\gamma,\delta))) = \delta_{-\alpha,\gamma} \, e^{-i\alpha(\delta+\beta)/2} \, e^{i\alpha^2t/2},
$$

where $\delta_{\alpha,\gamma} = 1$, if $\alpha = \gamma$ and zero otherwise; therefore positivity of the energy follows.

In conclusion, by the GNS construction, one has a nonregular representation of the Weyl algebra in a Hilbert space \mathcal{H} , with cyclic vector Ψ_{Ω} . Since eqs.(B.1) imply

$$
V(\beta)\Psi_{\Omega} = \Psi_{\Omega},\tag{B.3}
$$

 Ψ_{Ω} is also cyclic with respect to the algebra generated by the $U(\alpha)$'s. The occurrence of non regular representations should not be regarded as too bizarre, since they can be related to reasonable physical descriptions. In fact, if we consider a free particle in a bounded volume V , it is reasonable to consider the algebra of canonical variables \mathcal{A}_V generated by $\exp(i\beta p)$, $\beta \in \mathbf{R}$ and by the (continuous) functions of q, $f_V(q)$, with support contained in V.

There is a natural embedding of \mathcal{A}_V into the Weyl algebra \mathcal{A}_W , i.e., if

$$
f_V(x) = \sum c_n e^{ik_n x}, \qquad x \in V,
$$

then its periodic extension

$$
f(q) = \sum c_n e^{ik_n q} , \qquad q \in \mathbf{R},
$$

defines a corresponding element of \mathcal{A}_W . Then, if $\Psi_V(x) = const$ denotes the ground state in the volume V (with periodic or Neumann boundary conditions for the Hamiltonian), one has

$$
(\Psi_V, f_V(x) e^{i \beta p} \Psi_V) = \Omega(f(q) e^{i \beta p}),
$$

where Ω is the nonregular state characterized in the above Proposition. The nonregular representation provides therefore a volume independent mathematical description of the above concrete situation.

C Appendix C. Positive euclidean formulation and stochastic processes

In order to discuss the stochastic processes associated to the quantum free particle, it is convenient to derive the corresponding euclidean formulation.

For the representation characterized in the previous Appendix, we have for the two point (Wightman) function for the Weyl operators $U(\alpha)$

$$
\Omega(U(-\alpha) e^{iHt} U(\alpha')) = \delta_{\alpha,\alpha'} e^{i\alpha^2 t/2}.
$$
 (C.1)

The Fourier transform is

$$
\tilde{W}(\omega) = \sqrt{2\pi} \,\delta(\omega - \alpha^2/2) \,\delta_{\alpha,\alpha'}.
$$

The n-point (Wightman)functions of the Weyl operators $U(\alpha)$ are obtained by induction from

$$
\alpha_t(U(\alpha) V(\beta)) = U(\alpha) V(\beta + \alpha t) e^{i\alpha^2 t/2}
$$

using eq.(C.1), and are given by $(U(\gamma, t) \equiv \alpha_t(U(\gamma)))$

$$
\Omega((U(\alpha_1, t_1))...U(\alpha_n, t_n))) = \delta_{\sum \alpha_i, 0} e^{i \sum_{i=2}^n (t_i - t_{i-1})(\sum_{k=i}^n \alpha_k)^2/2}.
$$
 (C.2)

The corresponding n-point Schwinger functions are immediately obtained by analytic continuation to ordered imaginary times $\tau_1 \leq \tau_2 ... \leq \tau_n$:

$$
\mathcal{S}(\alpha_1 \tau_1, \ldots \alpha_n \tau_n) = \delta_{\sum \alpha_i, 0} e^{-\sum_{i=2}^n (\tau_i - \tau_{i-1})(\sum_{k=i}^n \alpha_k)^2/2}.
$$
 (C.3)

and extended by symmetry to all euclidean times $\tau_1, \tau_2...\tau_n \in \mathbb{R}$.

¿From the existence and positivity of the Hamiltonian in the representation defined by eq.(B.1), it follows that the above Schwinger functions can also be written as

$$
\mathcal{S}(\alpha_1 \tau_1, \ldots \alpha_n \tau_n) = (\Psi_{\Omega}, U(\alpha_1) e^{-(\tau_2 - \tau_1)H} U(\alpha_2) \ldots e^{-(\tau_n - \tau_{n-1})H} U(\alpha_n) \Psi_{\Omega})
$$
\n(C.4)

By standard arguments, one can introduce the corresponding Borchers algebra and euclidean fields $U_E(\alpha, \tau)$ so that the Schwinger functions define a linear functional E on the euclidean fields

$$
\mathcal{S}(\alpha_1\tau_1,\ldots\alpha_n\tau_n)=E(U_E(\alpha_1,\tau_1)\ldots U_E(\alpha_n,\tau_n)).
$$

Equation (4.3) implies the Osterwalder-Schrader (OS) positivity of the Schwinger functions, with the OS reflection operator θ defined by

$$
\theta U_E(\alpha, \tau) = U_E(\alpha, -\tau),
$$

i.e. one has

$$
E(\theta B B) \ge 0,
$$

 $\forall B$ belonging to the algebra generated by $U_E(\alpha, \tau)$, $\alpha \in \mathbf{R}$, $\tau \geq 0$, where $U_E(\alpha, \tau) \equiv U_E(-\alpha, \tau).$

It is a non trivial fact that also Nelson positivity holds. As a matter of fact, the above Schwinger functions can be expressed as expectations of fields $e^{i\alpha x(\tau)}$ with functional measure

$$
d\mu(x(\tau)) = dW_{0,x}(x(\tau)) d\nu(x),
$$

where $dW_{s,y}$ denotes the Wiener measure for trajectories starting at the point y at time $\tau = s$ and dv denotes the ergodic mean; in fact dv defines a measure on the Gelfand spectrum Σ of the Bohr algebra generated by $e^{i\alpha x}$ [\[5\]](#page-22-4) and

$$
\int d\nu(x) dW_{0,x}(x(\tau)) e^{i\alpha_1 x(\tau_1)} \dots e^{i\alpha_n x(\tau_n)}
$$

=
$$
\int d\nu(x) \int dW_{0,0}(y(\tau)) e^{i \sum_i \alpha_i y(\tau_i)} e^{i \sum_i \alpha_i x}
$$

=
$$
\delta(\sum_i \alpha_i) \int dW_{0,0}(y(\tau)) e^{i \sum_i \alpha_i y(\tau_i)} = \mathcal{S}(\alpha_1 \tau_1, \dots \alpha_n \tau_n), \quad y(\tau_i) \equiv x(\tau_i) - x.
$$

The measure $d\mu$ is invariant under time translations,

$$
dW_{0,y}(x(\tau))d\nu(y) = dW_{s,y}(x(\tau))d\nu(y)
$$

because $d\nu(x)$ is invariant under translations $x \to x + a$ and therefore stationary for the Brownian motion.

The measure $d\mu$ is invariant under translations $x(\tau) \to x(\tau) + a$; they can be given the meaning of gauge trasformations and can be regarded as the analog of the gauge group of traslations of step 2π for a particle on a circle [\[5\]](#page-22-4). In conclusion, the euclidean correlation functions can be obtained as the stochastic process with expectations defined by $d\mu$.

Since the ground state Ψ_{Ω} is cyclic with respect to the euclidean algebra at time zero in the space H , obtained by the OS recostruction theorem, such a space can be identified with $L^2(\Sigma, d\nu)$.

One can explicitly check that the Markov property holds; if $\tau_1 \leq 0 \leq \tau_2$ one has

$$
\int d\mu \, e^{i\alpha x(\tau_1)} \, e^{i\beta x(\tau_2)} = \int d\nu(x) e^{i(\alpha+\beta)x} \int dW_{0,x}(x(\tau)) e^{i\alpha(x(\tau_1)-x)} e^{i\beta(x(\tau_2)-x)}
$$
\n
$$
= \int d\nu(x) e^{i(\alpha+\beta)x)} dW_{0,0}^-(y(\tau)) \, e^{i\alpha y(\tau_1)} \, dW_{0,0}^+(y(\tau)) \, e^{i\beta y(\tau_2)}.
$$

D Appendix D. Markov property without Nelson positivity

For simplicity, we discuss the problem at the level of the two point function, which is assumed to define a non degenerate inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{S},$ satisfying the following properties:

- i) $\langle \theta f, \theta q \rangle = \langle f, q \rangle, \quad (\theta f)(\tau) \equiv f(-\tau),$
- ii) there exists an operator D on S, such that, $\forall f, g \in S$

$$
\langle f, Dg \rangle = (f, g)_{L^2}, \ D\mathcal{S}^{\pm} \subseteq \mathcal{S}^{\pm}, \ [D, \theta] = 0,
$$

iii) S is weakly dense in a nondegenerate weakly complete inner product space \overline{S} , which has the following decomposition

$$
\overline{\mathcal{S}} = \overline{\mathcal{S}^-} < + > \overline{\mathcal{DS}^+},\tag{D.1}
$$

with $\lt +$ > denoting a \lt , > orthogonal sum.

Then, by i), ii) one also has

$$
\overline{S} = \overline{S^+} < + > \overline{DS^-}.\tag{D.2}
$$

Furthermore, since $\overline{DS^{\pm}} \subseteq \overline{S^{\pm}}$, $\forall f \in S$, by eq.(D.1) one has

$$
f = f_{-} + f_{0+}, \quad f_{-} \in \mathcal{S}^{-}, \quad f_{0+} \in D\mathcal{S}^{+}
$$

and by eq.(D.2) $f_{-} = (f_{-})_{+} + (f_{-})_{0-}, \quad (f_{-})_{+} \in \overline{\mathcal{S}^{+}}, \quad (f_{-})_{0-} \in \overline{D\mathcal{S}^{-}}.$ Hence,
 $f = f_{0+} + (f_{-})_{0-} + (f_{-})_{+}, \quad (f_{-})_{+} \in \overline{\mathcal{S}^{+}} \cap \overline{\mathcal{S}^{-}}.$

In conclusion, one has

$$
\overline{S} = \overline{DS^-} < + > \overline{DS^+} < + > \mathcal{V}, \qquad \mathcal{V} = \overline{S^+} \cap \overline{S^-}. \tag{D.3}
$$

Since $\bar{\mathcal{S}}$ is non degenerate the decompositions (D.1-D.3) are unique; in fact, non uniqueness would imply a non zero intersection of orthogonal spaces, which implies degeneracy of the inner product. Thus, any vector $f \in \mathcal{S}$ has unique decompositions according to eqs.(D.1-D.3), which means that correspondingly there are everywhere defined idempotent operators, which are hermitean with respect to $\langle \rangle$. In particular, there are E_{\pm} , E_0 defined by $E_{\pm}\overline{\mathcal{S}} = \overline{\mathcal{S}^{\pm}}$, and $E_0\overline{\mathcal{S}} = \mathcal{V}$ and eqs.(D.1-D.3) give

$$
E_{+}E_{-}=E_{-}E_{+}=E_{0}.
$$
 (D.4)

Thus, we get the same operator formulation of the Markov property as in the positive case.

For quasi free states, the above construction extends in the usual way to the n-point Schwinger functions, in terms of the sum of the symmetric tensor products of the above indefinite spaces. Similarly, the euclidean field $x(f)$ has an extension to $f \in \overline{S}$.

In the example discussed in this note, $D = d^2/d\tau^2$ with the Krein structure given by eq. (3.7) . The decomposition $(D.3)$ reduces to (3.3) and it is also orthogonal in the positive scalar product. The lesson from the model is that the space $\mathcal V$ can be larger than the standard time zero space; in the model it contains a time translationally invariant variable, actually the (positive time) ergodic limit of the velocity.

Thus, the model supports the idea that, in the (indefinite) space defined by the correlation functions of the field algebra, the observable algebra has in general a reducible vacuum representation [\[8,](#page-22-7) [9,](#page-22-8) [11\]](#page-22-10).

References

- [1] R.F. Streater and A.S. Wightman, *PCT, Spin and Statistics and All That*, Benjamin-Cummings 1980
- [2] J. Glimm and A. Jaffe, *Quantum Physics. A Functional Integral Point of View*, Springer 1987
- [3] F. Acerbi, G. Morchio and F. Strocchi, Jour. Math. Phys. 34, 899 (1993)
- [4] F. Acerbi, G. Morchio and F. Strocchi, Lett. Math. Phys. 27,1 (1993)
- [5] J. Löffelholz, G. Morchio and F. Strocchi, Lett. Math. Phys. 35, 251 (1995)
- [6] J. Löffelholz, G. Morchio and F. Strocchi, Ann. Phys. $(N.Y.)$ 250, 367 (1996)
- [7] G. Morchio, D. Pierotti and F. Strocchi, Jour. Math. Phys. 31, 1467 (1990)
- [8] G. Morchio, D. Pierotti and F. Strocchi, Ann. (N.Y) 188, 217 (1988)
- [9] F. Strocchi, *Selected topics on the general properties of quantum field theory*, World Scientific 1993
- [10] M. Creutz and T.N. Tudron, Phys. Rev. D17, 2619 (1978); G. Rossi and M. Testa, Phys. Rev. D29, 2997 (1984); A. Slavnov ans S.A. Frolov, Theor. Math. Phys. 68 880 (1986)
- [11] J. Löffelholz, G. Morchio and F. Strocchi, Jour. Math. Phys. 44, 5095 (2003)
- [12] O. Bratteli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, Vol.2, Springer 1996
- [13] R. Haag, *Local Quantum Physics*, Springer 1996
- [14] M. Mnatsakanova, G. Morchio, F. Strocchi and Y. Vernov, Journ. Math. Phys. 39, 2969 (1998)
- [15] G. Morchio and F. Strocchi, Canadian Mathematical Society, Conference Proceedings, Vol. 29, 491 (2000)
- [16] E. Nelson, Probability and Euclidean Field Theory, in *Constructive Quantum Field theory*, "E. Majorana" School of Mathematical Physics, G. Velo and A.S. Wightman eds., Springer 1973
- [17] L. Jakobczyk and F. Strocchi, Comm. Math. Phys. 119, 529 (1988)

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