Solution of matrix Riemann-Hilbert problems with quasi-permutation monodromy matrices

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Abstract. In this paper we solve an arbitrary matrix Riemann-Hilbert (inverse monodromy) problem with quasi-permutation monodromy representations outside of a divisor in the space of monodromy data. This divisor is characterized in terms of the theta-divisor on the Jacobi manifold of an auxiliary compact Riemann surface realized as an appropriate branched covering of $\mathbb{C}P^1$. The solution is given in terms of a generalization of Szegö kernel on the Riemann surface. In particular, our construction provides a new class of solutions of the Schlesinger system. The isomonodromy tau-function of these solutions is computed up to a nowhere vanishing factor independent of the elements of monodromy matrices. Results of this work generalize the results of papers [14] and [5] where the 2×2 case was solved.

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1 Introduction

Apart from pure mathematical significance (see review of A.Bolibruch [4]), matrix Riemann-Hilbert (inverse monodromy) problems and related theory of isomonodromic deformations play an important role in mathematical physics. In particular, the RH problems are central in the theory of integrable systems (see for example [28, 7, 12]) and the theory of random matrices [6]. In applications the main object of interest is the so-called tau-function, which was first introduced by M.Jimbo, T.Miwa and their collaborators [13]; it was later shown by B.Malgrange [21] that the tau-function may be interpreted as determinant of certain Töplitz operator. The set of zeros of the tau-function in the space of singularities of the RH problem is called the Malgrange divisor (ϑ); it plays a crucial role in discussion of solvability of RH problem with given monodromy data.

For generic monodromy data neither the solution of a matrix RH problem nor the corresponding tau-function can be computed analytically in terms of known special functions [26, 27]. However, there are exceptional cases, when the RH problem can be solved explicitly; surprisingly enough, these

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cases often appear in applications. For example, the solution of 2×2 RH problem with an arbitrary set of off-diagonal monodromy matrices was successfully applied to the problem of finding physically meaningful solutions of stationary axially symmetric Einstein equations [19, 22, 15] and to complete classification of $SU(2)$-invariant self-dual Einstein manifolds [12, 2]. The solution of general 2×2 RH problem of this kind was given only in 1998 in the papers [14, 5] (however, some important ingredients of this solution were understood already three decades ago, see review [30]). In [14] it was also calculated the tau-function corresponding to this RH problem, which turned out to coincide with determinant of Cauchy-Riemann operator acting in tensor product of the spinor bundle and an appropriate flat line bundle on a hyperelliptic curve (see [29, 3, 16, 1]). In [20] a family of Riemann-Hilbert problems in arbitrary matrix dimension with quasi-permutation monodromies was solved in terms of Szegö kernel on compact Riemann surfaces; however, this family did not contain enough parameters to cover the whole set of quasi-permutation monodromy groups; also the Miwa-Jimbo tau-function was not computed for dimension higher than 2.

Results of present work generalize the results of papers [14, 5] and [20]; we present a complete solution of Riemann-Hilbert problems with an arbitrary quasi-permutation monodromy representation in any matrix dimension outside of the divisor of zeros of corresponding tau-function in the space of monodromy data (by monodromy data we mean the given monodromy representation and positions of singularities). For that purpose we use an appropriate generalization of Szegö kernel on associated Riemann surface. This leads to a new class of solutions of the Schlesinger system. We compute the Jimbo-Miwa tau-function up to a factor which depends only on positions of singularities of the RH problem and does not depend on the matrix elements of monodromy matrices; in some cases (for matrix dimension 2 and for RH problems in arbitrary matrix dimension corresponding to Riemann surfaces of genus 0 and 1) this factor can also be found explicitly. From the point of view of string theory [16] this factor can in some cases be interpreted as determinant of Cauchy-Riemann operator acting in trivial line bundle over $\mathcal{L}$; from the point of view of the theory of Frobenius manifolds this factor is equal to isomonodromic tau-function of Frobenius manifolds associated to Hurwitz spaces [18]. The divisor of zeros of the tau-function corresponding to our RH problem in the space of monodromy data can be characterized in terms of the theta-divisor on the Jacobi manifold of the Riemann surface.

The main technical tools used in this paper are kernel functions on Riemann surfaces, Fay identities and deformation theory of Riemann surfaces. The systematic description of these objects may be found in Fay's books [8, 9].

We expect present results to find an application to the problem of isolating the subclass of physically reasonable solutions of stationary axially symmetric Einstein-Maxwell system [19] in the spirit of works [19, 22, 15], devoted to vacuum Einstein equations. For Einstein-Maxwell system the matrix dimension of RH problem is equal to three. Other potential areas of application are the theory of Frobenius manifolds [7] and random matrices [6].

Let's describe the organization of this paper. In section 2 we remind the formulation of general Riemann-Hilbert (inverse monodromy problem), the isomonodromy deformation equations (Schlesinger system), and definition of Jimbo-Miwa tau-function. We further discuss quasi-permutation monodromy representations and their natural relationship to branched coverings of $\mathbb{C}P^1$.

In section 3 we review the necessary facts from the deformation theory of Riemann surfaces and adjust them to the situation when the Riemann surface is realized as a branched covering of the complex plane.

In section 4 we solve an arbitrary RH problem with irreducible quasi-permutation monodromy representation outside of a divisor in the space of monodromy data.

In section 5 we describe corresponding solutions of Schlesinger system.

Section 6 is devoted to computation of corresponding tau-function; the divisor of the zeros of tau-function is described in terms of theta-divisor on Jacobi manifold of an auxiliary branch covering.

2 Riemann-Hilbert problem with quasi-permutation monodromies and branched coverings of $\mathbb{C}P^1$

2.1 Riemann-Hilbert problem, isomonodromy deformations and tau-function

Consider a set of $M+1$ points $\lambda_0, \lambda_1, \dots, \lambda_M \in \mathbb{C}$, and a given $GL(N)$ monodromy representation $\mathcal{M}$ of $\pi_1[\mathbb{C}P^1 \setminus \{\lambda_1, \dots, \lambda_M\}]$. Let us formulate the following Riemann-Hilbert problem:

Find function $\Psi(\lambda) \in GL(N, \mathbb{C})$, defined on universal cover of $\mathbb{C}P^1 \setminus \{\lambda_1, \dots, \lambda_M\}$, which satisfies the following conditions:

1. *$\Psi(\lambda)$ is normalized at a point λ_0 on some sheet of the universal cover as follows:*

$$\Psi(\lambda_0) = I \; ; \tag{2.1}$$

2. *$\Psi(\lambda)$ has given right holonomy $\mathcal{M}_\gamma$ along each contour $\gamma \in \pi_1[\mathbb{C}P^1 \setminus \{\lambda_1, \dots, \lambda_M\}]$;*

3. *$\Psi(\lambda)$ has regular singularities at the points λ_n (i.e. function Ψ grows at a neighbourhood of λ_m not faster than some power of $\lambda - \lambda_m$).*

Consider the following set of standard generators $l_1, \dots, l_M$ of $\pi_1[\mathbb{C}P^1 \setminus \{\lambda_1, \dots, \lambda_M\}]$. Choose λ_0 to be the starting point and assume that the contour l_n starts and ends at λ_0 such that the interior of l_n contains only one marked point λ_n (our convention is that the point $\lambda = \infty$ belongs to the exterior of any closed contour on $\mathbb{C}P^1$). Moreover, we assume that these generators are ordered according to the following relation:

$$l_M l_{M-1} \dots l_1 = \mathbf{1} \; . \tag{2.2}$$

The matrices $\mathcal{M}_{\gamma_n} := \mathcal{M}_n$ are called monodromy matrices. As a corollary of (2.2) we have:

$$\mathcal{M}_M \mathcal{M}_{M-1} \dots \mathcal{M}_1 = I \; . \tag{2.3}$$

We shall consider only the monodromy groups for which the singularity of solution Ψ of the RH problem at the points λ_n has the following form:

$$\Psi(\lambda) = \{G_n + O(\lambda - \lambda_n)\}(\lambda - \lambda_n)^{T_n} C_n \; , \tag{2.4}$$

where $G_n, C_n \in GL(N)$; $T_n = \mathrm{diag}(t_n^{(1)}, \dots t_n^{(N)})$.

The monodromy matrices $\mathcal{M}_n$ are in this case related to coefficients of asymptotics (2.4) as follows:

$$\mathcal{M}_n = C_n^{-1} e^{2\pi i T_n} C_n \; . \tag{2.5}$$

i.e. all these matrices are diagonalizable (of course, not simultaneously in non-trivial cases). The set $\{\lambda_n, \; \mathcal{M}_n \; , \; T_n \; , \; n = 1, \dots, M\}$ is called the set of monodromy data.

Solution $\Psi(\lambda)$ of such RH problem satisfies the following matrix differential equation with meromorphic coefficients with simple poles:

$$\frac{d\Psi}{d\lambda} = \sum_{n=1}^{M} \frac{A_n}{\lambda - \lambda_n} \Psi \; , \tag{2.6}$$

where

$$A_n = G_n T_n G_n^{-1} \; . \tag{2.7}$$

Suppose now that matrices C_n and T_n (and, therefore, the monodromy matrices) don't depend on positions of singularities $\{\lambda_n\}$. Then function Ψ, in addition to (2.6), satisfies the equations with respect to positions of singularities λ_n:

$$\frac{d\Psi}{d\lambda_n} = \left(\frac{A_n}{\lambda_0 - \lambda_n} - \frac{A_n}{\lambda - \lambda_n}\right)\Psi \; . \tag{2.8}$$

Compatibility conditions of equations (2.6) and (2.8) imply Schlesinger equations for residues A_n:

$$\frac{\partial A_n}{\partial \lambda_m} = \frac{[A_n,\, A_m]}{\lambda_n - \lambda_m} - \frac{[A_n,\, A_m]}{\lambda_0 - \lambda_m}\,, \qquad m \neq n\;;$$

$$\frac{\partial A_m}{\partial \lambda_m} = -\sum_{n\neq m}\left(\frac{[A_n,\, A_m]}{\lambda_n - \lambda_m} - \frac{[A_n,\, A_m]}{\lambda_n - \lambda_0}\right)\;. \tag{2.9}$$

Once a solution of the Schlesinger system is given, one can define the tau-function [13] by the system of equations

$$\frac{\partial}{\partial \lambda_n}\ln\tau = H_n := \frac{1}{2}\mathrm{res}|_{\lambda=\lambda_n}\mathrm{tr}\left(\Psi_\lambda\Psi^{-1}\right)^2\;; \qquad \frac{\partial\tau}{\partial\overline{\lambda_n}} = 0\;. \tag{2.10}$$

According to Malgrange [21], the isomonodromic tau-function can be interpreted as determinant of certain Töplitz operator. The important role in the theory of RH problems is played by the divisor of zeros of the tau-function in the universal covering of the space $\{\{\lambda_m\}\in\mathbb{C}^M \,\big|\, \lambda_m\neq\lambda_n \;\; if \;\; m\neq n\}$. In analogy to the theta-divisor (Θ) on a Jacobi variety, Malgrange denoted this divisor by (ϑ). The divisor (ϑ) has the following meaning: if $\{\lambda_n\}\in(\vartheta)$, the Riemann-Hilbert problem with the given set of monodromy matrices and eigenvalues $t_n^{(j)}$ does not have a solution; the solution $\{A_m\}$ of Schlesinger system is singular on (ϑ).

2.2 Quasi-permutation monodromy representations and branched coverings

In this paper we shall consider two special kinds of $N\times N$ monodromy representations.

Definition 1 *Representation $\mathcal{M}$ is called the permutations representation if matrix $\mathcal{M}_\gamma$ is a permutation matrix for each $\gamma\in\pi_1[\mathbb{C}P^1\setminus\{\lambda_1,\dots,\lambda_M\}]$.*

Remind that a matrix is called the matrix of permutation if each raw and each column of this matrix contain exactly one non-vanishing entry and this entry equals to 1. Permutation matrices of size $N\times N$ are in natural one-to-one correspondence with elements of permutation group S_N. The definition (1) is self-consistent since the product of any two permutation matrices is again a permutation matrix.

Theorem 1 *There exists a one-to-one correspondence between $N\times N$ permutation representations of $\pi_1[\mathbb{C}P^1\setminus\{\lambda_1,\dots,\lambda_M\}]$ and compact Riemann surfaces (not necessarily connected) realized as N-fold ramified coverings of $\mathbb{C}P^1$ with projections of branch points on $\mathbb{C}P^1$ equal to $\lambda_1,\dots,\lambda_M$.*

Proof. Given a ramified covering $\mathcal{L}$ with projections $\lambda_1,\dots,\lambda_M$ of branch points on $\mathbb{C}P^1$, we construct the corresponding permutation representation as follows. Denote the projection of $\mathcal{L}$ on $\mathbb{C}P^1$ by Π. Generators $\mathcal{M}_n$ of permutation monodromy group are given by the following construction. Consider the lift $\Pi^{-1}(l_n)$ of the generator l_n on $\mathcal{L}$. This is a union of N (not necessary closed) non-intersecting contours on $\mathcal{L}$ which start and end at some of the points $\lambda_0^{(j)}$ (by $\lambda^{(j)}$ we denote the point of jth sheet of $\mathcal{L}$ which has projection λ on $\mathbb{C}P^1$). Denote by $l_n^{(j)}$ the component of $\Pi^{-1}(l_n)$ which starts at the point $\lambda_0^{(j)}$; the endpoint of this contour is $\lambda_0^{(j_n[j])}$ for some index $j_n[j]$. If $\lambda_n^{(j)}$ is not a branch point, then $j_n[j]=j$, and contour $l_n^{(j)}$ is closed; if $\lambda_n^{(j)}$ is a branch point, then $j_n[j]\neq j$ and contour $l_n^{(j)}$ is non-closed. Then the permutation matrix $\mathcal{M}_n$ has the following form:

$$(\mathcal{M}_n)_{jl}=\delta_{j_n[j],l} \tag{2.11}$$

and naturally corresponds to some element s_n of the permutation group S_N. On the other hand, starting from some permutation monodromy representation we can glue N copies of $\mathbb{C}P^1$ at the branch points $\{\lambda_n\}$ in such a way that the obtained compact Riemann surface corresponds to the permutation monodromies (2.11) (see [10], p.257).

◇

Definition 2 *Representation $\mathcal{M}$ is called the quasi-permutations representation if $\mathcal{M}_\gamma$ is a quasi-permutation matrix for any $\gamma\in\pi_1[\mathbb{C}P^1\setminus\{\lambda_1,\dots,\lambda_M\}]$.*

Again, this definition is natural since all quasi-permutation matrices form a subgroup in $GL(N)$. Remind that a matrix is called the quasi-permutation matrix if each raw and each column of this matrix contain only one non-vanishing entry.

We shall call two quasi-permutation representations $\mathcal{M}$ and $\mathcal{M}'$ equivalent if there exists some *diagonal* matrix D with $\det D=1$ such that

$$\mathcal{M}'_\gamma=D\mathcal{M}_\gamma D^{-1} \tag{2.12}$$

for all $\gamma\in\pi_1[\mathbb{C}P^1\setminus\{\lambda_1,\dots,\lambda_M\}]$.

To every quasi-permutation representation $\mathcal{M}$ we can naturally assign a permutation representation $\mathcal{M}^0$ substituting 1 instead of all non-vanishing entries of all monodromy matrices; then from $\mathcal{M}^0$ we reconstruct the branched covering $\mathcal{L}$.

We shall consider quasi-permutation monodromy representations $\mathcal{M}$ which satisfy the following additional conditions:

Condition 1 *Representation $\mathcal{M}$ can not be decomposed into direct sum of two other representations, both of whose are quasi-permutation representations with respect to the same basis in $\mathbb{C}^N$.*

Condition 2 *Monodromy matrices of representation $\mathcal{M}$ can not be simultaneously diagonalized.*

Condition 1 obviously implies that the permutation representation $\mathcal{M}^0$ also can not be decomposed into a direct sum of two representation both of which are permutation representations in the same basis; in turn, this implies connectedness of corresponding branched covering $\mathcal{L}$. The condition 1 is weaker than the standard condition of irreducibility of $\mathcal{M}$: there exist reducible quasi-permutation representations which are however irreducible into a product of two quasi-permutation representations (for example, any permutation representation is reducible in usual sense since there exists an invariant subspace $x_1+\dots+x_N=0$).

Condition 2 is imposed for convenience: it guarantees that the matrix Riemann-Hilbert problem is not trivially reducible to N independent scalar Riemann-Hilbert problems.

Definition 3 *Denote by $\mathcal{Q}(\mathcal{L})$ the space of orbits of the group (2.12) acting on the space of irreducible quasi-permutation monodromy representations corresponding to a given connected branched covering $\mathcal{L}$.*

Lemma 1 *The manifold $\mathcal{Q}(\mathcal{L})$ has dimension $MN-2N+1$; its universal covering $\widehat{\mathcal{Q}(\mathcal{L})}$ is isomorphic to $\mathbb{C}^{MN-2N+1}$.*

Proof. Let us first prove that $\mathcal{Q}(\mathcal{L})$ is a $MN-2N+1$-dimensional space. The space of $M-1$ quasi-permutation matrices has dimension $(M-1)N$ (matrix $\mathcal{M}_M$ can be expressed in terms of $\mathcal{M}_1,\dots,\mathcal{M}_{M-1}$ according to (2.3). Let us prove that the orbits of the action (2.12) by diagonal matrices D are $N-1$-dimensional. Infinitesimally, matrix D can be written as $D = I + \epsilon D_0$, where D_0 is a traceless diagonal matrix; the action (2.12) then takes the form $\mathcal{M}_\gamma \to \mathcal{M}_\gamma + [D_0, \mathcal{M}_\gamma]\epsilon$. If the orbits have dimension less than $N-1$, there must exist a non-vanishing diagonal traceless matrix D_0 commuting with all $\mathcal{M}_\gamma$, which contradicts the condition 2.

The space $\mathcal{Q}(\mathcal{L})$ can be covered by $\mathbb{C}^{MN-2N+1}$ as follows (the space $\mathcal{Q}(\mathcal{L})$ is non-simply-connected since each monodromy matrix must contain exactly N non-vanishing entries). Starting from an arbitrary $\mathcal{M} \in \mathcal{Q}(\mathcal{L})$, we define a point in $\mathbb{C}^{(M-1)N}$, whose coordinates are equal to the logarithms of non-vanishing entries of monodromy matrices $\mathcal{M}_1,\dots,\mathcal{M}_{M-1}$ (i.e. the covering is defined by exponentiation applied to each non-vanishing component). The transformations (2.12) act in this space as translations in $N-1$ independent directions; corresponding space of orbits is a $MN-2N+1$-dimensional affine space which universally covers $\mathcal{Q}(\mathcal{L})$.

◇

Denote the branch points of $\mathcal{L}$ by $P_1,\dots,P_L$, where $L \geq M$; the equality, $L = M$, takes place only if all branch points P_k have different projections on λ-plane. Denote the ramification indexes of the branch points (i.e. numbers of sheets glued at each point P_k) by $\mathbf{k}_1,\dots,\mathbf{k}_L$ respectively.

Remark 1 If some quasi-permutation monodromy matrix $\mathcal{M}_n$ is diagonal, then corresponding matrix $\mathcal{M}_n^0$ is equal to I, and λ_n is not a projection of any branch point on $\mathbb{C}P^1$. However, in the sequel we shall treat such points in the same fashion as all other λ_m's by assigning to all non-branch points the ramification index 1. All formulas below are written in such form that this does not lead to any inconveniences or inconsistencies.

Lemma 2 *Every quasi-permutation matrix is diagonalizable.*

The *proof* is simple: we can decompose $\mathbb{C}^N$ into a direct sum of orthogonal invariant subspaces such that in each subspace our quasi-permutation matrix acts as a power of elementary cyclic permutation multiplied by a diagonal matrix; it is easy to verify that each such matrix has different eigenvalues and, therefore, is diagonalizable. Therefore, the original quasi-permutation matrix acting in the whole $\mathbb{C}^N$ is also diagonalizable.

◇

According to the Riemann-Hurwitz formula, the genus of the connected Riemann surface $\mathcal{L}$ is equal to

$$g = \sum_{l=1}^{L} \frac{\mathbf{k}_l - 1}{2} - N + 1 \; . \tag{2.13}$$

Denoting the set of branch points by $\mathcal{P} := \{P_1,\dots,P_L\}$, we get the natural partition $\mathcal{P} = \mathcal{P}_1 \cup \dots \cup \mathcal{P}_M$, where $\mathcal{P}_m$ consists of s_m branch points $P_{s_1},\dots,P_{s_m}$ which project down to λ_m i.e. $\Pi(\mathcal{P}_m) = \lambda_m$.

Corresponding ramification indexes $\{\mathbf{k}_{s_1},\dots,\mathbf{k}_{s_m}\}$ assigned to each λ_m form the *passport* of the branch covering $\mathcal{L}$. The branched coverings with fixed passport form a stratum of the Hurwitz space $H_{g,N}$ of meromorphic functions of degree N on Riemann surfaces of genus g. The points P_l are the critical points of these maps, and λ_m are corresponding critical values. The critical values λ_m can be used as local coordinates on the stratum of Hurwitz space with given passport.

The stratum of highest dimension (i.e. the bulk of the Hurwitz space) corresponds to branch coverings with simple branch points (i.e. $\mathbf{k}_m = 2$ for all m).

3 Riemann surfaces. Variational formulas

3.1 Riemann surfaces

Here we collect some useful facts from the theory of Riemann surfaces and their deformations. Consider a canonical basis of cycles $(a_\alpha, b_\alpha),\ \alpha = 1,\dots,g$ on $\mathcal{L}$. Introduce the dual basis of holomorphic 1-forms w_α on $\mathcal{L}$ normalized by $\oint_{a_\alpha} w_\beta = \delta_{\alpha\beta}$. The matrix of b-periods $\mathbf{B}$ and the Abel map $U(P)\,,\ P\in\mathcal{L}$ are given by

$$\mathbf{B}_{\alpha\beta} = \oint_{b_\alpha} w_\beta\ , \qquad U_\alpha(P) = \int_{P_0}^{P} w_\alpha\ , \tag{3.1}$$

where P_0 is a basepoint. Consider theta-function with characteristics $\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\mathbf{z}|\mathbf{B})$, where $\mathbf{p},\mathbf{q}\in\mathbb{C}^g$ are vectors of characteristics; $\mathbf{z}\in\mathbb{C}^g$ is the argument. The theta-function is holomorphic function of variable $\mathbf{z}$ with the following periodicity properties:

$$\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\mathbf{z}+\mathbf{e}_\alpha) = \Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\mathbf{z}) e^{2\pi i p_\alpha}\ ;$$

$$\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\mathbf{z}+\mathbf{B}\mathbf{e}_\alpha) = \Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\mathbf{z}) e^{-2\pi i q_\alpha} e^{-2\pi i z_\alpha - \pi i \mathbf{B}_{\alpha\alpha}}\ , \tag{3.2}$$

where $\mathbf{e}_\alpha \equiv (0,\dots,1,\dots,0)$ is the standard basis in $\mathbb{C}^g$. The theta-function satisfies the heat equation:

$$\frac{\partial^2 \Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\mathbf{z})}{\partial z_\alpha \partial z_\beta} = 4\pi i \frac{\partial \Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\mathbf{z})}{\partial \mathbf{B}_{\alpha\beta}}\ . \tag{3.3}$$

Let us consider some non-singular odd half-integer characteristic $[\mathbf{p}^*,\mathbf{q}^*]$. The prime-form $E(P,Q)$ is defined as follows:

$$E(P,Q) = \frac{\Theta\left[\begin{smallmatrix}\mathbf{p}^*\\ \mathbf{q}^*\end{smallmatrix}\right](U(P)-U(Q))}{h(P)h(Q)}\ , \tag{3.4}$$

where the square of a section $h(P)$ of a spinor bundle over $\mathcal{L}$ is given by the following expression:

$$h^2(P) = \sum_{\alpha=1}^{g} \partial_{z_\alpha}\left\{\Theta\left[\begin{smallmatrix}\mathbf{p}^*\\ \mathbf{q}^*\end{smallmatrix}\right](0)\right\} w_a(P)\ . \tag{3.5}$$

Then $h(P)$ itself is a section of the spinor bundle corresponding to characteristic $\left[\begin{smallmatrix}\mathbf{p}^*\\ \mathbf{q}^*\end{smallmatrix}\right]$. The automorphy factors of the prime-form along all cycles a_α are trivial; the automorphy factor along cycle b_α equals to $\exp\{-\pi i B_{\alpha\alpha} - 2\pi i (U_\alpha(P) - U_\alpha(Q))\}$. The prime-form has the following local behavior as $P\to Q$:

$$E(P,Q) = \frac{x(P)-x(Q)}{\sqrt{dx(P)}\sqrt{dx(Q)}}(1+o(1))\ , \tag{3.6}$$

where $x(P)$ is a local parameter.

The Bergmann kernel is defined by the formula $\mathbf{w}(P,Q) = d_P d_Q \ln E(P,Q)$. It has a double pole with the following local behavior on the diagonal $P \to Q$:

$$\mathbf{w}(P,Q) = \left\{ \frac{1}{(x(P) - x(Q))^2} + H(x(P), x(Q)) \right\} dx(P) dx(Q) \; . \tag{3.7}$$

where $H(x(P), x(Q))$ is the non-singular part of the Bergmann kernel in each coordinate chart. The restriction of the function H on the diagonal gives the Bergmann projective connection $R(x)$:

$$R(x) = 6H(x(P), x(P)) \; , \tag{3.8}$$

which non-trivially depends on the chosen system of local coordinates on $\mathcal{L}$.

The Szegö kernel $S(P,Q)$ is the $(1/2,1/2)$-form on $\mathcal{L} \times \mathcal{L}$ defined by the formula

$$S(P,Q) = \frac{1}{\Theta \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} (0)} \frac{\Theta \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} (U(P) - U(Q))}{E(P,Q)} \; , \tag{3.9}$$

where $\mathbf{p}, \mathbf{q} \in \mathbb{C}^g$ are two vectors such that $\Theta \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} (0) \neq 0$. The Szegö kernel is the kernel of the integral operator $\overline{\partial}^{-1}$, where the operator $\overline{\partial}$ acts in the line bundle $\Delta \otimes \chi_{\mathbf{p},\mathbf{q}}$, which is the product of the spin bundle (with trivial automorphy factors along the basic cycles) Δ over $\mathcal{L}$ (divisor of Δ is equivalent to vector of Riemann constants which we denote by the same letter) and the flat line bundle $\chi_{\mathbf{p},\mathbf{q}}$ defined by the automorphy factors $e^{2\pi i p_\alpha}$ and $e^{-2\pi i q_\alpha}$ along basic cycles. The Szegö kernel itself has the automorphy factors $e^{2\pi i p_\alpha}$ and $e^{-2\pi i q_\alpha}$ along the cycles a_α and b_α, respectively, in its first argument; the automorphy factors of the Szegö kernel with respect to its second argument are the inverse (i.e. $S(P,Q)$ is a section of the line bundle $\Delta \otimes \chi_{\mathbf{p},\mathbf{q}}$ with respect to P and a section of $\Delta \otimes \chi_{\mathbf{p},\mathbf{q}}^{-1}$ with respect to Q). On the diagonal, as $Q \to P$, it behaves as follows:

$$S(P,Q) = \left(\frac{1}{x_P - x_Q} + a_0(P) + O(x_P - x_Q) \right) \sqrt{dx_P} \sqrt{dx_Q} \; , \tag{3.10}$$

where coefficient a_0 is given by ([9], p.29)

$$a_0(P) = \frac{1}{dx_P} \sum_{\alpha=1}^{g} \partial_\alpha \{ \ln \Theta \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} (0) \} w_\alpha(P) \; . \tag{3.11}$$

The Szegö kernel is related to the Bergmann kernel as follows ([8], p.26):

$$-S(P,Q)S(Q,P) = \mathbf{w}(P,Q) + \sum_{\alpha,\beta=1}^{g} \partial^2_{z_\alpha z_\beta} \{ \ln \Theta \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} (0) \} w_\alpha(P) w_\beta(Q) \; . \tag{3.12}$$

For any two sets $P_1, \dots, P_N$ and $Q_1, \dots, Q_N$ of points on $\mathcal{L}$ the following Fay identity takes place (see [8], p.33):

$$\det\{S(P_j, Q_k)\} = \frac{\Theta \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} \left(\sum_{j=1}^{N} (U(P_j) - U(Q_j)) \right)}{\Theta \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} (0)} \frac{\prod_{j<k} E(P_j, P_k) E(Q_k, Q_j)}{\prod_{j,k} E(P_j, Q_k)} \; . \tag{3.13}$$

In particular, for $N = 2$ this is Fay's trisecant identity.

3.2 Rauch variational formulas

The infinitesimal variation of the basic holomorphic 1-forms and matrix of b-periods with respect to a Beltrami differential μ is given by the Rauch formulas ([9], p.57):

$$\delta_\mu w_\alpha(P) = \frac{1}{2\pi i}\iint_{\mathcal{L}} \mu(Q) w_\alpha(Q)\mathbf{w}(P,Q)\;, \qquad \overline{\delta}_\mu w_\alpha(P) = 0\;; \tag{3.14}$$

$$\delta_\mu \mathbf{B}_{\alpha\beta} = \iint_{\mathcal{L}} \mu w_\alpha w_\beta\;, \qquad \overline{\delta}_\mu \mathbf{B}_{\alpha\beta} = 0\;. \tag{3.15}$$

Let us apply these formulas to a Riemann surface $\mathcal{L}$ realized as a branched covering of $\mathbb{C}P^1$.

Theorem 2 *Basic holomorphic differentials and matrix of b-periods of an N-fold covering $\mathcal{L}$ of $\mathbb{C}P^1$ satisfy the following equations:*

$$\partial_{\lambda_m}\{w_\alpha(P)\} = \mathrm{res}\Big|_{\lambda=\lambda_m}\left\{\frac{1}{(d\lambda)^2}\sum_j w_\alpha(\lambda^{(j)})\mathbf{w}(\lambda^{(j)},P)\right\}\;, \tag{3.16}$$

$$\partial_{\lambda_m}\{\mathbf{B}_{\alpha\beta}\} = -\mathrm{res}\Big|_{\lambda=\lambda_m}\left\{\frac{4\pi i}{(d\lambda)^2}\sum_{j<k} w_\alpha(\lambda^{(j)})w_\beta(\lambda^{(k)})\right\}\;, \tag{3.17}$$

$$\partial_{\overline{\lambda}_m}\{w_\alpha(P)\} = \partial_{\overline{\lambda}_m}\mathbf{B} = 0\;.$$

where $\lambda^{(j)}$ denotes the point of $\mathcal{L}$ which has projection λ on λ-plane and belongs to the jth sheet of $\mathcal{L}$ (under certain dissection of $\mathcal{L}$ into N sheets).

Proof. We first notice that the residue in (3.16), (3.17) and below is understood as the residue of function of variable λ, not the 1-form. We start from proving the theorem under assumption that all branch points of $\mathcal{L}$ have different projections on λ-plane i.e. there is a bijection between the set of branch points P_m and their projections λ_m; then each $\mathcal{P}_m$ contains only one point P_m.

Choose in the Rauch formulas (3.14), (3.15) the Beltrami differential as follows:

$$\mu = -\frac{1}{2\delta^{\mathbf{k}_m}}\left(\frac{|x_m|}{x_m}\right)^{\mathbf{k}_m-2}\mathbf{1}_{\{|x_m|\le\delta\}}\frac{d\,\overline{x_m}}{d\,x_m} \tag{3.18}$$

with sufficiently small $\delta > 0$, where $x_m \equiv (\lambda-\lambda_m)^{1/\mathbf{k}_m}$ is a local parameter around P_m; function $\mathbf{1}_{\{|x_m|\le\delta\}}$ is equal to 1 inside the disc of radius δ centered at λ_m and zero outside. If $\mathbf{k}_m = 2$, this is nothing but the Schiffer variation; this variation acts on the moduli of the Riemann surface in the same way as the delta-function with support at P_m.

Then formula (3.14) gives rise to (3.16). Computing the b-period of formula (3.16), we get

$$\frac{1}{2\pi i}\partial_{\lambda_m}\{\mathbf{B}_{\alpha\beta}\} = \mathrm{res}\Big|_{\lambda=\lambda_m}\left\{\frac{1}{(d\lambda)^2}\sum_{j=1}^{N} w_\alpha(\lambda^{(j)})w_\beta(\lambda^{(j)})\right\}\;. \tag{3.19}$$

In turn, this formula implies (3.17) if we take into account the following lemma:

Lemma 3 *An arbitrary holomorphic differential $w(P)$ on a compact Riemann surface $\mathcal{L}$, realized as N-fold covering of $\mathbb{C}P^1$, satisfies the following relation:*

$$\sum_{j=1}^{N} w(\lambda^{(j)}) = 0 \,. \tag{3.20}$$

Proof. It is sufficient to check that $\sum_{j=1}^{N} w_\alpha(\lambda^{(j)})$ is a holomorphic differential on $\mathbb{C}P^1$. The suspicious points are the branch points P_m. Consider Taylor series of $w(P)$ in the neighborhood of the branch point: $w(x_m) = \sum_{n=1}^{\infty} A_n x_m^n dx_m$. We have to check the regularity of the expression

$$\sum_{n=1}^{\infty} \left\{ \sum_{j=0}^{\mathbf{k}_m - 1} \gamma_m^{j(n+1)} \right\} A_n x_m^n \frac{dx_m}{d\lambda} \,,$$

where $\gamma = e^{2\pi i/\mathbf{k}_m}$, in a neighborhood of the point P_m. Taking into account that $d\lambda = \mathbf{k}_m x_m^{\mathbf{k}_m - 1} dx_m$, this regularity follows from the fact that $\sum_{j=0}^{\mathbf{k}_m - 1} \gamma^{j(n+1)} = 0$ for $n = 0, \dots, \mathbf{k}_m - 2$.

◇

Thus we proved the formulas (3.16), (3.17) for the case when each $\mathcal{P}_m$ consists of only one branch point. Any family of general coverings can be obtained in a smooth limit from these coverings if one assumes that some λ_m coincide; the formulas (3.16), (3.17) are already written in the form which is stable with respect to such limiting procedure.

◇

We notice that the non-trivial contributions in variational formulas (3.16), (3.17) arise only from the branch points contained in $\Pi^{-1}(\lambda_m)$.

To write down variational formula for the Szegö kernel it is convenient to introduce function

$$s(P, Q) = \frac{S(P, Q)}{\sqrt{dx_P}\sqrt{dx_Q}} \,, \tag{3.21}$$

which, obviously, depends on a choice of local parameters near points P and Q. We shall write down the variational formula for the Szegö kernel only in the partial case of simple branch points. For the case of arbitrary multiplicities the variational formulas can be deduced from the formulas of ([9], p.56) similarly to (3.16), (3.17).

Theorem 3 *Suppose that $\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](0) \neq 0$. Assume that all branch points of the covering $\mathcal{L}$ are simple and have different projections on λ-plane, i.e. $\Pi(P_m) = \lambda_m$. Suppose that the local parameters dx_P and dx_Q from (3.21) don't depend on (some) λ_m. Then*

$$\partial_{\lambda_m}\{s(P, Q)\} = \frac{1}{4}\left\{D_m[s(P, P_m)]s(P_m, Q) - s(P, P_m)D_m[s(P_m, Q)]\right\} \tag{3.22}$$

where

$$D_m[s(P, P_m)] := \frac{ds(P, Q)}{dx_m(Q)}\Big|_{Q=P_m} \tag{3.23}$$

Proof. The formula (3.22) can be deduced from general variational formula in ([9], p.56) by substitution of Schiffer variation. The simple independent proof looks as follows. The Szegö kernel $S(P, Q)$ behaves as follows as $P \to P_m$, when $x_P = \sqrt{\lambda - \lambda_m}$:

$$S(P, Q) = s(P, Q)\sqrt{dx_Q}\sqrt{d\sqrt{\lambda - \lambda_m}} \,,$$

where

$$s(P,Q) = s(P_m,Q) + D_m[s(P_m,Q)]\sqrt{\lambda-\lambda_m} + O(\lambda-\lambda_m)\ .$$

Differentiating

$$\sqrt{d\sqrt{\lambda-\lambda_m}} = \frac{\sqrt{d\lambda}}{\sqrt{2}(\lambda-\lambda_m)^{1/4}}$$

with respect to λ_m, we see that, as $P \to P_m$:

$$\partial_{\lambda_m}\{S(P,Q)\} = \frac{1}{4}\left\{\frac{s(P_m,Q)}{\lambda-\lambda_m} - \frac{D_m[s(P_m,Q)]}{\sqrt{\lambda-\lambda_m}}\right\}\sqrt{dx_Q}\sqrt{d\sqrt{\lambda-\lambda_m}}\ .$$

Analogous analysis of $\partial_{\lambda_m}\{S(P,Q)\}$ as $Q \to P_m$ allows to conclude that $\partial_{\lambda_m}\{S(P,Q)\}$ has the same set of singularities and singular parts as the expression

$$\frac{1}{4}\left\{s(P_m,Q)D_m[s(P,P_m)] - D_m[s(P_m,Q)]s(P,P_m)\right\}\sqrt{dx_P}\sqrt{dx_Q} \tag{3.24}$$

(the differentiation with respect to λ_m kills the pole of $S(P,Q)$ as $P \to Q$). Moreover, $S(P,Q)$ and expression (3.24) are meromorphic sections of the same line bundle $\Delta \otimes \chi_{\mathbf{p},\mathbf{q}}$ over $\mathcal{L}$ with respect to P and $\Delta \otimes \chi_{\mathbf{p},\mathbf{q}}^{-1}$ with respect to Q. Since, as long as the Szegö kernel is well-defined (i.e. $\Theta\begin{bmatrix}\mathbf{p}\\ \mathbf{q}\end{bmatrix}(0) \neq 0$), both of these bundles don't have holomorphic sections (see for example [9], p.29) we come to (3.22).

◇

4 Solution of Riemann-Hilbert problems with quasi-permutation monodromies

Here we solve Riemann-Hilbert problems with an arbitrary quasi-permutation monodromy representation $\mathcal{M}$ satisfying the non-triviality conditions 1 and 2 and an arbitrary set of regular singularities, except a divisor in the space of the monodromy data $\{\lambda_m,\ \mathcal{M}_m\}$. Consider some quasi-permutation monodromy representation $\mathcal{M}$ of $\pi_1[\mathbb{C}P^1 \setminus \{\lambda_1,\ldots,\lambda_M\}]$ satisfying conditions 1 and 2 and construct corresponding permutation representation $\mathcal{M}^0$; then construct the corresponding connected branched covering $\mathcal{L}$ of $\mathbb{C}P^1$. As before, denote the branch points of $\mathcal{L}$ by $P_1,\ldots,P_L$, where $L \geq M$, and their ramification indexes by $\mathbf{k}_1,\ldots,\mathbf{k}_L$ respectively. The genus g of $\mathcal{L}$ is given by the Riemann-Hurwitz formula (2.13). Introduce on $\mathcal{L}$ a canonical basis of cycles (a_α, b_α) such that the projections of the basic cycles on λ-plane don't pass through points $\lambda_0,\lambda_1,\ldots,\lambda_M$.

In the sequel it will be convenient to extend the notion of ramification index $\mathbf{k}_m^{(j)}$ to all of the points $\lambda_m^{(j)}$ assuming that the ramification index is equal to $\mathbf{k}_l$ if $\lambda_m^{(j)}$ coincides with the branch point P_l, and the ramification index is equal to 1 if $\lambda_m^{(j)}$ is not a branch point. Introduce the following set of parameters:

- Two vectors $\mathbf{p},\mathbf{q} \in \mathbb{C}^g$.
- Constants $r_m^{(j)} \in \mathbb{C}$ assigned to each point $\lambda_m^{(j)}$; we assume that the constants $r_m^{(j)} = r_m^{(j')}$ coincide if $\lambda_m^{(j)} = \lambda_m^{(j')}$ i.e. if $\lambda_m^{(j)}$ is a branch point. We require that

$$\sum_{m=1}^{M}\sum_{j=1}^{N} r_m^{(j)} = 0\ . \tag{4.1}$$

Therefore, among constants $r_m^{(j)}$ we have

$$MN - 1 - \sum_{l=1}^{L} (\mathbf{k}_l - 1) = MN - 2g - 2N + 1$$

independent parameters naturally assigned to non-coinciding points among $\lambda_m^{(j)}$.

Hence, altogether we introduced $MN - 2N + 1$ independent constants $\mathbf{p}, \mathbf{q}$ and $r_m^{(j)}$; according to lemma 1, this number exactly equals the number of non-trivial parameters carried by the non-vanishing entries of the quasi-permutation monodromy matrices of our RH problem.

Let us introduce on $\mathcal{L}$ a contour S, which connects some initial point P_0 with all points $\lambda_m^{(j)}$, including all branch points (we hope that the use of the same notation for this contour and the Szegö kernel does not lead to a confusion). Introduce also another contour S_0, which connects the point P_0 only with the branch points of odd multiplicity (i.e. the branch points with even ramification indexes $\mathbf{k}_m$); the number of such branch points must be even itself to get the integer genus via Riemann-Hurwitz formula. We assume that both contours S and S_0 don't intersect the basic cycles, i.e. they belong to the interior of fundamental polygon $\hat{\mathcal{L}}$ of $\mathcal{L}$.

Suppose that the point λ_0 does not belong to the set of projections of basic cycles (a_α, b_α) and contours S and S_0 on $\mathbb{C}P^1$. Let us define the intersection indexes of the contours $l_m^{(j)}$ with all basic cycles and the contour S:

$$I_{m\alpha}^{(j)} = l_m^{(j)} \circ a_\alpha \;, \qquad J_{m\alpha}^{(j)} = l_m^{(j)} \circ b_\alpha \;, \qquad K_m^{(j)} = l_m^{(j)} \circ S \;, \qquad L_m^{(j)} = l_m^{(j)} \circ S_0 \;, \tag{4.2}$$

$$\text{where} \quad m = 1, \dots, M \;; \;\; \alpha = 1, \dots, g \;; \;\; j = 1, \dots, N \;.$$

The contour S can always be chosen in such a way that $K_m^{(j)} = 1$ if $\lambda_m^{(j)}$ is not a branch point; if $\lambda_m^{(j)}$ is a branch point, then either $K_m^{(j)} = 1$ or $K_m^{(j)} = 0$.

Another auxiliary object we need to discuss is the lift of meromorphic spinor $\sqrt{d\lambda}$ from $\mathbb{C}P^1$ to $\mathcal{L}$. On $\mathbb{C}P^1$ this spinor has a single simple pole at $\lambda = \infty$. The differential $d\lambda$ on $\mathcal{L}$ has N second order poles at all infinities $\infty^{(k)}$ and zeros of order $\mathbf{k}_m - 1$ at all branch points P_m. If all ramification indexes $\mathbf{k}_m$ are odd, $\sqrt{d\lambda}$ must be a section of one of 4^g spinor bundles over $\mathcal{L}$ (this case was discussed in [24]); if some of $\mathbf{k}_m$ are even, $\sqrt{d\lambda}$ is a section of a spinor bundle on $\mathcal{L}$ with additional branch cut on the contour S_0, where $\sqrt{d\lambda}$ changes its sign.

To find the half-integer characteristic $[\mathbf{p}^0, \mathbf{q}^0]$ which corresponds to the spinor bundle defined by $\sqrt{d\lambda}$ we recall that the Abel map of the divisor of spinor bundle with all twists equal to $+1$ is equal to the vector of Riemann constants Δ. Therefore, the difference between Abel map of divisor of $\sqrt{d\lambda}$ and the vector of Riemann constants is equal to $\mathbf{B}\mathbf{p}^0 + \mathbf{q}^0$:

$$\mathbf{B}\mathbf{p}^0 + \mathbf{q}^0 = \frac{1}{2} \sum_{m=1}^{L} (\mathbf{k}_m - 1) U(P_m) - \sum_{j=1}^{N} U(\infty^{(j)}) - \Delta \tag{4.3}$$

The automorphy factors of $\sqrt{d\lambda}$ along cycles a_α and b_α are then equal to $e^{2\pi i p_\alpha^0}$ and $e^{-2\pi i q_\alpha^0}$, respectively.

Now we are in position to define the $N \times N$ matrix-valued function $\Psi(\lambda_0, \lambda)$ (we explicitly indicate dependence of Ψ on normalization point λ_0 for future convenience) which will later turn out to solve a Riemann-Hilbert problem. We define the germ of function $\Psi(\lambda_0, \lambda)$ in a small neighborhood of the normalization point λ_0 by the following formula:

$$\Psi_{kj}(\lambda_0, \lambda) = \psi(\lambda^{(j)}, \lambda_0^{(k)}) \ , \tag{4.4}$$

where the scalar function $\psi(P,Q)$ $(P,Q \in \hat{\mathcal{L}})$ is defined by

$$\psi(P,Q) = \widehat{S}(P,Q) E_0(\lambda, \mu) \ , \qquad \lambda = \Pi(P) \ , \quad \mu = \Pi(Q) \ , \tag{4.5}$$

and $\widehat{S}(P,Q)$ is the modified Szegö kernel, given by the following formula inside of the fundamental polygon of Riemann surface $\mathcal{L}$:

$$\widehat{S}(P,Q) := \frac{\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](U(P) - U(Q) + \Omega)}{\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\Omega) E(P,Q)} \prod_{m=1}^{M} \prod_{l=1}^{N} \left[\frac{E(P, \lambda_m^{(l)})}{E(Q, \lambda_m^{(l)})}\right]^{r_m^{(l)}} \ . \tag{4.6}$$

By E_0 we denote the prime-form on $\mathbb{C}P^1$

$$E_0(\lambda, \lambda_0) = \frac{\lambda - \lambda_0}{\sqrt{d\lambda d\lambda_0}} \ ,$$

lifted to $\mathcal{L}$ as we discussed above;

$$\Omega := \sum_{m=1}^{M} \sum_{j=1}^{N} r_m^{(j)} U(\lambda_m^{(j)}) \ . \tag{4.7}$$

The vector Ω does not depend on the choice of initial point of the Abel map due to assumption (4.1). The formula (4.4) makes sense if $\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\Omega) \neq 0$.

The following theorem gives a solution to RH problems with quasi-permutation monodromies satisfying non-triviality conditions 1 and 2 outside of divisor defined by equation $\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\Omega) = 0$. This is the main result of this section:

Theorem 4 *Suppose that $\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\Omega) \neq 0$. Let us analytically continue function $\Psi(\lambda)$ (4.4) from the neighborhood of the normalization point λ_0 to the universal covering $\hat{T}$ of $\mathbb{C}P^1 \setminus \{\lambda_1, \dots, \lambda_M\}$. Then the function $\Psi(\lambda)$ is non-singular and non-degenerate on $\hat{T}$. It has regular singularities at the points $\lambda = \lambda_m$ of the form (2.4), satisfies the normalization condition $\Psi(\lambda = \lambda_0) = I$ and solves the Riemann-Hilbert problem with the following quasi-permutation monodromies:*

$$(\mathcal{M}_n)_{jl} = \exp 2\pi i \left\{ \mathbf{k}_n^{(j)} r_n^{(j)} K_n^{(j)} - \sum_{\alpha=1}^{g} \{ J_{n\alpha}^{(j)}(p_\alpha + p_\alpha^0) + I_{n\alpha}^{(j)}(q_\alpha + q_\alpha^0) \} + \frac{1}{2} L_n^{(j)} \right\} \delta_{j_m[j],l} \ , \tag{4.8}$$

where all constants $\mathbf{p}, \mathbf{q}$ and $r_n^{(j)}$ were introduced above; half-integer characteristic $[\mathbf{p}^0, \mathbf{q}^0]$ is given by (4.3); the intersection indexes are defined by (4.2) ; $j_m[j]$ stands for the number of the sheet where the contour $l_m^{(j)}$ ends.

Proof. Choose in the Fay identity (3.13) $P_j = \lambda^{(j)}$ and $Q_k = \lambda_0^{(k)}$. Then, taking into account the holonomy properties of the prime-form and the asymptotics (3.6), we conclude that

$$\det\Psi = \prod_{m=1}^{M}\prod_{j,k=1}^{N}\left[\frac{E(\lambda^{(j)},\lambda_m^{(k)})}{E(\lambda_0^{(j)},\lambda_m^{(k)})}\right]^{r_m^{(k)}} ,$$

which, being considered as function of λ, does not vanish outside of the points $\lambda_m^{(k)}$; thus Ψ is non-degenerate and holomorphic if λ does not coincide with any of λ_m. The normalization condition $\Psi_{jk}(\lambda_0) = \delta_{jk}$ is an immediate corollary of the asymptotic expansion of the prime form (3.6).

Expressions (4.8) for the monodromy matrices of function Ψ follow from the simple consideration of the components of function Ψ. Suppose for a moment that the function $\widehat{S}(P,\lambda_0^{(k)})E_0(\lambda,\lambda_0)$, defined by (4.6), would be a single-valued function on $\mathcal{L}$ (as function of $P\in\mathcal{L}$). Then all monodromy matrices would be matrices of permutation: the analytical continuation of the matrix element $\widehat{S}(\lambda^{(j)},\lambda_0^{(k)})E_0(\lambda,\lambda_0)$ along contour $l_m^{(j)}$ would simply give the matrix element $\widehat{S}(\lambda^{(j_m[j])},\lambda_0^{(k)})E_0(\lambda,\lambda_0)$. However, since in fact the function $\widehat{S}(P,\lambda_0^{(k)})E_0(\lambda,\lambda_0)$ gains some non-trivial multipliers from crossing the basic cycles a_α, b_α and contour S, we get in (4.8) an additional exponential factor. Its explicit form is a corollary of the definition of intersection indexes which enter this expression, and periodicity properties of the theta-function and the prime-form.

Function Ψ is singular at the points λ_m due to, first, the product of the prime-forms in (4.6), and, second, due to different local parameters on λ-plane and on $\mathcal{L}$ which leads to additional singularity in (4.4). Obviously, the singularity of Ψ at λ_m is regular. $\diamond$

To elucidate the role of constants $\mathbf{p}$, $\mathbf{q}$ and $r_m^{(j)}$ we shall compute the matrices T_m from (2.4) which are the logarithms of the diagonal form of matrices $\mathcal{M}_m$ according to (2.5). For simplicity we consider the following "model" situation:

Theorem 5 *Suppose that $\mathcal{P}_m$ contains only one branch point P_m and this branch point has degree $\mathbf{k}_m$. Assume that sheets number $j = 1,\dots,\mathbf{k}_m$ are glued at P_m i.e. points $\lambda_m^{(j)}$ for $j = \mathbf{k}_m+1,\dots,N$ are non-branch points. Then the elements of diagonal matrix T_m are given by:*

$$t_m^{(j)} = r_m^{(j)} - \frac{1}{2} + \frac{1}{\mathbf{k}_m}(j-\frac{1}{2}) , \qquad j = 1,\dots,\mathbf{k}_m , \tag{4.9}$$

$$t_m^{(j)} = r_m^{(j)} , \qquad j = \mathbf{k}_m+1,\dots,N \tag{4.10}$$

(we recall that $r_m^{(1)} = \dots = r_m^{(\mathbf{k}_m)}$).

Proof. To verify (4.9) we first put all $r_m^{(j)} = 0$. Then the singular part $(\lambda-\lambda_m)^{T_m}$ from (2.4) of matrix Ψ (4.4) at λ_m has the form

$$\sqrt{\frac{dx_m}{d\lambda}}\mathrm{diag}(1,\, x_m,\dots,x_m^{\mathbf{k}_m-1},1,\dots,1) , \tag{4.11}$$

where $x_m = (\lambda-\lambda_m)^{1/\mathbf{k}_m}$, which leads to (4.9) with $r_m^{(j)} = 0$ after computing $dx_m/d\lambda$. Coefficients $t_m^{(j)}$, $j = \mathbf{k}_m+1,\dots,N$, which correspond to non-branch points $\lambda_m^{(j)}$ are in this case vanishing.

If we now introduce the non-trivial constants $r_m^{(j)}$, formulas (4.10) are obvious from (4.4), (4.5), (4.6). To check (4.9) we recall that, according to (4.4), (4.6), the components Ψ_{kj} for $j = 1,\dots,\mathbf{k}_m$ get at P_m an additional singularity of the form $x_m^{\mathbf{k}_m r_m^{(j)}}$ i.e. $(\lambda-\lambda_m)^{r_m^{(j)}}$, which leads to (4.9).

◇

The form of matrix T_m for an arbitrary branching structure over λ_m is a straightforward generalization of (4.9),(4.10).

Theorem 6 *The theorem 4 provides the solution of the Riemann-Hilbert problem with an arbitrary set of singularities $\{\lambda_m\}$ and an arbitrary (up to equivalence (2.12)) quasi-permutation monodromy representation, satisfying conditions 1 and 2, outside of the divisor in the $\{\mathcal{M}_m,\,\lambda_m\}$-space defined by the equation $\Theta\left[^{\mathbf{p}}_{\mathbf{q}}\right](\Omega)=0$.*

Proof. Denote the vector space of dimension $MN-2N+1$ with coordinates p_α, q_α and $r_m^{(j)}$ by $\mathcal{H}$; its $2g$-dimensional subspace defined by equations $r_m^{(j)}=0$ we denote by $\mathcal{H}_1$. The orthogonal subspace $\mathbf{p}=\mathbf{q}=0$ of dimension $MN-2N+1-2g$ is denoted by $\mathcal{H}_2$. The formulas (4.8) for $\ln\{(\mathcal{M}_n)_{jl}\}$ define an affine map (denote it by $\mathcal{F}$) from $\mathcal{H}$ to the space $\widehat{\mathcal{Q}(\mathcal{L})}$ (according to lemma 1, its dimension is also equal to $MN-2N+1$).

To show that an arbitrary quasi-permutation monodromy representation satisfying conditions 1 and 2 is covered by the theorem 4 unless $\Theta\left[^{\mathbf{p}}_{\mathbf{q}}\right](\Omega)=0$, it is sufficient to show that the affine map $\mathcal{F}$ (4.8) is non-degenerate. Non-degeneracy of $\mathcal{F}$ on $\mathcal{H}_2$ follows from the fact that, according to proposition 5, constants $r_m^{(j)}$ determine the same number of different eigenvalues of monodromy matrices.

On the other hand, vectors $\mathbf{p}$ and $\mathbf{q}$ don't enter the eigenvalues at all, i.e. these vectors influence only matrices C_n from (2.5). Thus $\mathcal{F}(\mathcal{H}_1)$ and $\mathcal{F}(\mathcal{H}_2)$ have only one common point - the image of the origin. Therefore, it remains to verify that the map $\mathcal{F}$ is non-degenerate on $\mathcal{H}_1$. The simplest way to verify this non-degeneracy is to observe that equivalent monodromy representations always correspond to coinciding (up to a constant factor) isomonodromic tau-functions. As we shall see below (6.10), for $r_m^{(j)}=0$ the vectors $\mathbf{p}$ and $\mathbf{q}$ enter the tau-function only via characteristics of the theta-function $\Theta\left[^{\mathbf{p}}_{\mathbf{q}}\right](0)$. This theta-function obviously can not remain invariant on any non-trivial linear subspace (independent of $\{\lambda_m\}$) in the $2g$-dimensional vector space spanned by the vectors $\mathbf{p}$ and $\mathbf{q}$.

◇

Remark 2 If we assume that all constants $r_m^{(j)}$ vanish, the formula (4.4) may be rewritten in terms of the Szegö kernel (3.9) as follows:

$$\Psi(\lambda_0,\lambda)_{kj}=S(\lambda^{(j)},\lambda_0^{(k)})E_0(\lambda,\lambda_0) \tag{4.12}$$

where $E_0(\lambda,\lambda_0)=(\lambda-\lambda_0)/\sqrt{d\lambda}\sqrt{d\lambda_0}$ is the prime-form on $\mathbb{C}P^1$.

The solution (4.4) of the Riemann-Hilbert problem satisfies the equation (2.6) with some matrices A_j. Below we shall give compact expressions for the residues A_j. Now we write down a formula for $\Psi_\lambda\Psi^{-1}$ using a simple procedure of inversion of matrix Ψ. Namely, if as before we explicitly indicate dependence of matrix Ψ on the argument λ and the normalization point λ_0 i.e. we write it as $\Psi(\lambda_0,\lambda)$, then for an arbitrary set of three points λ, μ and ν, we have the well-known relation:

$$\Psi(\mu,\lambda)\Psi(\lambda,\nu)=\Psi(\mu,\nu)\;. \tag{4.13}$$

In particular, for $\mu=\nu=\lambda_0$, we get

$$\Psi^{-1}(\lambda_0,\lambda)=\Psi(\lambda,\lambda_0)\;; \tag{4.14}$$

this relation in our case can be also verified via Fay's identity. Therefore, we have

$$\left(\Psi_\lambda \Psi^{-1}\right)_{lj} = \sum_{k=1}^{N} \psi_\lambda(\lambda^{(k)}, \lambda_0^{(l)}) \psi(\lambda_0^{(j)}, \lambda^{(k)}) \; , \tag{4.15}$$

where ψ is given by (4.5), (4.6). It is easy to see directly, using the formulas for $\psi(P,Q)$, that this expression has simple poles at all λ_m. Consider, for example, the contribution of a branch point P_m. In a neighbourhood of P_m we have

$$\psi_\lambda(P,Q)\psi(Q,P) = \frac{r_m^{(j)}}{\lambda - \lambda_m}(1 + O(x_m)) + \frac{1}{\lambda - \lambda_m} x_m^{1-\mathbf{k}_m}(a_o + a_1 x_m + \dots) \; , \tag{4.16}$$

as $P \to P_m$, where $r_m^{(j)}$ is a constant corresponding to the branch point P_m. Taking into account that $\sum_{s=0}^{\mathbf{k}_m - 1} \gamma_m^{sn} = 0$, where $\gamma_m = e^{2\pi i/\mathbf{k}_m}$, for any $n = 1, \dots, \mathbf{k}_m - 1$, we conclude that $\Psi_\lambda \Psi^{-1}$ has indeed a simple pole at $\lambda = \lambda_m$.

5 Isomonodromic deformations and solutions of Schlesinger system

If we now assume that vectors $\mathbf{p}$, $\mathbf{q}$ and constants $r_m^{(j)}$ don't depend on $\{\lambda_m\}$ then the monodromy matrices M_j also don't carry any $\{\lambda_m\}$-dependence and the isomonodromy deformation equations take place.

Theorem 7 *Assume that vectors* $\mathbf{p}$ *and* $\mathbf{q}$ *and constants* $r_m^{(j)}$ *don't depend on* $\{\lambda_m\}$. *Then the functions*

$$A_n(\{\lambda_m\}) := \operatorname{res}|_{\lambda=\lambda_n} \left\{ \Psi_\lambda \Psi^{-1} \right\} \; , \tag{5.1}$$

where $\Psi(\lambda)$ *is defined in (4.4), satisfy the Schlesinger system (2.9) outside of the hyperplanes* $\lambda_n = \lambda_m$ *and a submanifold* (ϑ) *of codimension one in the* $\{\lambda_m\}$*-space defined by the condition*

$$\mathbf{B}\mathbf{p} + \mathbf{q} + \Omega \in (\Theta) \; , \tag{5.2}$$

where (Θ) *denotes the theta-divisor on Jacobian* $J(\mathcal{L})$.

Proof. We can verify validity of deformation equations (2.8) directly for any choice of $r_m^{(j)}$ (as long as the set $\{\lambda_m, \mathcal{M}_m\}$ stays away from the divisor (5.2)). One way of proving (2.8) is the direct computation which uses an expression for $\Psi_{\lambda_m}\Psi^{-1}$ looking like (4.15), where derivative with respect to λ is substituted by derivative with respect to λ_m. The analysis of behaviour of $\Psi_{\lambda_m}\Psi^{-1}$ in a neighbourhood of λ_m is then parallel to analysis of $\Psi_\lambda \Psi^{-1}$ near this point; it shows that $\Psi_{\lambda_m}\Psi^{-1}$ has a simple pole at λ_m with the residue equal to $-A_m$, according to (2.8). Non-singularity of $\Psi_{\lambda_m}\Psi^{-1}$ at all other points λ_n for $n \neq m$ can be shown analogously, which leads to (2.8).

Another, and simpler way to prove the deformation equations (2.8) is to choose $\lambda_0 = \infty$, and use the fact that $\Psi_{\lambda_m}\Psi^{-1}$ is singular only at λ_m. The function Ψ is obviously invariant with respect to simultaneous shift of all λ_m and λ by small constant ϵ: $\Psi(\lambda + \epsilon, \{\lambda_m + \epsilon\}) = \Psi(\lambda, \{\lambda_m\})$ (this is true only if the normalization point is taken to be ∞, otherwise we have to shift λ_0, too). Differentiating this relation with respect to ϵ at $\epsilon = 0$, we get

$$\Psi_\lambda + \Psi_{\lambda_1} + \dots + \Psi_{\lambda_M} = 0 \; ,$$

which implies (2.8) with $\lambda_0 = \infty$. Then equations (2.8) with arbitrary λ_0 are obtained by gauge transformation of function Ψ.

$\diamond$

Introduce the function

$$\hat{s}(P,Q) = \frac{\widehat{S}(P,Q)}{\sqrt{dx_P}\sqrt{dx_Q}} \tag{5.3}$$

where the modified Szegö kernel $\widehat{S}(P,Q)$ is defined by (4.6), x_P and x_Q are local parameters at points P and Q respectively. If λ is used as local coordinate near P and Q (i.e. P and Q don't coincide with branch points and points at infinity), then the function $\hat{s}$ is related to the function ψ (4.5) as follows:

$$\hat{s}(P,Q) = \frac{\psi(P,Q)}{\lambda - \mu} , \tag{5.4}$$

where $\lambda = \Pi(P)$, $\mu = \Pi(Q)$.

The next proposition gives compact expressions for the solutions $\{A_m\}$ of the Schlesinger system.

Theorem 8 *The solutions (5.1) of the Schlesinger system (2.9) can be expressed as follows:*

$$(A_m)_{kj} = (\lambda_0 - \lambda_m)^2 \partial_{\lambda_m} \{\hat{s}(\lambda_0^{(j)}, \lambda_0^{(k)})\} , \qquad j \neq k \tag{5.5}$$

$$(A_m)_{kk} = (\lambda_0 - \lambda_m)^2 \partial_{\lambda_m} \{a_0^{(k)}\} , \tag{5.6}$$

where function $\hat{s}(P,Q)$ is defined by (5.3) with λ used as local parameter in a neighbourhood of $\lambda_0^{(j)}$ on every sheet; $a_0^{(k)}$ is defined as a coefficient in the Laurent series:

$$\hat{s}(\lambda^{(k)}, \lambda_0^{(k)}) = \frac{1}{\lambda - \lambda_0} + a_0^{(k)} + O(\lambda - \lambda_0) , \quad \text{as} \quad \lambda \to \lambda_0 . \tag{5.7}$$

Proof. From (2.8) we find

$$A_m = (\lambda_0 - \lambda_m)^2 \left(\Psi_{\lambda_m} \Psi^{-1}\right)_\lambda \Big|_{\lambda=\lambda_0} ,$$

which implies (since $\Psi(\lambda_0, \lambda_0) = I$) that

$$A_m = (\lambda_0 - \lambda_m)^2 \Psi_{\lambda\lambda_m} \Big|_{\lambda=\lambda_0} .$$

This relation immediately leads to (5.6), (5.5) if we use the expression of Ψ (4.4) and the link (5.4) between ψ and $\hat{s}$.

$\diamond$

The derivative with respect to λ_m in (5.5), (5.6) can be computed using variational formulas for all ingredients of function ψ (4.5). In general (for arbitrary multiplicities of branch points and non-vanishing $r_m^{(j)}$) the result turns out to be rather complicated. Therefore, we write the final formulas only in the following partial case.

Corollary 1 *Assume that all branch points are simple and have different projections on λ-plane, and that all constants $r_m^{(j)}$ vanish. Then solution (5.5), (5.6) of the Schlesinger system (2.9) can be written as follows:*

$$(A_m)_{kj} = \frac{1}{4} \left\{ D_m[s(\lambda_0^{(j)}, P_m)] s(P_m, \lambda_0^{(k)}) - s(\lambda_0^{(j)}, P_m) D_m[s(P_m, \lambda_0^{(k)})] \right\} \tag{5.8}$$

for any $j, k = 1, \ldots, N$, where $s(P,Q)$ is given by (3.9), (3.21).

Remark 3 Our solutions $\{A_m(\{\lambda_n\})\}$ (5.5), (5.6) of the Schlesinger system is singular on the Malgrange divisor (ϑ) (it has a pole whose order is equal to the order of the zero of the tau-function at the point of the divisor). Since this divisor carries a non-trivial dependence on monodromy matrices, which parametrise the initial conditions for the Schlesinger system, this singularity depends on initial conditions i.e. it is "movable".

The next section is devoted to computation of tau-function (2.10) corresponding to solutions of Schlesinger system given by the theorem 8.

6 Isomonodromic tau-function

6.1 Tau-function and Bergmann projective connection

According to the definition of the tau-function (2.10), let us start with calculation of expression $\mathrm{tr}\left(\Psi_\lambda \Psi^{-1}\right)^2$. Notice that this object is independent of the choice of normalization point λ_0 [substitution of λ_0 by another point $\tilde{\lambda}_0$ corresponds to the λ-independent "gauge" transformation $\Psi(\lambda) \to \tilde{\Psi}(\lambda) = \Psi^{-1}(\tilde{\lambda}_0)\Psi(\lambda)$].

Consider the limit $\lambda_0 \to \lambda$ in the formula (4.4) for Ψ_{jk}, where $\widehat{S}(P,Q)$ is given by expression (4.6). In this limit matrix elements of the function Ψ behave as follows:

$$\Psi_{kj}(\lambda,\lambda_0) = \frac{\lambda_0 - \lambda}{d\lambda}\widehat{S}(\lambda^{(j)},\lambda^{(k)}) + O\{(\lambda_0-\lambda)^2\}\ , \qquad k \neq j \tag{6.1}$$

$$\Psi_{jj}(\lambda,\lambda_0) = 1 + \frac{\lambda_0-\lambda}{d\lambda}\left\{W_1(\lambda^{(j)}) - W_2(\lambda^{(j)})\right\}\ , \tag{6.2}$$

where $W_1(P)$ is a linear combination of the basic holomorphic 1-forms on $\mathcal{L}$:

$$W_1(P) = \frac{1}{\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\Omega)}\sum_{\alpha=1}^{g}\partial_{z_\alpha}\{\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\Omega)\}w_\alpha(P)\ , \tag{6.3}$$

and $W_2(P)$ is the following meromorphic 1-form with simple poles at the points $\lambda_m^{(j)}$ and the residues $r_m^{(j)}$:

$$W_2(P) = \sum_{m=1}^{M}\sum_{j=1}^{N} r_m^{(j)} d_P \ln E(P,\lambda_m^{(j)})\ . \tag{6.4}$$

Taking into account independence of the expression $\mathrm{tr}\left(\Psi_\lambda\Psi^{-1}\right)^2$ on position of the normalization point λ_0, we have

$$\mathrm{tr}\left(\Psi_\lambda\Psi^{-1}\right)^2(d\lambda)^2 = 2\sum_{j<k}\widehat{S}(\lambda^{(j)},\lambda^{(k)})\widehat{S}(\lambda^{(k)},\lambda^{(j)}) + \sum_{j=1}^{N}\left(W_1(\lambda^{(j)}) - W_2(\lambda^{(j)})\right)^2\ .$$

To transform this expression we first notice that, according to (3.12),

$$\widehat{S}(P,Q)\widehat{S}(Q,P) = -\mathbf{w}(P,Q) - \sum_{\alpha,\beta=1}^{g}\partial^2_{z_\alpha z_\beta}\{\ln\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\Omega)\}w_\alpha(P)w_\beta(Q)\ .$$

Furthermore, since $W_1(P)$ is a holomorphic 1-form on $\mathcal{L}$, the expression $\sum_{j=1}^N W_1(\lambda^{(j)})$ vanishes identically according to Lemma 3; hence

$$\sum_{j=1}^N \{W_1(\lambda^{(j)})\}^2 = -2\sum_{\substack{j,k=1\\ j<k}}^N \sum_{\alpha,\beta=1}^g \partial_{z_\alpha}\{\ln\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\Omega)\}\partial_{z_\beta}\{\ln\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\Omega)\}w_\alpha(\lambda^{(j)})w_\beta(\lambda^{(k)})\ .$$

Similarly, we can conclude that $\sum_{j=1}^N\{W_2(\lambda^{(j)})\}^2$ is a meromorphic 2-form on $\mathbb{C}P^1$ which has poles only at the points λ_m; calculation of corresponding residues gives

$$\sum_{j=1}^N \{W_2(\lambda^{(j)})\}^2 = \sum_{m,n=1}^M \frac{r_{mn}(d\lambda)^2}{(\lambda-\lambda_n)(\lambda-\lambda_m)}\ , \tag{6.5}$$

where

$$r_{mn} = \sum_{j=1}^N r_m^{(j)} r_n^{(j)}\ . \tag{6.6}$$

Therefore, as the first step of our calculation, we get the following expression:

$$\frac{1}{2}\mathrm{tr}\left(\Psi_\lambda\Psi^{-1}\right)^2(d\lambda)^2 = -\sum_{j<k}\mathbf{w}(\lambda^{(j)},\lambda^{(k)}) \tag{6.7}$$

$$-\frac{1}{\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\Omega)}\sum_{j<k}\sum_{\alpha,\beta}\partial^2_{z_\alpha z_\beta}\{\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\Omega)\}w_\alpha(\lambda^{(j)})w_\beta(\lambda^{(k)}) + \frac{1}{2}\sum_{m,n}\frac{r_{mn}(d\lambda)^2}{(\lambda-\lambda_n)(\lambda-\lambda_m)}$$

$$-\frac{1}{\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\Omega)}\sum_\alpha \partial_{z_\alpha}\{\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\Omega)\}\sum_m\sum_j r_m^{(j)} w_\alpha(\lambda^{(j)}) d_P \ln E(P,\lambda_m^{(j)})\ .$$

Let us now analyze the Hamiltonians

$$H_m \equiv \frac{1}{2}\mathrm{res}|_{\lambda=\lambda_m}\left\{\mathrm{tr}\left(\Psi_\lambda\Psi^{-1}\right)^2\right\}$$

(to avoid confusion we notice that in this section we use only the notion of residue of *function* of variable λ at finite points of the complex plane). Using the heat equation for theta-function (3.3), and Rauch's formula (3.17), we can represent H_m in the following form:

$$H_m = -\mathrm{res}|_{\lambda=\lambda_m}\frac{1}{(d\lambda)^2}\left\{\sum_{j<k}\mathbf{w}(\lambda^{(j)},\lambda^{(k)})\right\} + \frac{1}{2}\sum_{n\neq m}\frac{r_{mn}}{\lambda_m-\lambda_n} \tag{6.8}$$

$$+\frac{1}{\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\Omega)}\sum_{\alpha,\beta}\frac{\partial\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\Omega)}{\partial\mathbf{B}_{\alpha\beta}}\partial_{\lambda_m}\{\mathbf{B}_{\alpha\beta}\} + \frac{1}{\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\Omega)}\sum_\alpha\partial_{z_\alpha}\{\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\Omega)\}\partial_{\lambda_m}\{\Omega_\alpha\}\ ,$$

or, equivalently,

$$H_m = -\mathrm{res}|_{\lambda=\lambda_m}\left\{\frac{1}{(d\lambda)^2}\sum_{j<k}\mathbf{w}(\lambda^{(j)},\lambda^{(k)})\right\} + \partial_{\lambda_m}\ln\left\{\prod_{l<n}(\lambda_l-\lambda_n)^{r_{ln}}\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\Omega)\right\}\ . \tag{6.9}$$

Therefore, we come to the following

Theorem 9 *The tau-function corresponding to solution (5.1) of Schlesinger system, is given by*

$$\tau(\{\lambda_n\}) = F(\{\lambda_n\}) \prod_{m,n=1}^{M} (\lambda_m - \lambda_n)^{r_{mn}} \Theta \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} (\Omega|\mathbf{B}) \ , \tag{6.10}$$

where function $F(\{\lambda_n\})$ does not depend on constants $\mathbf{p}, \mathbf{q}$ and $r_n^{(j)}$, and satisfies the following system of compatible equations

$$\partial_{\lambda_m}\{\ln F\} = \sum_{P_l \in \mathcal{P}_m} \frac{1}{12\mathbf{k}_l} \frac{R_l^{(\mathbf{k}_l-2)}(P_l)}{(\mathbf{k}_l-2)!} \ , \tag{6.11}$$

where $R_l(P)$ is Bergmann projective connection corresponding to our choice of system of local parameters near branch points $P_l \in \mathcal{P}_m$: $x_l = (\lambda - \lambda_m)^{1/\mathbf{k}_l}$.

Proof. According to the expressions (6.9), we need to prove the following lemma.

Lemma 4 *The following identity holds:*

$$\sum_{P_l \in \mathcal{P}_m} \frac{1}{12\mathbf{k}_l} \frac{R_l^{(\mathbf{k}_l-2)}(P_l)}{(\mathbf{k}_l-2)!} = -\text{res}|_{\lambda=\lambda_m} \left\{ \sum_{j \neq k} \frac{\mathbf{w}(\lambda^{(j)}, \lambda^{(k)})}{(d\lambda)^2} \right\} . \tag{6.12}$$

Proof. The right hand side of the formula (6.12) can be rewritten in terms of non-singular part H of the Bergmann kernel (3.7) as follows:

$$-\text{res}|_{\lambda=\lambda_m} \sum_{P_l \in \mathcal{P}_m} \left\{ \sum_{j,k=1}^{\mathbf{k}_l} H(\gamma_l^j x_l, \gamma_l^k x_l) \gamma_l^{j+k} \left(\frac{dx_l}{d\lambda} \right)^2 \right\} , \tag{6.13}$$

where $\gamma_l = exp \frac{2\pi i}{\mathbf{k}_l}$. In terms of coefficients of the Taylor series of $H(x_l, y_l)$ around P_l we have:

$$H(x_l, y_l) = \sum_{s=0}^{\infty} \sum_{p=0}^{s} \frac{H^{(p,s-p)}(0,0)}{p!(s-p)!} x_l^p y_l^{s-p} \ ,$$

and expression (6.13) looks as follows:

$$-\sum_{P_l \in \mathcal{P}_m} \frac{1}{\mathbf{k}_l^2} \sum_{p=0}^{\mathbf{k}_l-2} \frac{H^{(p,\mathbf{k}_l-2-p)}(0,0)}{p!(\mathbf{k}_l-2-p)!} \sum_{j,k=1\,,\, j<k}^{\mathbf{k}_l} \gamma^{(p+1)k + (\mathbf{k}_l-p-1)j} \ .$$

Summing up the geometrical progression, we get:

$$\sum_{P_l \in \mathcal{P}_m} \frac{1}{2\mathbf{k}_l} \sum_{p=0}^{\mathbf{k}_l-2} \frac{H^{(p,\mathbf{k}_l-2-p)}(0,0)}{p!(\mathbf{k}_l-2-p)!} = \sum_{P_l \in \mathcal{P}_m} \frac{1}{12\mathbf{k}_l} \frac{R_l^{(\mathbf{k}_l-2)}(0)}{(\mathbf{k}_l-2)!} \ .$$

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One can check that only the non-singular part of the Bergmann kernel contributes to the residue in (6.12); therefore, we can further express $\partial_{\lambda_m} \ln F$ in terms of the Bergmann projective connection corresponding to the natural choice of local coordinates on $\mathcal{L}$ on the branched covering $\mathcal{L}$.

Remark 4 Suppose that all projections of branch points P_m on $\mathbb{C}P^1$ are different. Then equations (6.11) for function $F(\{\lambda_n\})$ can be written as follows:

$$\partial_{\lambda_m}\{\ln F\} = -\frac{1}{12\pi}\int_{\mathcal{L}} \mu_m R_m (dx)^2 \;, \tag{6.14}$$

where μ_m is the Beltrami differential (3.18) corresponding to variation of the branch point P_m.

Theorem 10 *The following equations for Bergmann projective connection on the branch covering $\mathcal{L}$ are fulfilled:*

$$\frac{\partial}{\partial\lambda_n}\left\{\sum_{P_l\in\mathcal{P}_m}\frac{1}{(\mathbf{k}_l-2)!\,\mathbf{k}_l}\left(\frac{d}{dx_l}\right)^{\mathbf{k}_l-2} R_l(x_l)\big|_{x_l=0}\right\}$$
$$=\frac{\partial}{\partial\lambda_m}\left\{\sum_{P_l\in\mathcal{P}_n}\frac{1}{(\mathbf{k}_l-2)!\,\mathbf{k}_l}\left(\frac{d}{dx_l}\right)^{\mathbf{k}_l-2} R_l(x_l)\big|_{x_l=0}\right\}\;. \tag{6.15}$$

Proof. Equations (6.15) provide integrability of equations (6.11) for the function F which follows from integrability of the equations (2.10) for the isomonodromic tau-function.

◇

Corollary 2 *Let all branch points of $\mathcal{L}$ be simple and have different projections on λ-plane. Then values of Bergmann projective connection computed with respect to the natural system of local parameters on $\mathcal{L}$ (i.e. $x_m = \sqrt{\lambda-\lambda_m}$ at the branch point P_m) satisfy the following equations:*

$$\frac{\partial R_m(P_m)}{\partial\lambda_n} = \frac{\partial R_n(P_n)}{\partial\lambda_m}\;. \tag{6.16}$$

These equations are analogous to equations for accessory parameters which appear in the uniformization problem of punctured sphere [25].

Since the Bergmann projective connection $R_m(P_m)$ is finite and holomorphic function of $\{\lambda_m\}$ as long as the Riemann surface $\mathcal{L}$ remains non-degenerate, we conclude that the function F does not vanish and remains finite outside of the hyperplanes $\lambda_m = \lambda_n$. This allows to claim that the divisor of zeros of the tau-function (6.10) coincides with the divisor of zeros of the theta-function $\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\Omega|\mathbf{B})$:

Theorem 11 *The set of singularities $\{\lambda_m\}$ lies in the Malgrange divisor $(\vartheta)\subset\mathbb{C}^M$ iff the vector $\mathbf{Bp}+\mathbf{q}+\Omega$ belongs to the theta-divisor (Θ) in the Jacobi manifold $J(\mathcal{L})$ of the Riemann surface $\mathcal{L}$.*

We remind that in the expression $\mathbf{Bp}+\mathbf{q}+\Omega$ the $\{\lambda_m\}$-dependence is hidden inside the matrix of b-periods and the vector Ω.

Remark 5 It turns out [18] that function F itself coincides with the isomonodromic tau-function of another RH problem introduced by Dubrovin [7] in the context of Frobenius manifolds associated with Hurwitz spaces. It would be interesting to obtain the explicit link between this RH problem and the one studied in this paper on the level of monodromy representations and function Ψ.

6.2 Riemann-Hilbert problems with off-diagonal 2×2 monodromy matrices

Here we consider the simplest case of $N = 2$, when any matrix of quasi-permutation is either diagonal or off-diagonal. We shall consider monodromy groups such that all monodromies M_m are off-diagonal; the insertion of additional diagonal monodromies according to the general scheme is straightforward. In this case the branched covering $\mathcal{L}$ corresponds to hyperelliptic algebraic curve with branch points $\lambda_1, \dots, \lambda_M$ and function F may be calculated explicitly [14]. We have $M = 2g + 2$, where g is the genus of the hyperelliptic curve $\mathcal{L}$:

$$w^2 = \prod_{m=1}^{2g+2} (\lambda - \lambda_m) \; . \tag{6.17}$$

Let us put all $r_m^{(j)} = 0$; in this case the formula (4.12) gives the solution $\Psi(\lambda) \in SL(2)$ of the RH problem with arbitrary off-diagonal $SL(2)$-valued monodromies:

$$M_m = \begin{pmatrix} 0 & d_m \\ -d_m^{-1} & 0 \end{pmatrix} ,$$

where constants d_m may be expressed in terms of the elements of vectors $\mathbf{p}, \mathbf{q}$. Let us count the number of essential parameters in the monodromy matrices and in the construction of function Ψ. The matrices M_m contain altogether $2g + 2$ constants; however, there is one relation (product of all monodromies gives I). One more parameter is non-essential due to possibility of simultaneous conjugation of all monodromies with an arbitrary diagonal constant matrix. Therefore, the set of monodromy matrices contains $2g$ non-trivial constants in accordance with the number of non-trivial constants contained in the vectors $\mathbf{p}$ and $\mathbf{q}$.

To integrate the remaining equations

$$\partial_{\lambda_m} \ln F = \frac{1}{24} R(\lambda_m) \tag{6.18}$$

on hyperelliptic curve (6.17) we use the following formula ([8], p.20) for the Bergmann projective connection at arbitrary point of P of the hyperelliptic curve $\mathcal{L}$ (where x is the local parameter in the neighborhood of the point P, $\lambda = \Pi(P)$ is the projection of P on λ-plane):

$$R(P) = \{\lambda(x), x\}(P) + \frac{3}{8} \left(\frac{d}{dx} \ln \frac{\prod_{\lambda_m \in T} (\lambda - \lambda_m)}{\prod_{\lambda_m \notin T} (\lambda - \lambda_m)} \right)^2 (P) \tag{6.19}$$

$$- \frac{6}{\Theta \begin{bmatrix} \mathbf{p}^T \\ \mathbf{q}^T \end{bmatrix} (0)} \sum_{\alpha,\beta=1}^{g} \partial^2_{z_\alpha z_\beta} \left\{ \Theta \begin{bmatrix} \mathbf{p}^T \\ \mathbf{q}^T \end{bmatrix} (0) \right\} \frac{w_\alpha}{dx}(P) \frac{w_\beta}{dx}(P) \; .$$

Here $\{\lambda, x\}$ is the Schwarzian derivative of λ with respect to x; T is an arbitrary divisor consisting of $g+1$ branch points, which satisfies certain non-degeneracy condition. Characteristic $\begin{bmatrix} \mathbf{p}^T \\ \mathbf{q}^T \end{bmatrix}$ is the even half-integer characteristic corresponding to the divisor T according to the following equation:

$$\mathbf{B}\mathbf{p}^T + \mathbf{q}^T = \sum_{\lambda_m \in T} U(\lambda_m) - \Delta \; , \tag{6.20}$$

where Δ is a vector of Riemann constants; the initial point of the Abel map is chosen to be, say, λ_1. In this case the r.h.s. of (6.20) is a linear combination, with integer or half-integer coefficients, of the vectors $\mathbf{e}_\alpha$ and $\mathbf{B}\mathbf{e}_\alpha$. These coefficients are composed in vectors $\mathbf{p}^T$ and $\mathbf{q}^T$. The non-degeneracy requirement imposed on the divisor T gives rise to the condition that the vector $\mathbf{B}\mathbf{p}^T + \mathbf{q}^T$ does not belong to the theta-divisor on $J(\mathcal{L})$, i.e. $\Theta \begin{bmatrix} \mathbf{p}^T \\ \mathbf{q}^T \end{bmatrix} (0) \neq 0$.

The Bergmann projective connection R, as well as the function F, are independent of the choice of the divisor T. If in (6.19) we choose $P = \lambda_m$, the local parameter is $x = \sqrt{\lambda - \lambda_m}$. Then all terms in $R_m(\lambda_m)$ which don't contain theta-function can be integrated explicitly; the terms containing theta-function can be represented as logarithmic derivative with respect to λ_m by making use of the heat equation for theta-function (3.3) and Rauch formula (3.17). These terms are equal to $-6\partial_{\lambda_m} \ln \Theta \begin{bmatrix} \mathbf{p}^T \\ \mathbf{q}^T \end{bmatrix} (0)$. In turn, this expression may be rewritten using the Thomae formula

$$\Theta \begin{bmatrix} \mathbf{p}^T \\ \mathbf{q}^T \end{bmatrix} (0) = \pm (\det \mathcal{A})^2 \prod_{\lambda_m, \lambda_n \in T} (\lambda_m - \lambda_n) \prod_{\lambda_m, \lambda_n \notin T} (\lambda_m - \lambda_n),$$

where $\mathcal{A}_{\alpha_\beta} = \oint_{a_\alpha} \frac{\lambda^{\beta-1}}{w}$ is the $g \times g$ matrix of a-periods of non-normalized holomorphic differentials on $\mathcal{L}$. Collecting all factors arising from the Thomae formula and from the expression (6.19), we get the following answer for the function F:

$$F = [\det \mathcal{A}]^{-\frac{1}{2}} \prod_{m<n} (\lambda_m - \lambda_n)^{-\frac{1}{8}} \,, \tag{6.21}$$

which is equal to $\{\det \overline{\partial}_0\}^{-1/2}$, where $\det \overline{\partial}_0$ can be interpreted as (defined and computed heuristically [29, 16]) determinant of Cauchy-Riemann operator acting in trivial bundle over $\mathcal{L}$. For the tau-function itself we get the following expression

$$\tau(\{\lambda_m\}) = [\det \mathcal{A}]^{-\frac{1}{2}} \prod_{m<n} (\lambda_m - \lambda_n)^{-\frac{1}{8}} \Theta \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} (0|\mathbf{B}) \,,$$

which can be interpreted as determinant of the Cauchy-Riemann operator acting on the sections of line bundle $\Delta \otimes \chi_{\mathbf{p},\mathbf{q}}$ over $\mathcal{L}$.

6.3 Tau-function in genus 0 and 1

The function F (6.12), and, therefore, the Jimbo-Miwa tau-function can be also calculated for monodromy groups corresponding to arbitrary coverings of genus 0 and 1.

Theorem 12 *Let the branched covering $\mathcal{L}$ corresponding to a Riemann-Hilbert problem with quasi-permutation monodromies have genus 0. Then the Jimbo-Miwa tau-function has the following form:*

$$\tau = \left\{ \frac{\prod_{m=1}^{M} (\frac{dU}{dx_m}(P_m))^{\frac{\mathbf{k}_m - 1}{2}}}{\prod_{k=2}^{N} (\frac{dU}{d\zeta_k}(\infty^{(k)})} \right\}^{1/12} \prod_{m,n=1}^{M} (\lambda_m - \lambda_n)^{r_{mn}} \,, \tag{6.22}$$

where $U(P) : \mathcal{L} \to \mathbb{C}P^1$ is the uniformization map of the branched covering $\mathcal{L}$ fixed by the condition $U(\infty^{(1)}) = \infty$; $x_m = (\lambda - \lambda_m)^{1/\mathbf{k}_m}$ are local parameters near the branch points; $\zeta_k \equiv 1/\lambda$ are the local parameters around infinities of $\mathcal{L}$. Constants r_{mn} are given by (6.6).

Analogous statement is valid in genus 1 case:

Theorem 13 *Let the branched covering $\mathcal{L}$ corresponding to a Riemann-Hilbert problem with quasi-permutation monodromies have genus 1. Then the Jimbo-Miwa tau-function has the following form:*

$$\tau = \left\{ \frac{\prod_{m=1}^{M} (\frac{dU}{dx_m}(P_m))^{\frac{\mathbf{k}_m - 1}{2}}}{\prod_{k=1}^{N} (\frac{dU}{d\zeta_k}(\infty^{(k)})} \right\}^{1/12} \prod_{m,n=1}^{M} (\lambda_m - \lambda_n)^{r_{mn}} \frac{\Theta \left[\begin{smallmatrix} \mathbf{p} \\ \mathbf{q} \end{smallmatrix} \right] (\Omega|\mu)}{[\Theta_1'(0|\mu)]^{1/3}} , \tag{6.23}$$

where $U(P) = \int^P w(P)$ is the uniformization map of the branched covering $\mathcal{L}$ to its fundamental parallelogramm with periods 1 *and μ ($w(P)$ is the normalized holomorphic 1-form on $\mathcal{L}$); θ_1 is the odd Jacobi theta-function on $\mathcal{L}$; $x_m = (\lambda - \lambda_m)^{1/\mathbf{k}_m}$ are local parameters near the branch points; $\zeta_k \equiv 1/\lambda$ are the local parameters around infinities of $\mathcal{L}$. Constants r_{mn} are given by (6.6); argument Ω of the theta-function is defined by (4.7).*

The proofs of both theorems are contained in [17]; they are based on the properties of Dirichlet action corresponding to the metric $d\lambda d\overline{\lambda}$ in genus zero, and the metric $w\overline{w}$ in genus 1.

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