A maximally superintegrable system on an n-dimensional space of nonconstant curvature

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Abstract

A novel Hamiltonian system in n dimensions which admits the maximal number $2n - 1$ of functionally independent, quadratic first integrals is presented. This system turns out to be the first example of a maximally superintegrable Hamiltonian on an n-dimensional Riemannian space of nonconstant curvature, and it can be interpreted as the intrinsic Smorodinsky–Winternitz system on such a space. Moreover, we provide three different complete sets of integrals in involution and solve the equations of motion in closed form.

1 Introduction

A Hamiltonian system on a $2n$-dimensional phase space is *maximally superintegrable* (MS) if it is Liouville integrable [2] and there exist further $n - 1$ functionally independent (global) first integrals. Throughout this paper we shall restrict ourselves to the case of *natural* Hamiltonians, whose kinetic term defines a Riemannian metric on an n-dimensional configuration space.

The intrinsic relevance of MS systems has resulted in a wealth of results concerning MS Hamiltonians [11,20,21,28,34]. MS models with integrals of motion quadratic in the momenta are of particular interest because of their connections with generalized symmetries [19,35], isochronic potentials [10,18], and separability of the associated Hamilton–Jacobi (HJ) equation and Schrödinger equations [8,22,27]. A systematic study of MS systems on certain low-dimensional spaces is currently being developed by Kalnins et al. (cf. [24] and references therein), and in fact in two dimensions they managed to obtain a (local) classification of all MS Hamiltonians with integrals at most quadratic in the momenta [25,26]. This result, which hinges on a classical theorem by Koenigs [29],

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yielded the first examples of MS systems not defined on constant curvature spaces.

Unfortunately, most examples of MS Hamiltonians are restricted to low-dimensional spaces. To the best of our knowledge, the only known examples of n-dimensional MS systems with quadratic integrals are given by the (generalized) Kepler problem [14], the Smorodinsky–Winternitz (SW) system [15,17], their generalizations to the simply connected spaces of constant sectional curvature (cf. [4] and references therein), and the geodesic flow on these spaces. Note that the allowed potentials correspond essentially to those appearing in the extension to space forms of Bertrand's theorem [7,31]. If one drops the requirement that the integrals be quadratic in momenta, the list can be enlarged with the rational and hyperbolic Calogero–Sutherland–Moser models of type A [9,32,36, 38], the nonisotropic oscillator with rational frequencies and the nonperiodic Toda lattice [1,37].

In this paper we show that the Hamiltonian

$$\mathcal{H}(\mathbf{p},\mathbf{q}) = \frac{\mathbf{p}^2 + \omega^2\,\mathbf{q}^2 + \sum_j b_j q_j^{-2}}{\kappa + \mathbf{q}^2}\,, \tag{1}$$

where $\kappa > 0$ and $\omega^2, b_j \geq 0$, is a MS system with quadratic first integrals. Here $\mathbf{p},\mathbf{q} \in \mathbb{R}^n$, and we shall always assume that $\mathbb{R}^{2n}$ is endowed with its standard symplectic form $\mathrm{d}\mathbf{q} \wedge \mathrm{d}\mathbf{p}$. For any $\mathbf{u} \in \mathbb{R}^n$ we use the notation

$$\mathbf{u}^2 = \mathbf{u} \cdot \mathbf{u} = \sum_i u_i^2$$

and all the sum indices run from 1 to n unless otherwise stated.

The Hamiltonian (1) yields the motion under the potential

$$V(\mathbf{q}) = \frac{\omega^2\,\mathbf{q}^2 + \sum_j b_j q_j^{-2}}{\kappa + \mathbf{q}^2} \tag{2}$$

on the conformally flat Riemannian manifold $\mathcal{M}^n := (\mathbb{R}^n, \mathrm{d}s^2)$ whose metric is given by

$$\mathrm{d}s^2 = (\kappa + \mathbf{q}^2)\,\mathrm{d}\mathbf{q}^2\,. \tag{3}$$

The Hamiltonian (1) provides the first example of a Hamiltonian system on a Riemannian space of nonconstant curvature which is MS in any dimension. The analysis of the geometry of $\mathcal{M}^n$ presented in Sec. 4 shows that this model can be in fact regarded as the intrinsic SW system on this space. When $n = 2$ this model is listed in Kalnins et al.'s classification of MS systems in the Darboux space of type III [26]. This model is singled out among the others in this list because it is the only one readily amenable to the Poisson coalgebra treatment developed in Sec. 2.

The paper is organized as follows. In Sec. 2 we show that the Hamiltonian (1) is indeed MS and briefly study the separability of the HJ equation. The proof is greatly simplified by the use of the $\mathfrak{sl}(2)$ Poisson coalgebra symmetry [5,6] of the model. In Sec. 3 we study the equations of motion and find their general solution in closed form using the previously computed first integrals. Quite remarkably, the radial motion is essentially that of the Kepler problem, whereas the HJ equation essentially coincides with the one for the SW system. In Sec. 4 we analyze the geometry of the underlying Riemannian space and relate the Hamiltonian (1) to the intrinsic harmonic oscillator in $\mathcal{M}^n$.

2 Maximal superintegrability and separability of the HJ equation

Instead of dealing with the model (1) directly, in the following two sections we prefer to analyze the Hamiltonian

$$H(\mathbf{p}, \mathbf{q}) = \frac{\mathbf{p}^2 - c + \sum_j b_j q_j^{-2}}{2(\kappa + \mathbf{q}^2)}, \qquad c := \kappa \omega^2, \tag{4}$$

which is related to (1) via $\mathcal{H} = 2H + \omega^2$. Obviously the maximal superintegrability of H is equivalent to that of $\mathcal{H}$, whereas the trajectories of H can be mapped into those of $\mathcal{H}$ via the time dilation $t \mapsto t/2$. In particular, H and $\mathcal{H}$ possess the same orbits.

A first observation is that there is an $\mathfrak{sl}(2)$ Poisson coalgebra symmetry of the Hamiltonian (4) that immediately provides $2n - 2$ quadratic integrals. This underlying coalgebra structure is shared by all the other known examples of quadratically MS models [3,4], as well as by uncountably many other integrable Hamiltonian systems which are not MS. Therefore, the crucial result in this section is the derivation of an additional first integral, functionally independent of the previous $2n - 2$.

The $\mathfrak{sl}(2)$ Poisson coalgebra is defined by the basis $\{J_\epsilon : \epsilon = \pm, 0\}$ together with the following Lie–Poisson brackets, Casimir C and primitive coproduct Δ:

$$\{J_0, J_+\} = 2J_+, \qquad \{J_0, J_-\} = -2J_-, \qquad \{J_-, J_+\} = 4J_0, \tag{5}$$

$$C = J_- J_+ - J_0^2, \qquad \Delta(J_\epsilon) = J_\epsilon \otimes 1 + 1 \otimes J_\epsilon. \tag{6}$$

A symplectic (one-dimensional) realization of the algebra (5) on $(\mathbb{R}^2, \mathrm{d}q_1 \wedge \mathrm{d}p_1)$ is given by

$$J_- = q_1^2, \qquad J_0 = q_1 p_1, \qquad J_+ = p_1^2 + b_1 q_1^{-2},$$

where b_1 is a real parameter labeling the representation which corresponds to the value of the (one-dimensional) Casimir. Under this realization, the Lie-Poisson product of the $\mathfrak{sl}(2)$ algebra is recovered by computing the corresponding Poisson bracket in $(\mathbb{R}^2, \mathrm{d}q_1 \wedge \mathrm{d}p_1)$. Moreover, its n-th coproduct yields a symplectic realization in $\mathbb{R}^{2n}$ via

$$J_- = \mathbf{q}^2, \qquad J_0 = \mathbf{p} \cdot \mathbf{q}, \qquad J_+ = \mathbf{p}^2 + \sum_j b_j q_j^{-2}, \tag{7}$$

with $b_j \in \mathbb{R}$. Again, the Lie-Poisson product is given by the standard Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial \mathbf{q}} \frac{\partial g}{\partial \mathbf{p}} - \frac{\partial f}{\partial \mathbf{p}} \frac{\partial g}{\partial \mathbf{q}}$$

and the Casimir (6) is simply

$$C = \mathbf{L}^2 + \sum_j \frac{b_j \mathbf{q}^2}{q_j^2},$$

where $\mathbf{L}^2$ denotes the angular momentum. From this expression it is apparent that C is a homogenous function of degree 0.

This $\mathfrak{sl}(2) \otimes \cdots \otimes \mathfrak{sl}(2)$ symmetry endows any n-dimensional Hamiltonian in the enveloping algebra with $2n - 3$ integrals other than the Hamiltonian given by the left and right partial Casimirs. More precisely we have the following result, which we quote from Ref. [3].

Theorem 1. *Any Hamiltonian $H_0(\mathbf{p}, \mathbf{q}) = h(J_+, J_0, J_-)$ possesses $2n - 3$ first integrals given by the left and right partial Casimirs*

$$C^{(m)} = \sum_{1 \leq i < j \leq m} \left[(q_i p_j - q_j p_i)^2 + \frac{b_i q_j^2}{q_i^2} + \frac{b_j q_i^2}{q_j^2} \right] + \sum_{i=1}^{m} b_i \,,$$

$$C_{(m)} = \sum_{n-m < i < j \leq n} \left[(q_i p_j - q_j p_i)^2 + \frac{b_i q_j^2}{q_i^2} + \frac{b_j q_i^2}{q_j^2} \right] + \sum_{i=n-m+1}^{n} b_i \,,$$

where $1 < m \leq n$. Here $C^{(n)} = C_{(n)} = C$ is the Casimir (6) and the functions

$$\left\{ H_0, C^{(l)}, C_{(m)} : 1 < l < n, 1 < m \leq n \right\}$$

are functionally independent. Moreover, the subsets $\{ H_0, C^{(m)} : 1 < m \leq n \}$ and $\{ H_0, C_{(m)} : 1 < m \leq n \}$ are in involution.

The previous result obviously applies to the Hamiltonian (4), which corresponds to

$$H = \frac{J_+ - c}{2(\kappa + J_-)} \,.$$

The main result of this paper is that the remaining first integral, which makes H MS, can indeed be found easily.

Theorem 2. *The remaining first integral can be chosen as*

$$I_i(\mathbf{p}, \mathbf{q}) = p_i^2 - 2H(\mathbf{p}, \mathbf{q})\, q_i^2 + b_i q_i^{-2} \,, \tag{8}$$

for any $1 \leq i \leq n$. Moreover, the set $\{ I_i : 1 \leq i \leq n \}$ is also in involution.

Proof. The equations of motion under H are given by

$$\dot{q}_i = \frac{p_i}{\kappa + \mathbf{q}^2} \,, \tag{9}$$

$$\dot{p}_i = \frac{2Hq_i + b_i q_i^{-3}}{\kappa + \mathbf{q}^2} \,. \tag{10}$$

Combining both equations one can write

$$(\kappa + \mathbf{q}^2)\ddot{q}_i + 2(\mathbf{q} \cdot \dot{\mathbf{q}})\dot{q}_i - \frac{2Hq_i + b_i q_i^{-3}}{(\kappa + \mathbf{q}^2)} = 0 \,.$$

Multiplying this equation by $2(\kappa + \mathbf{q}^2)\dot{q}_i$ one immediately finds

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(\kappa + \mathbf{q}^2)^2 \dot{q}_i^2 - 2Hq_i^2 + b_i q_i^{-2} \right] = 0 \,,$$

4

which yields the desired result. The functional independence of I_i is easily established through a tedious but straightforward computation. The fact that $\{I_i\}$ are in involution follows from

$$
\begin{aligned}
\{I_i, I_j\} &= -2q_i^2\left(\{H, p_j^2\} - 2H\{H, q_j^2\} + b_j\{H, q_j^{-2}\}\right) \\
&\qquad + 2q_j^2\left(\{H, p_i^2\} - 2H\{H, q_i^2\} + b_i\{H, q_i^{-2}\}\right) \\
&= -2q_i^2\{H, I_j\} + 2q_j^2\{H, I_i\} = 0 \,.
\end{aligned}
$$

$\square$

Quadratic integrability is linked to the separability of the HJ equation by a theorem due to Kalnins and Miller [27]. In fact, it is not difficult to show that this equation is actually superseparable. If we write $S(t, \mathbf{q}) = W(\mathbf{q}) - \frac{1}{2}Et$ the HJ equation for H reads

$$
\left(\frac{\partial W}{\partial \mathbf{q}}\right)^2 - E\mathbf{q}^2 + \sum_j b_j q_j^{-2} = c + \kappa E \,.
$$

Thus one essentially recovers the HJ equation for the SW system with a different set of constants, namely

$$
H_{\mathrm{SW}}(\mathbf{p}, \mathbf{q}) = \mathbf{p}^2 - E\mathbf{q}^2 + \sum_j b_j q_j^{-2} \,.
$$

Therefore,

Proposition 3. *H separates in the same coordinate systems as the SW system.*

In three dimensions, e.g., it separates in 8 out of the 11 possible orthogonal coordinate systems [16, 23]. As a matter of fact, it has long been conjectured that any MS system should separate in multiple coordinate systems, but to our best knowledge this general claim has not been proved or disproved.

3   Integration of the equations of motion

In this section we shall compute the trajectories of the Hamiltonian system (4) in closed form. It is convenient to start by exploiting the $\mathfrak{sl}(2)$ symmetry to compute the evolution of the radial variable

$$
x = \kappa + J_- = \kappa + \mathbf{q}^2 \,.
$$

We use the notation

$$
E := 2H = \frac{J_+ - c}{x}
$$

and assume $c \neq 0$.

The evolution of the $\mathfrak{sl}(2)$ generators (7) is given by the following set of equations:

$$
\dot{x} = \{J_-, H\} = \frac{2J_0}{x} \,, \tag{11a}
$$

$$
\dot{J}_0 = \{J_0, H\} = \frac{J_+ + E(x - \kappa)}{x} \,, \tag{11b}
$$

$$
\dot{J}_+ = \{J_+, H\} = \frac{2EJ_0}{x} \,. \tag{11c}
$$

Using these equations one can write the Casimir (6) as

$$C = -\tfrac{1}{4}x^2\dot{x}^2 + Ex^2 + (c - E\kappa)x - c\kappa \geq 0\,.$$

For the sake of concreteness, let us assume that $E > 0$ and set

$$\alpha := \frac{1}{2}\left(\kappa - \frac{c}{E}\right)\,, \tag{12}$$

$$\gamma^2 := \frac{1}{4}\left(\kappa + \frac{c}{E}\right)^2 + \frac{C}{E}\,. \tag{13}$$

In this case

$$x^2\dot{x}^2 = 4E\big[(x - \alpha)^2 - \gamma^2\big]\,, \tag{14}$$

and one can easily integrate this equation as

$$\pm 2\sqrt{E}(t - \tau) = \sqrt{(x - \alpha)^2 - \gamma^2} + \alpha\,\cosh^{-1}\left(\frac{x - \alpha}{\gamma}\right)\,. \tag{15}$$

Here τ is an arbitrary constant and γ is the positive square root of γ^2. Note that the equation (14) for the variable x (the squared radius) coincides with the radial equation in the Kepler problem.

The above expression yields t as a monotonic function of x, so this relation is globally invertible (in each half-orbit). However, it is not possible to obtain the inverse function $x(t)$ in closed form. Hence we prefer to parametrize the trajectory by the radial variable x. In terms of this variable, the first integral (8) reads

$$I_i = 4E\big[(x - \alpha)^2 - \gamma^2\big]\left(\frac{\mathrm{d}q_i}{\mathrm{d}x}\right)^2 - Eq_i^2 + b_iq_i^{-2}\,. \tag{16}$$

Let us set

$$Q_i = q_i^2\,, \qquad \alpha_i = -\frac{I_i}{2E}\,, \qquad \gamma_i^2 = \alpha_i^2 + E^{-1}b_i\,, \tag{17}$$

so that

$$Eq_i^4 + I_iq_i^2 - b_i = E\big[(Q_i - \alpha_i)^2 - \gamma_i^2\big]\,.$$

Then one can perform the integration of Eq. (16) to obtain

$$\int \frac{\mathrm{d}Q_i}{\sqrt{(Q_i - \alpha_i)^2 - \gamma_i^2}} = \varphi_i + \int \frac{\mathrm{d}x}{\sqrt{(x - \alpha)^2 - \gamma^2}}\,,$$

i.e.,

$$\cosh^{-1}\left(\frac{Q_i - \alpha_i}{\gamma_i}\right) = \varphi_i + \cosh^{-1}\left(\frac{x - \alpha}{\gamma}\right)\,,$$

where φ_i are constants and $\gamma_i \geq 0$. The full solution of this equation is therefore given by

$$Q_i = \alpha_i + \gamma_i\,\cosh(X + \varphi_i)$$

$$= \alpha_i + \gamma^{-1}\gamma_i\,\cosh\varphi_i\,(x - \alpha) + \gamma_i\,\sinh\varphi_i\left|1 - \left(\frac{x - \alpha}{\gamma}\right)^2\right|^{1/2}\,, \tag{18}$$

with $X := \cosh^{-1}(\gamma^{-1}(x - \alpha))$. Eqs. (15) and (18) are the main results of this section.

6

All the orbits with positive energy are recovered through an appropriate choice of the $2n$ parameters

$$\left\{\tau, \alpha_i, \varphi_j : 1 \le i \le n-1, \, 1 \le j \le n\right\}, \tag{19}$$

where τ was defined in Eq. (15). In what follows we shall express the remaining quantities in terms of this fundamental set.

If we sum over i in Eq. (18) and use that $x = \kappa + \sum_i Q_i$, we immediately find

$$x - \kappa = \sum_i \alpha_i + \gamma^{-1}(x - \alpha) \sum_i \gamma_i \cosh \varphi_i + \left|1 - \left(\frac{x - \alpha}{\gamma}\right)^2\right|^{1/2} \sum_i \gamma_i \sinh \varphi_i\,.$$

Therefore we reach the following compatibility conditions:

$$\sum_i \alpha_i + \kappa = \alpha\,, \tag{20a}$$

$$\sum_i \gamma_i \cosh \varphi_i = \gamma\,, \tag{20b}$$

$$\sum_i \gamma_i \sinh \varphi_i = 0\,. \tag{20c}$$

Eq. (20a) gives the value of α in terms of $\{\alpha_j\}$ and Eq. (20b) consistently provides the value of the Casimir by means Eq. (13). Eq. (20c), which imposes the constraint

$$\sum_i \sinh \varphi_i \left(\alpha_i^2 + E^{-1} b_i\right)^{1/2} = 0$$

on the the parameters $\{\alpha_i, \varphi_j\}$ by virtue of Eq. (17), is responsible for the fact that only $2n - 1$ among these parameters have been included in the set (19). Note that Eqs. (12) and (20a) can be combined to express E as a function of $\{\alpha_i\}$ as

$$E = -\frac{c}{\kappa + 2\sum_i \alpha_i}\,.$$

Finally, this permits to express γ_i in terms of $\{\alpha_j\}$ by means of Eq. (17). We omit the discussion of the case $E \le 0$, which goes along the same lines.

4 The geometric content of $\mathcal{H}$

It is apparent that $\mathcal{H}$ is a (nonconstant) multiple of the SW system. What we want to stress in this section is that their connection does not end here: $\mathcal{H}$ is in fact the *intrinsic* SW Hamiltonian in the manifold $\mathcal{M}^n$. And what makes this space remarkable is that its intrinsic SW model is also MS.

It is apparent from Eq. (3) that the Riemannian manifold $\mathcal{M}^n$ is spherically symmetric, this $SO(n)$ symmetry being a consequence of the $\mathfrak{sl}(2)$ coalgebra structure outlined in Sec. 2. Its curvature is certainly nonconstant; for the sake of completeness we note that its scalar curvature is negative and given by

$$R = -(n-1)\frac{3(n-2)\mathbf{q}^2 + 2\kappa n}{(\kappa + \mathbf{q}^2)^3}\,.$$

One can define intrinsic versions of the Kepler and harmonic oscillator potentials on $\mathcal{M}^n$. To this end, denote by $\Delta_{\mathcal{M}^3}$ the Laplace–Beltrami operator on $\mathcal{M}^3$ and let $V_3 \in C^\infty(\mathcal{M}^3 \backslash \{\mathbf{0}\})$ be its (minimal symmetric) Green function, i.e., a function $V_3(\mathbf{q}) = v(|\mathbf{q}|)$ satisfying

$$-\Delta_{\mathcal{M}^3} V_3 = \delta_{\mathcal{M}^3}\,.$$

Here $v \in C^\infty(\mathbb{R}^+ \backslash \{0\})$ and $\delta_{\mathcal{M}^3}$ stands for the delta distribution in $\mathcal{M}^3$ supported at the origin. It can be proved [13] that such a function exists and is unique.

Definition 4. The *Kepler and harmonic oscillator potentials* in $\mathcal{M}^n$ are

$$V_{\text{Kepler}}(\mathbf{q}) := K\, v(|\mathbf{q}|)\,, \qquad V_{\text{Harm}}(\mathbf{q}) := K\, v(|\mathbf{q}|)^{-2}\,,$$

where $K \in \mathbb{R}$ is an arbitrary constant.

Remark 5. The above definition, which is based on the potential theory of these spaces, can be easily extended to any spherically symmetric Riemannian space with the appropriate behavior at infinity [30]. Such a prescription reproduces the intrinsic Kepler and harmonic potentials on constant curvature spaces (see [12] and references therein). The case of U(2)-symmetric Kähler 4-manifolds, which had already appeared in the literature [33], is based on the same ideas but does not fit into this framework. Certainly the above prescription does not generally lead to MS potentials.

Let $\mathbf{q} \in \mathcal{M}^3$ and define $r = |\mathbf{q}|$. It can be readily verified that the action of the Laplace–Beltrami operator in $\mathcal{M}^3$ on some function $f(r)$ is

$$\Delta_{\mathcal{M}^3} f = \frac{1}{r^2(\kappa + r^2)} \frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \sqrt{\kappa + r^2}\, \frac{\mathrm{d}f}{\mathrm{d}r}\right)\,.$$

By setting the above expression equal to zero for $r > 0$ it is straightforward to find that the function v defined above is

$$v(r) = \frac{\sqrt{\kappa + r^2}}{r}$$

up to a multiplicative constant. Hence the Kepler and harmonic oscillator potentials in $\mathcal{M}^n$ are respectively given by

$$V_{\text{Kepler}}(\mathbf{q}) = K\frac{\sqrt{\kappa + \mathbf{q}^2}}{|\mathbf{q}|}\,, \qquad V_{\text{Harm}}(\mathbf{q}) = \frac{K\,\mathbf{q}^2}{\kappa + \mathbf{q}^2}\,.$$

This shows that the MS Hamiltonian (1) is in fact the intrinsic SW systems, i.e.,

$$\mathcal{H}(\mathbf{p}, \mathbf{q}) = \|\mathbf{p}\|_{\mathcal{M}^n}^2 + V_{\text{Harm}}(\mathbf{q}) + (\kappa + \mathbf{q}^2)^{-1}\sum_j b_j q_j^{-2}\,,$$

with $\|\cdot\|_{\mathcal{M}^n}$ representing the metric on the cotangent bundle of $\mathcal{M}^n$.

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