Twist deformations for Yangians

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Abstract

It is demonstrated how chains of twists for classical Lie algebras induce the new twist deformations (the deformed Yangians) that quantize the generalized rational solutions of the classical Yang-Baxter equation. For the case of ${Y(g)|g = so(2N + 1)}$ the explicit expression for the corresponding $R_{\mathcal{Y}}$ -matrix in the defining representation is given.

1 Introduction

Yangians $Y(q)$ were introduced by V.G.Drinfeld [[2\]](#page-7-0) as the quantizations of rational solutions of the classical Yang-Baxter equation (CYBE). It was demonstrated by A.Stolin[[3\]](#page-7-0) that rational solutions with values in the tensor square of the Lie algebra g space can be written in the form

$$
\frac{C^2}{u-v} + a_0 + b_1u + b_2v + cuv
$$

with $a_0, b_i, c \in g$ and C^2 – the second Casimir of the algebra g. Notice that this does not necessarily signify the simplicity of g. The existence of Yangians for nonsemisimple Lie algebras was predicted in[[4\]](#page-7-0) and explicitly demonstrated in $|5|$.

The important class of rational solutions is of the form

$$
\frac{C^2}{u-v} + r_0,
$$

where the additional summand is a constant solution of the CYBE. As it was proved in [\[6](#page-7-0)] the quantization of such solutions leads to the twist deformations $Y_{\mathcal{F}}(g)$ of the Yangians $Y(g)$ (the latter refer to the canonical rational solutions \check{C}^2 $\frac{C^2}{u-v}$. The deformed Yangian $Y_{\mathcal{F}}(g)$ has the same multiplication as in $Y(g)$, the twisted coproduct

$$
\Delta_{\mathcal{F}}(y) = \mathcal{F}\Delta(y)\,\mathcal{F}^{-1},
$$

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and the transformed R-matrix

$$
\mathcal{R}_{\mathcal{F}} = (\mathcal{F})_{21} \, \mathcal{R} \mathcal{F}^{-1}.
$$

The twisting element $\mathcal F$ has to satisfy the equations [\[1\]](#page-7-0):

$$
\mathcal{F}_{12}(\Delta \otimes id)(\mathcal{F}) = \mathcal{F}_{23}(id \otimes \Delta)(\mathcal{F}),
$$

\n
$$
(\epsilon \otimes id)(\mathcal{F}) = (id \otimes \epsilon)(\mathcal{F}) = 1.
$$
\n(1)

When A and B are the universal enveloping algebras: $A = U(g) \supset B =$ $U(1)$ with $g \supset I$ and $U(1)$ is the minimal subalgebra on which F is completely defined as $\mathcal{F} \in U(1) \otimes U(1)$ then l is called the **carrier** algebra for \mathcal{F} .

The existence of a twist can be formulated in terms of a nondegenerate bilinear form on the carrier algebra. The generators dual to the PBW basic elements of $U(g)$ are very important. They provide the explicit presentation of the twisting element[[9\]](#page-7-0).

The two known examples of twists that were written explicitly correspond to the two-dimensional carrier subalgebra $B(2)$ with the generators H and E,

$$
[H,E]=E,
$$

and the four-dimensional carrier subalgebra L:

$$
[H, E] = E, [H, A] = \alpha A, [H, B] = \beta B,
$$

$$
[A, B] = E, [E, A] = [E, B] = 0, \alpha + \beta = 1.
$$
 (2)

The first one is called the **Jordanian twist** $|7|$ and has the twisting element

$$
\Phi_{\mathcal{J}} = e^{H \otimes \sigma}, \qquad \sigma = \ln(1 + E). \tag{3}
$$

The second one is the extended Jordanian twist suggested in[[9\]](#page-7-0) . The corresponding twisting element

$$
\mathcal{F}_{\mathcal{E}(\alpha,\beta)} = \Phi_{\mathcal{E}(\alpha,\beta)} \Phi_{\mathcal{J}} \tag{4}
$$

contains the Jordanian factor $\Phi_{\mathcal{J}}$ and the extension

$$
\Phi_{\mathcal{E}(\alpha,\beta)} = \exp\{A \otimes B e^{-\beta \sigma}\}.
$$
\n(5)

In general the composition of two twists is not a twist. But there are some important examples of the opposite behaviour. When L is a subalgebra **L** ⊂g there may exist several pairs of generators of the type (A, B) arranged so that the Jordanian twist can acquire several similar extensions[[9](#page-7-0)]. This demonstrates that some twistings can be applied successively to the initial Hopf algebra even in the case when their carrier subalgebras are nontrivially linked. In the universal enveloping algebras for classical Lie algebras there exists the possibility to construct systematically the special sequences of twists called chains [[10\]](#page-7-0):

$$
\mathcal{F}_{\mathcal{B}_{p\prec 0}} \equiv \mathcal{F}_{\mathcal{B}_p} \mathcal{F}_{\mathcal{B}_{p-1}} \dots \mathcal{F}_{\mathcal{B}_0}.
$$
 (6)

The factors $\mathcal{F}_{\mathcal{B}_k} = \Phi_{\mathcal{E}_k} \Phi_{\mathcal{J}_k}$ of the chain are the twisting elements of the extended Jordanian twists for the initial Hopf algebra A_0 . Here the extensions $\{\Phi_{\mathcal{E}_k}, k=0,\ldots p-1\}$ contain the fixed set of normalized factors $\Phi_{\mathcal{E}(\alpha,\beta)}=$ $\exp\{A\otimes Be^{-\beta\sigma}\}\,$, the full set. It was proved that in the classical Lie algebras that conserve symmetric invariant forms such chains can be made maximal and proper. This means that for the algebras $U(A_N)$, $U(B_N)$ and $U(D_N)$ of the three classical series there exist chains $\mathcal{F}_{\mathcal{B}_{p\prec 0}}$ that cannot be reduced to a chain for a simple subalgebra and their full sets of extensions are the maximal sets in the sense described below.

To construct a maximal proper chain for $\mathcal{A} = U(g)$ (where g is a classical Lie algebra with the root system $\Lambda_{\mathcal{A}}$) the sequences $\mathcal{A} \equiv \mathcal{A}_0 \supset \mathcal{A}_1 \supset \ldots \supset$ $\mathcal{A}_{p-1} \supset \mathcal{A}_p$ of Hopf subalgebras are to be fixed in \mathcal{A} . For each element \mathcal{A}_k of the sequence there must exist the so called **initial root** λ_0^k and the set π_k of its constituent roots,

$$
\pi_k = \left\{ \lambda', \lambda'' | \lambda' + \lambda'' = \lambda_0^k; \quad \lambda' + \lambda_0^k, \lambda'' + \lambda_0^k \right\} \tag{7}
$$

For any $\lambda' \in \pi_k$ there must be an element $\lambda'' \in \pi_k$ that $\lambda' + \lambda'' = \lambda_0^k$. So, π_k is naturally decomposed as

$$
\pi_k = \pi'_k \cup \pi''_k, \qquad \pi'_k = \{\lambda'\}, \quad \pi''_k = \{\lambda''\}. \tag{8}
$$

The triples $(\mathcal{A}_k, \lambda_0^k, \pi_k)$ are subject to the following conditions:

- 1. λ_0^k must be orthogonal to the roots of the subalgebra A_{k-1} ,
- 2. the subsets π'_k and π''_k must form the diagrams of conjugate representations for A_{k-1} .

In these terms the factors $\mathcal{F}_{\mathcal{B}_k}$ of the chain (6) are fixed as follows:

$$
\mathcal{F}_{\mathcal{B}_k} = \Phi_{\mathcal{E}_k} \Phi_{\mathcal{J}_k} \tag{9}
$$

with

$$
\Phi_{\mathcal{J}_k} = \exp\{H_{\lambda_0^k} \otimes \sigma_0^k\}, \qquad \sigma_0^k = \ln(1 + L_{\lambda_0^k}); \tag{10}
$$

$$
\Phi_{\mathcal{E}_k} = \prod_{\lambda' \in \pi'_k} \Phi_{\mathcal{E}_{\lambda'}} = \prod_{\lambda' \in \pi'_k} \exp\{L_{\lambda'} \otimes L_{\lambda_0^k - \lambda'} e^{-\frac{1}{2}\sigma_0^k}\} \tag{11}
$$

(here L_{λ} is the generator associated with the root λ).

Quantizations $\mathcal{A}_{\mathcal{F}_{\mathcal{B}_{p\prec 0}}}$ of classical universal enveloping algebras produce the chains of $\mathcal{R}_\mathcal{F}$ -matrices:

$$
\mathcal{R}_{\mathcal{B}_{p\prec 0}} = (\mathcal{F}_{\mathcal{B}_p})_{21} (\mathcal{F}_{\mathcal{B}_{p-1}})_{21} \dots (\mathcal{F}_{\mathcal{B}_0})_{21} \mathcal{F}_{\mathcal{B}_0}^{-1} \dots \mathcal{F}_{\mathcal{B}_{p-1}}^{-1} \mathcal{F}_{\mathcal{B}_p}^{-1}.
$$
 (12)

The deformation parameters can be introduced in chains by rescaling the generators in the subalgebras \mathcal{B}_k . Each \mathcal{B}_k can be rescaled separately with an independent variable ξ_k . When all these scaling factors are proportional to the deformation parameter ξ , i.e. $\xi_k = \xi \eta_k$, then in the classical limit the parameters η_k appear as the multipliers in the classical r-matrix:

$$
r_{\mathcal{B}_{p\prec 0}} = \sum_{k=0,1,\dots,p} \eta_k \left(H_{\lambda_0^k} \wedge L_{\lambda_0^k} + \sum_{\lambda' \in \pi_k} L_{\lambda'} \wedge L_{\lambda_0^k - \lambda'} \right). \tag{13}
$$

To obtain the necessary background for the integrable models with deformed Yangian symmetry the simplest way is to construct the defining representation of the universal $\mathcal{R}_{\mathcal{Y}}$ -matrix, $R = d(\mathcal{R}_{\mathcal{Y}})$. In some special situations the quantum R_F -matrix corresponding to the twisted algebra can be obtained directly from its classical counterpart r_F [[6\]](#page-7-0) (in particular this happens when $R = \exp(r)$ $R = \exp(r)$ $R = \exp(r)$ [[8\]](#page-7-0)). Such simplifications are unavailable when the r_F -matrices of the type (13) are considered for the orthogonal classical Lie algebras g. In such cases more information about the $\mathcal{R}_{\mathcal{F}}$ -matrix is necessary.

In this report it will be shown how the chains of extended twists([6\)](#page-2-0) lead to the series of deformed Yangians. To illustrate the situation the explicit formulas will be presented for the case of $g = so(2N + 1)$.

2 Chains and twisted R-matrices

Let g be a classical Lie algebra of the type B_N or D_N and $\mathcal{F}_{\mathcal{B}_{p\prec 0}} \equiv \mathcal{F}_{\mathcal{B}_p} \mathcal{F}_{\mathcal{B}_{p-1}} \ldots$ \ldots $\mathcal{F}_{\mathcal{B}_0}$ – its full proper chain of extended twists.

To construct a maximal proper chain for $\mathcal{A} = U(g)$ the following sequences $\mathcal{A} \equiv \mathcal{A}_0 \supset \mathcal{A}_1 \supset \ldots \supset \mathcal{A}_{p-1} \supset \mathcal{A}_p$ of Hopf subalgebras are to be fixed:

$$
U(so(2N)) \supset U(so(2(N-2)) \supset \ldots \supset U(so(2(N-2k)) \supset \ldots \text{ for } D_N \text{ (14)}
$$

$$
U(so(2N+1)) \supset U(so(2(N-2)+1) \supset \dots \supset U(so(2(N-2k)+1) \supset \dots \text{ for } B_N
$$
\n(15)

In both cases the initial roots for A_k can be chosen to be $\lambda_0^k = e_1 + e_2$ (here the root subsystems are considered for A_k separately and all the roots are written in the standard e-basis).

The twisting element for a chain can be written explicitly as

$$
\mathcal{F}_{\mathcal{B}_{p\prec 0}} = \Pi_{\lambda' \in \pi'_p} \left(\exp\{L_{\lambda'} \otimes L_{\lambda_0^p - \lambda'} e^{-\frac{1}{2}\sigma_0^p} \} \right) \cdot \exp\{H_{\lambda_0^p} \otimes \sigma_0^p \} \cdot \Pi_{\lambda' \in \pi'_{p-1}} \left(\exp\{L_{\lambda'} \otimes L_{\lambda_0^{p-1} - \lambda'} e^{-\frac{1}{2}\sigma_0^{p-1}} \} \right) \cdot \exp\{H_{\lambda_0^{p-1}} \otimes \sigma_0^{p-1} \} \cdot \Pi_{\lambda' \in \pi'_0} \left(\exp\{L_{\lambda'} \otimes L_{\lambda_0^0 - \lambda'} e^{-\frac{1}{2}\sigma_0^0} \} \right) \cdot \exp\{H_{\lambda_0^0} \otimes \sigma_0^0 \}
$$
\n
$$
(16)
$$

Let us introduce the deformation parameters $\xi_k = \xi \eta_k$ and rescale the generators $\left\{L_{\lambda_0^k}, L_{\lambda_0^k-\lambda'}\right\}$ by ξ_k in each subalgebra \mathcal{A}_k . Using the expressions ([10\)](#page-3-0) and [\(11\)](#page-3-0) one can get the first terms of the expansion for the twisting element

$$
\mathcal{F}_{\mathcal{B}_{p\prec 0}}(\xi)=I\otimes I+\xi\rho_{\mathcal{B}}+\mathcal{O}\left(\xi^2\right).
$$

Here

$$
\rho_{\mathcal{B}} = \sum_{k=0,1,\dots,p} \eta_k \left(H_{\lambda_0^k} \otimes L_{\lambda_0^k} + \sum_{\lambda' \in \pi_k} L_{\lambda'} \otimes L_{\lambda_0^k - \lambda'} \right) \tag{17}
$$

is "a half" of the $r_{\mathcal{B}_{p\prec 0}}$ -matrix:

$$
r_{\mathcal{B}_{p\prec 0}} = \rho_{\mathcal{B}} - \tau \circ \rho_{\mathcal{B}}.
$$

The carrier subalgebras for chains of extended twists are solvable. As it was already mentioned in Section 1 the generators of the carrier form dual sets. In the case of extended twists one of these sets $\mathcal{L} = \left\{ L_{\lambda_0^k}, L_{\lambda_0^k - \lambda'} | \lambda' \in \pi'_k \right\}$ forms a nilpotent subalgebra. In the defining representation $d(q)$ the corresponding matrixring is nilpotent with the index $\kappa = 3$. As it was demonstrated in [[9\]](#page-7-0) the twisting element can always be presented in the form $\mathcal{F} = \exp(\mathcal{L}_i^* \otimes \psi^i(\mathcal{L}))$ and the expansions for the elements $\psi^i(\mathcal{L})$ starts with the term $\xi \mathcal{L}^i$. Applying these results to the case of chains we get the following property.

Lemma. Let g be a Lie algebra of the type A_N , B_N or D_N and $d(g)$ – the defining representation of g . Then for the full proper chain of extended twists $\mathcal{F}_{\mathcal{B}_{p\prec 0}}(\xi) = I \otimes I + \xi \rho_{\mathcal{B}} + \mathcal{O}(\xi^2)$ the following relations are true:

- 1. $d(\rho_{\mathcal{B}}^3) = 0,$
- 2. $d\left(\mathcal{F}_{\mathcal{B}_{k}}\left(\xi\right)\right)=\exp\left(\xi d\left(\rho_{\mathcal{B}_{k}}\right)\right),$

3.
$$
R_{\mathcal{B}_{p\prec 0}}(\xi) = d\left(\mathcal{R}_{\mathcal{B}_{p\prec 0}}(\xi)\right) = I \otimes I - \xi d\left(\rho_{\mathcal{B}}\right) + \frac{\xi^2}{2} \left(d^2\left(\rho_{\mathcal{B}}\right) + d^2\left(\tau \circ \rho_{\mathcal{B}}\right)\right) - \xi^2 d\left(\tau \circ \rho_{\mathcal{B}}\right) d\left(\rho_{\mathcal{B}}\right).
$$

The construction of the R-matrix differs for linear and orthogonal algebras. The most interesting is the case $g = so(M)$, i.e. the series B_N or D_N . Let us fix g to be of B_N type, $g = so(2N+1)$. Thus C^{2N+1} is the space of the defining representation. Let P be the permutation matrix acting in $C^{2N+1} \otimes C^{2N+1}$

$$
P = M' \otimes M'' \in \text{Mat}(2N + 1, \mathsf{C})^{\otimes 2}
$$

Define also the matrix

$$
K = (M')^{\mathsf{T}} \otimes M''.
$$

Itwas shown in [[6\]](#page-7-0) that if F is the twisting element for $U(so(2N+1))$ with the twisted R-matrix $R_{\mathcal{F}}$ the corresponding rational solution of the quantum Yang-Baxter equation in the defining representation has the form

$$
R = d\left(\mathcal{R}_{\mathcal{Y}}\right) = uR_{\mathcal{F}} + P - \frac{u}{u + N - 1/2}d\left(\mathcal{F}_{21}\right)Kd\left(\mathcal{F}^{-1}\right). \tag{18}
$$

Applying the Lemma proved above we get the following final expression for R.

$$
R = u \left(I \otimes I - \xi d \left(\rho_{\mathcal{B}} \right) + \frac{\xi^2}{2} d \left(\rho_{\mathcal{B}}^2 + \left(\tau \circ \rho_{\mathcal{B}} \right)^2 - 2 \left(\tau \circ \rho_{\mathcal{B}} \right) \rho_{\mathcal{B}} \right) \right) + P -
$$

$$
-\frac{u}{u + N - 1/2} d \left(I \otimes I + \tau \circ \rho_{\mathcal{B}} + \frac{1}{2} \left(\tau \circ \rho_{\mathcal{B}} \right)^2 \right) K d \left(-I \otimes I - \rho_{\mathcal{B}} - \frac{1}{2} \left(\rho_{\mathcal{B}} \right)^2 \right).
$$
(19)

This R-matrix describes the Yangians $Y_{\mathcal{F}_{\mathcal{B}_{p\prec 0}}}(so(2N+1))$ deformed by the full chains of extended twists. To get the final answer the only term that must be calculated is the defining representation for the ρ_B -matrix [\(17\)](#page-4-0).

3 Example. QYBE solutions for $\text{so}(2N+1)$

Let $g = so(2N+1)$. To make the illustration maximally visual we shall consider the simplest nontrivial chain, i. e. put $p = 0$ and consider a chain with a single factor $\mathcal{F}_{\mathcal{B}_0}$ in [\(6](#page-2-0)). The algorithm for other factors $\mathcal{F}_{\mathcal{B}_k}$ is similar to that of the first one and the case of a full chain can be reconstructed using the formulas ([9-](#page-2-0)[11\)](#page-3-0) and the expressions presented below.

The matrices of the defining representation $d(g)$ are written in terms of basic antisymmetric Okubo matrices M_{ij} :

$$
d\left(H_{e_i+e_j}\right) = -i\left(M_{2i-1,2i} + M_{2j-1,2j}\right) \equiv H_{i+j},
$$

$$
d(L_{e_k}) = M_{2k,2N+1} - iM_{2k-1,2N+1} \equiv E_k, \qquad k = 1, ..., N
$$

$$
d(L_{e_i \pm e_j}) = \frac{1}{2} (-M_{2i,2j} \pm iM_{2i,2j-1} + iM_{2i-1,2j} \pm M_{2i-1,2j-1})
$$

$$
\equiv E_{i+j}, \qquad i < j
$$

In this representation the factors of the sequence

$$
d\left(\mathcal{F}_{\mathcal{B}_0}\left(\xi\right)\right) = d\left(\Phi_{\mathcal{E}_k}\left(\xi\right)\right) d\left(\Phi_{\mathcal{J}_0}\left(\xi\right)\right)
$$

have the following form:

$$
d\left(\Phi_{\mathcal{J}_0}(\xi)\right) = \exp\{\xi H_{1+2} \otimes E_{1+2}\},
$$

$$
d\left(\Phi_{\mathcal{E}_k}(\xi)\right) = \exp\left\{\xi \left(E_1 \otimes E_2 + 2\sum_{j>2}^N E_{1\pm j} \otimes E_{2\mp j}\right)\right\}.
$$

TheR-matrix can be easily obtained using the general formula (12) (12) (12) ,

$$
R_{\mathcal{B}_0} = I \otimes I - \xi \left(H_{1+2} \wedge E_{1+2} + E_1 \otimes E_2 + 2 \sum_{j>2}^N E_{1\pm j} \otimes E_{2\mp j} \right) +
$$

+ $\frac{1}{2} \xi^2 (E_1^2 \otimes E_2^2 + E_2^2 \otimes E_1^2 + 2E_{1+2} \otimes E_{1+2} - 2E_2 E_1 \otimes E_1 E_2) +$
+ $2 \xi^2 \sum_{j>2}^N (E_{1\pm j} E_{1\mp j} \otimes E_{2\mp j} E_{2\pm j} + E_{2\mp j} E_{2\pm j} \otimes E_{1\pm j} E_{1\mp j} -$
- $2E_{2\mp j} E_{1\pm j} \otimes E_{1\pm j} E_{2\mp j}$)

The corresponding matrix $d(\rho)$ is

$$
d(\rho_{\mathcal{B}_0}) = H_{1+2} \otimes E_{1+2} + E_1 \otimes E_2 + 2 \sum_{j>2}^N E_{1\pm j} \otimes E_{2\mp j}.
$$

This expression together with the general formula([19\)](#page-5-0) defines the final result – the set of $R_{\mathcal{Y}}$ -matrices for the deformed Yangians $Y_{\mathcal{B}_0}(so(2N+1)).$

4 Conclusions

We have demonstrated that for some types of simple Lie algebras the Yangians deformed by chains of extended twists are completely determined by the matrix $d(\rho)$ which is the logarithm of the twisting element in the defining representation of an algebra. This gives rise to a set R_y of solutions to the quantum Yang-Baxter equation and, therefore, to a series of integrable models with the corresponding local Hamiltonians.

When the chain has the index $p > 1$ the R_y -matrices naturally acquire the parameters η_k . Moreover, in chains of extended twists one can cut any number of factors from the left, i.e. use the discrete parametrization of the last nontrivial sequence $\mathcal{F}_{\mathcal{B}_p}$.

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