Minimal submanifolds of Kähler-Einstein manifolds with equal Kähler angles

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Abstract: We consider $F:M\to N$ a minimal oriented compact real 2n-submanifold M, immersed into a Kähler-Einstein manifold N of complex dimension 2n, and scalar curvature R. We assume that $n\geq 2$ and F has equal Kähler angles. Our main result is to prove that, if n=2 and $R\neq 0$, then F is either a complex submanifold or a Lagrangian submanifold. We also prove that, if $n\geq 3$ and F has no complex points, then : (A) If R<0, then F is Lagrangian; (B) If R=0, the Kähler angle must be constant. We also study pluriminimal submanifolds with equal Kähler angles, and prove that, if they are not complex submanifolds, N must be Ricci-flat and there is a natural parallel homothetic isomorphism between TM and the normal bundle.

Key Words: Minimal, pluriharmonic, Lagrangian submanifold, Kähler-Einstein manifold, Kähler angles

MSC 1991: 53A10, 53C42, 58E20, 53C55, 32C17, 53C15, 58F05

1 Introduction

Let (N, J, g) be a Kähler manifold of complex dimension 2n and $F: M \to N$ an immersed submanifold of real dimension 2n. We denote by ω the Kähler form of N, $\omega(X,Y) = g(JX,Y)$. On M we take the induced metric $g_M = F^*g$. N is Kähler-Einstein if its Ricci tensor is a multiple of the metric, $Ricci^N = Rg$. At each point $p \in M$, we identify $F^*\omega$ with a skew-symmetric operator of T_pM by using the musical isomorphism with respect to g_M , namely $g_M(F^*\omega(X),Y) = F^*\omega(X,Y)$. We take its polar decomposition

$$F^*\omega = \tilde{g}J_\omega \tag{1.1}$$

where $J_{\omega}: T_pM \to T_pM$ is a (in fact unique) partial isometry with the same kernel \mathcal{K}_{ω} as of F^*w , and where \tilde{g} is the positive semidefinite operator $\tilde{g} = |F^*\omega| = \sqrt{-(F^*\omega)^2}$. It turns

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out that $J_{\omega}: \mathcal{K}_{\omega}^{\perp} \to \mathcal{K}_{\omega}^{\perp}$ defines a complex structure on $\mathcal{K}_{\omega}^{\perp}$, the orthogonal compliment of \mathcal{K}_{ω} in T_pM . Moreover, it is g_M -orthogonal. If we denote by Ω_{2k}^0 the largest open set of M where $F^*\omega$ has constant rank 2k, $0 \leq k \leq n$, then $\mathcal{K}_{\omega}^{\perp}$ is a smooth sub-vector bundle of TM on Ω_{2k}^0 . Moreover, \tilde{g} and J_{ω} are both smooth on these open sets. The tensor \tilde{g} is continuous on all M and locally Lipschitz, for the map $P \to |P|$ is Lipschitz in the space of normal operators. Let $\{X_{\alpha}, Y_{\alpha}\}_{1 \leq \alpha \leq n}$ be a g_M -orthonormal basis of T_pM , that diagonalizes $F^*\omega$ at p, that is

$$F^*\omega = \bigoplus_{1 \le \alpha \le n} \begin{bmatrix} 0 & -\cos\theta_\alpha \\ \cos\theta_\alpha & 0 \end{bmatrix}, \tag{1.2}$$

where $\cos \theta_1 \ge \cos \theta_2 \ge ... \ge \cos \theta_n \ge 0$. The angles $\{\theta_\alpha\}_{1 \le \alpha \le n}$ are the Kähler angles of F at p. Thus, $\forall \alpha, F^*\omega(X_\alpha) = \cos\theta_\alpha Y_\alpha$, $F^*\omega(Y_\alpha) = -\cos\theta_\alpha X_\alpha$ and if $k \geq 1$, where 2kis the rank of $F^*\omega$ at p, $J_{\omega}X_{\alpha}=Y_{\alpha}$ $\forall \alpha \leq k$. The Weyl's perturbation theorem applied to the eigenvalues of the symmetric operator $|F^*\omega|$ shows that, ordering the $\cos\theta_{\alpha}$ in the above way, the map $p \to cos\theta_{\alpha}(p)$ is locally Lipschitz on M, for each α . A complex direction of F is a real two-plane P of T_pM such that dF(P) is a complex line of $T_{F(p)}N$, i.e., $JdF(P) \subset dF(P)$. Similarly, P is said to be a Lagrangian direction of F if ω vanishes on dF(P), that is, $JdF(P)\perp dF(P)$. The immersion F has no complex directions iff $\cos \theta_{\alpha} < 1 \ \forall \alpha$. M is a complex submanifold iff $\cos \theta_{\alpha} = 1 \ \forall \alpha$, and is a Lagrangian submanifold iff $\cos\theta_{\alpha} = 0 \ \forall \alpha$. We say that F has equal Kähler angles if $\theta_{\alpha} = \theta \ \forall \alpha$. Complex and Lagrangian submanifolds are examples of such case. If F is a complex submanifold, then J_{ω} is the complex structure induced by J of N. The Kähler angles are some functions that at each point p of M measure the deviation of the tangent plane T_pM of M from a complex or a Lagrangian subspace of $T_{F(p)}N$. This concept was introduced by Chern and Wolfson [Ch-W] for surfaces, namely $F^*\omega = \cos\theta Vol_M$. This $\cos\theta$ may have negative values and is smooth on all M. In our definition, for n=1, we demanded $\cos\theta \geq 0$, that is, it is the modulus of the $\cos \theta$ given for surfaces. This may make our $\cos \theta$ do not be smooth. We have chosen this definition, because in higher dimensions we do not have a preferential orientation assigned to the real planes $span\{X_{\alpha}, Y_{\alpha}\}.$

Our main aim is to find conditions for a minimal submanifold F to be Lagrangian or complex, or M to be a Kähler manifold with respect to J_{ω} . The first result in this direction is due to Wolfson, for the case n = 1:

Theorem 1.1 [W] If M is a real compact surface and N is a complex Kähler-Einstein surface with R < 0, and if F is minimal with no complex points, then F is Lagrangian.

Some results of [S-V] are a generalization of the above theorem to higher dimensions. In this paper we study the case of equal Kähler angles. Let us denote by $\nabla_X dF(Y) =$

 $\nabla dF(X,Y)$ the second fundamental form of F. It is a symmetric tensor and takes values in the normal bundle $NM = (dF(TM))^{\perp}$. F is minimal iff $trace_{g_M} \nabla dF = 0$. Let $(\)^{\perp}$ denote the orthogonal projection of $F^{-1}TN$ onto the normal bundle. If F is an immersion with no complex directions at p and $\{X_{\alpha}, Y_{\alpha}\}$ diagonalizes $F^*\omega$ at p, then $\{dF(Z_{\alpha}), dF(Z_{\bar{\alpha}}), (JdF(Z_{\alpha}))^{\perp}, (JdF(Z_{\bar{\alpha}}))^{\perp}\}$ constitutes a complex basis of $T_{F(p)}^c N$, where

$$Z_{\alpha} = \frac{X_{\alpha} - iY_{\alpha}}{2} = \alpha, \qquad Z_{\bar{\alpha}} = \overline{Z_{\alpha}} = \frac{X_{\alpha} + iY_{\alpha}}{2} = \bar{\alpha}.$$
 (1.3)

are complex vectors of the complexified tangent space of M at p. We extend to the complexified vector bundles the Riemannian tensor metric g_M (sometimes denoted by \langle,\rangle), the curvature tensors of M and N, and any other tensors that will occur, always by \mathcal{C} -multilinearity. On M the Ricci tensor of N can be described by the following expression ([S-V]): for $U, V \in T_{F(p)}N$,

$$Ricci^{N}(U, V) = \sum_{1 \le \mu \le n} \frac{4}{\sin^{2} \theta_{\mu}} R^{N}(U, JV, dF(\mu), (JdF(\bar{\mu}))^{\perp}), \tag{1.4}$$

where R^N denotes the Riemannian curvature tensor of N. An application of Codazzi equation to the above expression proves that, if N is Kähler-Einstein with $R \neq 0$, Theorem 1.1 can be generalized to any dimension for totally geodesic maps ([S-V]).

We can also obtain the same conclusion to "broadly-pluriminimal" immersions for n=2, and N Kähler-Einstein with negative Ricci tensor ([S-V]). A minimal immersion F is said to be broadly-pluriminimal, if, for each $p \in \Omega_{2k}^0$, with $k \geq 1$, F is pluriharmonic with respect to any g_M -orthogonal complex structure $\tilde{J} = J_\omega \oplus J'$ on T_pM where J' is any g_M -orthogonal complex structure of \mathcal{K}_{ω} at p, that is, $(\nabla dF)^{(1,1)} = 0$. The (1,1)-part of ∇dF is just given by $(\nabla dF)^{(1,1)}(X,Y) = \frac{1}{2} (\nabla dF(X,Y) + \nabla dF(\tilde{J}X,\tilde{J}Y)) \quad \forall X,Y \in T_pM.$ If $\mathcal{K}_{\omega} = 0$, this means that F is pluriharmonic with respect to the almost complex structure J_{ω} (see for example [O-V]). In this case, we say that F is pluriminimal in the usual sense, or simply pluriminimal. Pluriharmonic immersions are obviously minimal. If F has equal Kähler angles, then only Ω_{2n}^0 is considered, where $\mathcal{K}_{\omega} = 0$ and $\tilde{J} = J_{\omega}$. Products of minimal real surfaces of Kähler surfaces, totally geodesic submanifolds, minimal Lagrangian submanifolds, and complex submanifolds are examples of broadly-pluriminimal submanifolds. We will see in sections 2 and 3 that the concept of broadly-pluriminimality, for immersions without complex directions and with equal Kähler angles, may have a geometric interpretation in terms of the torsion of a new Riemannian connection on TM. described through an isomorphism Φ from the tangent bundle of M into the normal bundle. Pluriminimal immersions with equal Kähler angles immersed into Kähler-Einstein manifolds, that are not complex submanifolds, have constant Kähler angle, and only exist on Ricci-flat manifolds. In this case, Φ defines a parallel homothetic isomorphism between TM and NM.

For a minimal immersion F with no complex directions we consider the locally Lipschitz map, symmetric on the Kähler angles,

$$\kappa = \sum_{1 \le \alpha \le n} \log \left(\frac{1 + \cos \theta_{\alpha}}{1 - \cos \theta_{\alpha}} \right). \tag{1.5}$$

This map is smooth on each Ω_{2k}^0 , non-negative, and vanishes at Lagrangian points. It is an increasing map on each $\cos \theta_{\alpha}$. In [S-V] we have given an expression for $\Delta \kappa$ at a point $p_0 \in \Omega_{2k}^0$, which we prove in the appendix of this paper, namely,

$$\Delta \kappa = 4i \sum_{\beta} Ricci^{N}(JdF(\beta), dF(\bar{\beta}))$$

$$+ \sum_{\beta,\mu} \frac{32}{\sin^{2}\theta_{\mu}} Im(R^{N}(dF(\beta), dF(\mu), dF(\bar{\beta}), JdF(\bar{\mu}) + i\cos\theta_{\mu}dF(\bar{\mu})))$$

$$- \sum_{\beta,\mu,\rho} \frac{64(\cos\theta_{\mu} + \cos\theta_{\rho})}{\sin^{2}\theta_{\mu}\sin^{2}\theta_{\rho}} Re(g(\nabla_{\beta}dF(\mu), JdF(\bar{\rho}))g(\nabla_{\bar{\beta}}dF(\rho), JdF(\bar{\mu})))$$

$$+ \sum_{\beta,\mu,\rho} \frac{32(\cos\theta_{\rho} - \cos\theta_{\mu})}{\sin^{2}\theta_{\mu}\sin^{2}\theta_{\rho}} (|g(\nabla_{\beta}dF(\mu), JdF(\rho))|^{2} + |g(\nabla_{\bar{\beta}}dF(\mu), JdF(\rho))|^{2})$$

$$+ \sum_{\beta,\mu,\rho} \frac{32(\cos\theta_{\mu} + \cos\theta_{\rho})}{\sin^{2}\theta_{\mu}} (|\langle\nabla_{\beta}\mu,\rho\rangle|^{2} + |\langle\nabla_{\bar{\beta}}\mu,\rho\rangle|^{2}),$$

$$(1.6)$$

where $\{X_{\alpha}, Y_{\alpha}\}_{1 \leq \alpha \leq n}$ is a g_M -orthonormal local frame of M, with $Y_{\alpha} = J_{\omega}X_{\alpha}$ for $\alpha \leq k$, $\{X_{\alpha}, Y_{\alpha}\}_{\alpha \geq k+1}$ any g_M -orthonormal frame of \mathcal{K}_{ω} , and which at p_0 diagonalizes $F^*\omega$. For F pluriminimal on Ω^0_{2n} and N Kähler-Einstein , we can get the following very simple final expression on Ω^0_{2n} ([S-V])

$$\Delta \kappa = -2R \Big(\sum_{1 \le \beta \le n} \cos \theta_{\beta} \Big). \tag{1.7}$$

If F has equal Kähler angles, then the expression of $\Delta \kappa$ given in (1.6) can also be substantially simplified. Minimal surfaces with constant curvature and constant Kähler angle in complex space forms have been classified in [O]. Conditions on the curvature of M, N, and/or constant equal Kähler angles lead to some conclusions in our case as well, as we show in the theorems below. Henceforth, we assume N is Kähler-Einstein. The expression for $\Delta \kappa$, where the Ricci tensor of N appears, and the Weitzenböck formula for $F^*\omega$, leading to an integral equation involving the scalar curvature R, some trigonometric functions of the common Kähler angle, and the gradient of its cosine (Proposition 4.2), are our tools to obtain the results of this paper. In section 4 we prove our main results, namely:

Theorem 1.2 Let F be a minimal immersion of a compact oriented manifold M, into a Kähler-Einstein manifold N, with equal Kähler angles.

(i) If n=2 and $R \neq 0$, then F is either a complex or a Lagrangian submanifold.

- (ii) If $n \geq 3$, R < 0, and F has no complex points, then F is Lagrangian.
- (iii) If $n \geq 3$, R = 0, and F has no complex points, then the common Kähler angle must be constant.

The conclusions in (i) and (ii) give a generalization of Theorem 1.1 to higher dimensions and equal Kähler angles. The case n=2 is the most special, because, in this dimension, immersions with equal Kähler angles have harmonic $F^*\omega$, as we will see in section 3. The case n=3 also has special properties. If the angle is constant we may allow R>0:

Theorem 1.3 Let F be minimal with constant equal Kähler angles, M compact, orientable, and $R \neq 0$. Then, F is either a complex or a Lagrangian submanifold.

Theorem 1.4 Let F be minimal with equal Kähler angles, and M compact, orientable, with non-negative isotropic scalar curvature. If n = 2 or 3, then one of the following cases holds:

- (i) M is a complex submanifold of N.
- (ii) M is a Lagrangian submanifold of N.
- (iii) R = 0 and $\cos \theta = constant \neq 0, 1$, J_{ω} is a complex integrable structure, with (M, J_{ω}, g_M) a Kähler manifold.

For any $n \geq 1$, any R, and constant equal Kähler angle, (i), (ii) or (iii) hold as well.

This theorem can be applied, for instance, to flat minimal tori on Calabi-Yau manifolds, or to spheres or products of S^2 with S^2 or with flat tori minimaly immersed into Kähler-Einstein manifolds with positive scalar curvature.

2 The morphism Φ

We consider the following morphism of vector bundles

$$\begin{array}{cccc} \Phi: & TM & \to & NM \\ & X & \to & (JdF(X))^{\perp} \end{array}$$

We easily verify that

$$\Phi(X) = JdF(X) - dF(F^*\omega(X)). \tag{2.1}$$

Both TM and NM are real vector bundles of the same dimension 2n. F has no complex directions iff Φ is an isomorphism. In fact $\Phi(X) = 0$, iff JdF(X) = dF(Y) for some Y, i.e., $span\{X, Y = "JX"\}$ is a complex direction of F. Assume there are no complex directions. Then,

$$\hat{q}(X,Y) = q_M(X,Y) - q_M(F^*\omega(X), F^*\omega(Y))$$
(2.2)

defines a Riemannian metric on M. With this metric, $\Phi:(TM,\hat{g})\to (NM,g)$ is an isomorphism of Riemannian vector bundles. Let us denote by ∇ , $\hat{\nabla}$, ∇^{\perp} , and ∇' , respectively, the Levi-Civita connection of (M,g_M) , the Levi-Civita connection of (M,\hat{g}) , the usual connection of NM induced by the Levi-Civita connection of N, and the connection on TM that makes the isomorphism Φ parallel, namely $\nabla' = \Phi^{-1*}\nabla^{\perp}$. We will also denote by ∇ the Levi-Civita connection of N and the induced connection on $F^{-1}TN$, as well. Thus, if U is a smooth section of $NM \subset F^{-1}TN$, and X, Y are smooth vector fields on M, we have

 $\nabla_X^{\perp} U = (\nabla_X U)^{\perp}$ $\Phi(\nabla_X' Y) = \nabla_X^{\perp} (\Phi(Y)).$

The connections ∇ and $\hat{\nabla}$ have no torsion, because they are Levi-Civita, but ∇' may have non-zero torsion T'. Since both $\hat{\nabla}$ and ∇' are Riemannian connections of TM for the same Riemannian metric \hat{g} , then T'=0 iff $\hat{\nabla}=\nabla'$ iff Φ is parallel. Note that, if F is Lagrangian, then $\Phi(X)=JdF(X)\in NM$, J(NM)=dF(TM), and $\hat{g}=g_M$, $\hat{\nabla}=\nabla$. Therefore, $\nabla_X\Phi(Y)=\left(J\nabla_XdF(Y)\right)^\perp=0$, that is, Φ is parallel, and so $\nabla'=\nabla$, as well. In the next section (Corollary 3.2), we will see a converse of this. We extend $\Phi:TM^c\to NM^c$ to the complexified spaces by \mathcal{C} -linearity.

Lemma 2.1 If $\{X_{\alpha}, Y_{\alpha}\}$ is a diagonalizing g_M -orthonormal basis of $F^*\omega$ at p, then at p, and for each α, β

$$\Phi(T'(Z_{\alpha}, Z_{\bar{\beta}})) = i(\cos \theta_{\alpha} + \cos \theta_{\beta}) \nabla_{Z_{\alpha}} dF(Z_{\bar{\beta}})
\Phi(T'(Z_{\alpha}, Z_{\beta})) = i(\cos \theta_{\alpha} - \cos \theta_{\beta}) \nabla_{Z_{\alpha}} dF(Z_{\beta}).$$

Proof.

$$\Phi(\nabla'_X Y) = \nabla_X^{\perp}(\Phi(Y)) = (\nabla_X(\Phi(Y)))^{\perp} = (\nabla_X(JdF(Y) - dF(F^*\omega(Y))))^{\perp}$$

$$= (J\nabla_X dF(Y) + JdF(\nabla_X Y) - \nabla_X dF(F^*\omega(Y)))^{\perp}.$$

Therefore, using the symmetry of the ∇dF and the fact that ∇ is torsionless,

$$\Phi(T'(X,Y)) = \Phi(\nabla'_X Y - \nabla'_Y X - [X,Y]) = -\nabla_X dF(F^*\omega(Y)) + \nabla_Y dF(F^*\omega(X)). \tag{2.3}$$
 The lemma follows now immediately.

For each $U \in NM_p$, let us denote by $A^U : T_pM \to T_pM$ the symmetric operator $g_M(A^U(X), Y) = g(\nabla dF(X, Y), U)$. From Lemma 2.1 and (2.3) we have

Proposition 2.1 If F is an immersion without complex directions, then:

- (i) Φ is parallel iff $F^*\omega$ anti-commutes with A^U , $\forall U \in NM$.
- (ii) If F has equal Kähler angles, on Ω_{2n}^0 , T' is of type (1,1) with respect to J_{ω} .
- (iii) On Ω_{2n}^0 , F is pluriminimal iff T' is of type (2,0) + (0,2) with respect to J_{ω} .
- (iv) If F is broadly-pluriminimal, then, for $p \in \Omega^0_{2k}$ with $k \geq 1$, T' is of type (2,0)+(0,2) with respect to any g_M -orthogonal complex structure $\tilde{J} = J_\omega \oplus J'$ on T_pM , where J' is any g_M -orthogonal complex structure of \mathcal{K}_ω .

Remark 1. If we call ω_{NM} the restriction of the Kähler form ω to the normal bundle NM, we see that, if $\{X_{\alpha}, Y_{\alpha}\}$ is a diagonalizing g_{M} -orthonormal basis of $F^{*}\omega$ at a point p, then $\{U_{\alpha} = \Phi(\frac{Y_{\alpha}}{\sin\theta_{\alpha}}), V_{\alpha} = \Phi(\frac{X_{\alpha}}{\sin\theta_{\alpha}})\}$ is a diagonalizing g-orthonormal basis of ω_{NM} . Moreover, NM has the same Kähler angles as F. Let J_{NM} denote the complex structure on NM defined by this basis, that is, the one that comes from the polar decomposition of ω_{NM} . Then, $\Phi J_{\omega} = -J_{NM}\Phi$.

We should also remark the following:

Proposition 2.2 If F is an immersion with parallel 2-form $F^*\omega$, then the Kähler angles are constant and, in particular, $M = \Omega_{2k}^0$ for some k. In this case, considering TM with the Levi-Civita connection ∇ , \mathcal{K}_{ω} and $\mathcal{K}_{\omega}^{\perp}$ are parallel sub-vector bundles of TM, and $J_{\omega} \in C^{\infty}(\mathcal{K}_{\omega}^{\perp *} \otimes \mathcal{K}_{\omega}^{\perp})$, \tilde{g} , $\hat{g} \in C^{\infty}(\mathbb{O}^2 T^*M)$ are parallel sections. Furthermore, $(X, Y, Z) \rightarrow g(\nabla_Z dF(X), JdF(Y))$ is symmetric on TM, and, if F has no complex directions, $\hat{\nabla} = \nabla$. Moreover, if $\cos \theta_{\alpha_1} > \ldots > \cos \theta_{\alpha_r}$ are the distinct eigenvalues of $F^*\omega$, the corresponding eigenspaces E_{α_t} define a smooth integrable distribution of TM whose integral submanifolds are parallel submanifolds of M. The integral submanifolds of E_{α_r} are isotropic in N if $\cos \theta_{\alpha_r} = 0$, and the ones of E_{α_1} are complex submanifolds of N if $\cos \theta_{\alpha_1} = 1$. The other ones are Kähler manifolds with respect to J_{ω} , and F restricted to each one of them is an immersion of constant equal Kähler angles θ_{α_t} with respect to J.

Proof. If X, Y are smooth vector fields on M and $Z \in T_pM$, an elementary computation gives

$$\nabla_{Z}F^{*}\omega(X,Y) = -g(\nabla_{Z}dF(X), JdF(Y)) + g(\nabla_{Z}dF(Y), JdF(X)), \tag{2.4}$$

which proves the symmetry of $(X,Y,Z) \to g(\nabla_Z dF(X), JdF(Y))$. From (2.2) we see that \hat{g} is parallel. Consequently, outside complex directions, $\nabla = \hat{\nabla}$. If we parallel transport a diagonalizing orthonormal basis $\{X_{\alpha}, Y_{\alpha}\}$ of $F^*\omega$ at p_0 along geodesics, on a neighbourhood of p_0 , since $F^*\omega$ is parallel we get a diagonalizing orthonormal frame on a whole neighbourhood with the property $\nabla X_{\alpha}(p_0) = \nabla Y_{\alpha}(p_0) = 0$. It also follows that $\cos \theta_{\alpha}$ remains constant along geodesics, so it is constant, and $J_{\omega}(X_{\alpha}) = Y_{\alpha}$ on a neighbourhood of p_0 , with $\nabla J_{\omega} = 0$ at p_0 , and so J_{ω} is parallel. Similarly we see that \tilde{g} is parallel. If we extend $F^*\omega$ to the complexified tangent space $T_{p_0}^c M$, then $F^*\omega(Z_{\alpha}) = i\cos\theta_{\alpha}Z_{\alpha}$, and $F^*\omega(Z_{\bar{\alpha}}) = -i\cos\theta_{\alpha}Z_{\bar{\alpha}}$. Obviouly, the corresponding eigenspaces of $F^*\omega$, are parallel sub-vector bundles of T^cM .

3 Immersions with equal Kähler angles

If F has equal Kähler angles, then

$$F^*\omega = \cos\theta J_\omega$$
 and $\hat{g} = \sin^2\theta g_M$,

with $\cos\theta$ a locally Lipschitz map on M, smooth on the open set where it does not vanish, and $\Omega_{2k}^0 = \emptyset \ \forall k \neq 0, n$. Note that $\sin^2\theta$ and $\cos^2\theta$ are smooth on all M. The set $\mathcal{L} = \cos\theta^{-1}(\{0\})$ is the set of Lagrangian points, for, at these points, the tangent space of M is a Lagrangian subspace of the tangent space of N. Its subset of interior points is Ω_0^0 . Similarly, we say that $\mathcal{C} = \cos\theta^{-1}(\{1\})$ is the set of complex points. On the open set $\Omega_{2n}^0 = \cos\theta^{-1}(\mathbb{R} \sim \{0\}) = M \sim \mathcal{L}$, J_{ω} defines a smooth almost complex structure g_M -orthogonal. On the open set $\cos\theta^{-1}(\mathbb{R} \sim \{1\}) = M \sim \mathcal{C}$, \hat{g} is a smooth metric conformally equivalent to g_M . Thus, if $n \geq 2$, $\hat{\nabla} = \nabla$ iff θ is constant. Since the Kähler angles are equal, any smooth local orthonormal frame of the type $\{X_{\alpha}, Y_{\alpha} = J_{\omega}X_{\alpha}\}$ diagonalizes $F^*\omega$ on the whole set where it is defined. From $F^*\omega = \cos\theta J_{\omega}$, we get $\nabla_X F^*\omega = d\cos\theta(X)J_{\omega} + \cos\theta\nabla_X J_{\omega}$, with J_{ω} orthogonal to $\nabla_X J_{\omega}$ with respect to the Hilbert-Schmidt inner product (because $||J_{\omega}||^2 = 2n$ is constant). Hence, considering $F^*\omega$ an operator on TM, on $\Omega_{2n}^0 \cup \Omega_0^0$

$$\|\nabla F^*\omega\|^2 = 2n\|\nabla\cos\theta\|^2 + \cos^2\theta\|\nabla J_\omega\|^2. \tag{3.1}$$

We observe that $M \sim (\Omega_{2n}^0 \cup \Omega_0^0)$ is a set of Lagrangian points with no interior. On Ω_{2n}^0 , we have then, $\nabla F^*\omega = 0$ iff $\nabla J_\omega = 0$ and θ is constant. Note that $\|\nabla F^*\omega\|^2$, considering $F^*\omega$ an operator on TM, is twice the square norm when considering $F^*\omega$ a 2-form. From (2.3) we get, on $M \sim \mathcal{C}$,

$$\Phi(T'(X,Y)) = 2\cos\theta(\nabla dF)^{(1,1)}(J_{\omega}X,Y). \tag{3.2}$$

The right-hand side of (3.2) is defined to be zero at a Lagrangian point. Consequentely

Proposition 3.1 If F is an immersion with equal Kähler angles and without complex points, then T' = 0, that is, $\nabla' = \hat{\nabla}$ iff Φ is parallel iff F is Lagrangian or pluriminimal. In particular, if F is minimal, Φ is parallel iff F is broadly-pluriminimal.

Let Re(u + iv) = u, for $u, v \in NM$.

Proposition 3.2 If F is any immersion with equal Kähler angles, then, outside complex and Lagrangian points,

$$\Phi\left(\frac{1-n}{4}\nabla\log\sin^2\theta\right) = \frac{4\cos\theta}{\sin^2\theta}Re\left(i\sum_{\beta,\mu}\left(g(\nabla_{\bar{\mu}}dF(\mu),JdF(\beta)) - g(\nabla_{\bar{\mu}}dF(\beta),JdF(\mu))\right)\Phi(\bar{\beta})\right),$$
where $\nabla\log\sin^2\theta$ is the gradient with respect to g_M .

If F is a complex submanifold on a open set, then J_{ω} is the induced complex structure on M and ∇dF is of type (2,0). Applying Proposition 2.2 on Ω_0^0 , and Proposition 3.1 on open sets without complex and Lagrangian points, and noting that $\{\Phi(\beta), \Phi(\bar{\beta}) = \overline{\Phi(\beta)}\}_{1 \leq \beta \leq n}$ multiplied by $\frac{\sqrt{2}}{\sin \theta}$ constitutes an unitary basis of NM^c , we immediately conclude

Corollary 3.1 If F is an immersion with equal Kähler angles, and $n \geq 2$, then θ is constant iff

$$\sum_{\mu} g(\nabla_{\bar{\mu}} dF(\mu), JdF(\beta)) = \sum_{\mu} g(\nabla_{\bar{\mu}} dF(\beta), JdF(\mu)) \quad \forall \beta.$$
 (3.3)

Note that (3.3) is a sort of symmetry property, and the first term is just $\frac{n}{2}g(H,JdF(\beta))$, where $H = \frac{1}{2n}trace_{g_M}\nabla dF = \frac{2}{n}\sum_{\mu}\nabla dF(\bar{\mu},\mu)$ is the mean curvature of F.

Theorem 3.1 If $n \geq 2$ and F is a pluriminimal immersion with equal Kähler angles then $\cos \theta = constant$. Moreover, if it is not a complex submanifold, then $\nabla = \hat{\nabla} = \nabla'$, and N must be Ricci-flat. In particular, Φ defines a parallel homothetic isomorphism from (TM, g_M) onto (NM, g).

Proof. On a neighbourhood of a non-complex point, from Proposition 3.1, $\hat{\nabla} = \nabla'$, and from Corollary 3.1, $\cos \theta$ is constant. Then $\hat{\nabla} = \nabla$, as well. So if F is not a complex submanifold, it has no complex points anywhere. Finally, (1.7) for pluriminimal immersions with $\kappa = constant$ gives R = 0.

The above theorem and Proposition 3.1 lead to:

Corollary 3.2 If F is a minimal immersion with equal Kähler angles, without complex points, $n \geq 2$, and $R \neq 0$, then F is Lagrangian iff Φ is parallel.

To prove Proposition 3.2 we will need to relate the three connections of M, ∇ , $\hat{\nabla}$ and ∇' . Let $\{e_1, \ldots, e_{2n}\} = \{X_{\mu}, Y_{\mu} = J_{\omega}X_{\mu}\}_{1 \leq \mu \leq n}$ be a local g_M -orthonormal frame outside the Lagrangian and complex set, and $\partial_1, \ldots, \partial_{2n}$ a local frame of M defined by a coordinate chart. Set $g_{ij} = g_M(\partial_i, \partial_j)$, $\hat{g}_{ij} = \hat{g}(\partial_i, \partial_j) = \sin^2\theta g_{ij}$, and $e_s = \sum_i \lambda_{si}\partial_i$. The Christofel symbols are given by $2\hat{\Gamma}_{ij}^k = \sum_s \hat{g}^{ks}(\partial_i\hat{g}_{sj} + \partial_j\hat{g}_{is} - \partial_s\hat{g}_{ij}) = \delta_{kj}\partial_i\log\sin^2\theta + \delta_{ki}\partial_j\log\sin^2\theta - \sum_s g^{ks}g_{ij}\partial_s\log\sin^2\theta + 2\Gamma_{ij}^k$. Hence

$$\hat{\nabla}_{\partial_i}\partial_j - \nabla_{\partial_i}\partial_j = \sum_k (\hat{\Gamma}_{ij}^k - \Gamma_{ij}^k)\partial_k = \frac{1}{2} \Big(\partial_i (\log \sin^2 \theta)\partial_j + \partial_j (\log \sin^2 \theta)\partial_i - g_{ij}\nabla (\log \sin^2 \theta) \Big)$$

Since $\sum_{ij} g_{ij} \lambda_{si} \lambda_{sj} = 1$, $\sum_{s} \hat{\nabla}_{e_s} e_s - \nabla_{e_s} e_s = \sum_{sij} \lambda_{si} \lambda_{sj} (\hat{\nabla}_{\partial_i} \partial_j - \nabla_{\partial_i} \partial_j) = (1-n) \nabla \log \sin^2 \theta$. Therefore,

$$\sum_{\mu} \hat{\nabla}_{\bar{\mu}} \mu - \nabla_{\bar{\mu}} \mu = \frac{1}{4} \sum_{\mu} (\hat{\nabla}_{X_{\mu}} X_{\mu} + \hat{\nabla}_{Y_{\mu}} Y_{\mu} - \nabla_{X_{\mu}} X_{\mu} - \nabla_{Y_{\mu}} Y_{\mu}) - i (\hat{\nabla}_{X_{\mu}} Y_{\mu} - \hat{\nabla}_{Y_{\mu}} X_{\mu} - \nabla_{X_{\mu}} Y_{\mu} + \nabla_{Y_{\mu}} X_{\mu})$$

$$= \frac{1}{4} \sum_{s} (\hat{\nabla}_{e_{s}} e_{s} - \nabla_{e_{s}} e_{s}) + \frac{i}{4} \sum_{\mu} ([Y_{\mu}, X_{\mu}] - [Y_{\mu}, X_{\mu}]) = \frac{(1 - n)}{4} \nabla \log \sin^{2} \theta. \tag{3.4}$$

Set
$$S'(X,Y) = \nabla'_X Y - \hat{\nabla}_X Y$$
. Then $S'(X,Y) - S'(Y,X) = T'(X,Y)$. Similarly we get
$$\sum_{\mu} \nabla'_{\bar{\mu}} \mu - \hat{\nabla}_{\bar{\mu}} \mu = \frac{1}{4} trace_{g_M} S' - \frac{i}{4} \sum_{\mu} T'(X_{\mu}, Y_{\mu}). \tag{3.5}$$

Lemma 3.1 $\forall X \in T_p M$, $\sum_i \hat{g}(S'(e_i, e_i), X) = -\sum_i \hat{g}(T'(e_i, X), e_i)$.

Proof. We may assume that the local referencial ∂_i is \hat{g} -orthonormal at a fixed poit p_0 . On a neighbourhood of p_0 , we define Γ'^k_{ij} and S'^k_{ij} as

$$\nabla'_{\partial_i}\partial_j = \sum_k \Gamma'^k_{ij}\partial_k \qquad S'(\partial_i,\partial_j) = \sum_k S'^k_{ij}\partial_k = \sum_k (\Gamma'^k_{ij} - \hat{\Gamma}^k_{ij})\partial_k.$$

 $\nabla'_{\partial_i}\partial_j = \sum_k \Gamma'^k_{ij}\partial_k \qquad S'(\partial_i,\partial_j) = \sum_k S'^k_{ij}\partial_k = \sum_k (\Gamma'^k_{ij} - \hat{\Gamma}^k_{ij})\partial_k.$ Then $T'^k_{ij} = \Gamma'^k_{ij} - \Gamma'^k_{ji}$, and, at p_0 , $\Gamma'^k_{ij} = \hat{g}(\nabla'_{\partial_i}\partial_j,\partial_k)$, $S'^k_{ij} = \hat{g}(S'(\partial_i,\partial_j),\partial_k) = \sum_k (\Gamma'^k_{ij} - \hat{\Gamma}^k_{ij})\partial_k$. $\Gamma_{ij}^{\prime k} - \hat{\Gamma}_{ij}^{k}$. ∇' is a Riemannian connection with respect to \hat{g} . Then

$$\partial_i \hat{g}_{jk}(p_0) = \hat{g}(\nabla'_{\partial_i} \partial_j, \partial_k) + \hat{g}(\partial_j, \nabla'_{\partial_i} \partial_k) = \Gamma'^{k}_{ij} + \Gamma'^{j}_{ik}$$

Hence, at p_0

$$2\hat{\Gamma}_{ij}^{k} = \sum_{s} \hat{g}^{ks} (\partial_{i}\hat{g}_{sj} + \partial_{j}\hat{g}_{is} - \partial_{s}\hat{g}_{ij}) = \Gamma_{ik}^{'j} + \Gamma_{ij}^{'k} + \Gamma_{ji}^{'k} + \Gamma_{jk}^{'i} - \Gamma_{ki}^{'j} - \Gamma_{kj}^{'i}$$
$$= (\hat{\Gamma}_{ij}^{'k} + \hat{\Gamma}_{ji}^{'k}) + (\hat{\Gamma}_{ik}^{'j} - \hat{\Gamma}_{ki}^{'j}) + (\hat{\Gamma}_{jk}^{'i} - \hat{\Gamma}_{kj}^{'i}) = (\hat{\Gamma}_{ij}^{'k} + \hat{\Gamma}_{ji}^{'k}) + \hat{T}_{ik}^{'j} + \hat{T}_{jk}^{'i}$$

But $\Gamma'^{k}_{ij} + \Gamma'^{k}_{ji} = 2\Gamma'^{k}_{ij} + (\Gamma'^{k}_{ji} - \Gamma'^{k}_{ij}) = 2\Gamma'^{k}_{ij} + T'^{k}_{ji}$. Thus

$$S'_{ij}^{k} = \Gamma'_{ij}^{k} - \hat{\Gamma}_{ij}^{k} = \frac{1}{2} (T'_{ij}^{k} - T'_{ik}^{j} + T'_{kj}^{i}).$$

That is, at p_0 , $\hat{g}(S'(\partial_i, \partial_j), \partial_k) = \frac{1}{2} \Big(\hat{g}(T'(\partial_i, \partial_j), \partial_k) - \hat{g}(T'(\partial_i, \partial_k), \partial_j) + \hat{g}(T'(\partial_k, \partial_j), \partial_i) \Big)$. We may assume that, at p_0 , $\partial_i(p_0) = \frac{e_i}{\sin \theta}$, leading to the Lemma.

Following the proof of Lemma 2.1, $\Phi(\nabla'_X \mu - \nabla_X \mu) =$ Proof of Proposition 3.2. $= ((J - i\cos\theta)\nabla_X dF(\mu))^{\perp}$. Hence, from (3.4).

$$\Phi(\frac{(1-n)}{4}\nabla \log \sin^2 \theta) = \Phi(\sum_{\mu} \hat{\nabla}_{\bar{\mu}}\mu - \nabla_{\bar{\mu}}\mu) = \left((J - i\cos\theta)\frac{nH}{2}\right)^{\perp} - \sum_{\mu} \Phi(\nabla'_{\bar{\mu}}\mu - \hat{\nabla}_{\bar{\mu}}\mu).$$

But, from (3.5), $\sum_{\mu} \Phi(\nabla'_{\bar{\mu}}\mu - \hat{\nabla}_{\bar{\mu}}\mu) = \frac{1}{4}\Phi(trace_{g_M}S') - \frac{i}{4}\Phi(\sum_{\mu}T'(X_{\mu},Y_{\mu}))$. The skewsymmetry of T' and (3.2) implies that

$$\Phi(\sum_{\mu} T'(X_{\mu}, Y_{\mu})) = -2i \sum_{\mu} \Phi(T'(\mu, \bar{\mu})) = 4\cos\theta \nabla_{\mu} dF(\bar{\mu}) = 2n\cos\theta H.$$

Thus, $\sum_{\mu} \Phi(\nabla'_{\bar{\mu}}\mu - \hat{\nabla}_{\bar{\mu}}\mu) = \frac{1}{4}\Phi(trace_{g_M}S') - \frac{ni}{2}\cos\theta H$. Therefore,

$$\Phi\left(\frac{(1-n)}{4}\nabla\log\sin^2\theta\right) = \frac{1}{4}\left(2n(JH)^{\perp} - \Phi(Trace_{g_M}S')\right). \tag{3.6}$$

Using Lemma 3.1, (3.2), and $\Phi(\mu) = JdF(\mu) - i\cos\theta dF(\mu)$, we have

$$\begin{split} &\Phi(Trace_{g_M}S') = \sum_{j,k} \hat{g}\Big(S'(e_j,e_j), \frac{e_k}{\sin\theta}\Big) \Phi(\frac{e_k}{\sin\theta}) = \sum_{j,k} -\hat{g}\Big(T'(e_j, \frac{e_k}{\sin\theta}), e_j\Big) \Phi(\frac{e_k}{\sin\theta}) \\ &= \frac{-4}{\sin^2\theta} \sum_{\mu,\beta} \Big(\Big(\hat{g}(T'(\mu,\beta),\bar{\mu}) + \hat{g}(T'(\bar{\mu},\beta),\mu)\Big) \Phi(\bar{\beta}) + \Big(\hat{g}(T'(\mu,\bar{\beta}),\bar{\mu}) + \hat{g}(T'(\bar{\mu},\bar{\beta}),\mu)\Big) \Phi(\beta) \Big) \\ &= -\frac{4}{\sin^2\theta} \sum_{\mu,\beta} \Big(g\Big(\Phi(T'(\bar{\mu},\beta)), \Phi(\mu)\Big) \Phi(\bar{\beta}) + g\Big(\Phi(T'(\mu,\bar{\beta})), \Phi(\bar{\mu})\Big) \Phi(\beta) \Big) \\ &= \frac{8i\cos\theta}{\sin^2\theta} \sum_{\mu,\beta} \Big(g\Big(\nabla_{\bar{\mu}}dF(\beta), JdF(\mu)\Big) \Phi(\bar{\beta}) - g\Big(\nabla_{\mu}dF(\bar{\beta}), JdF(\bar{\mu})\Big) \Phi(\beta) \Big). \end{split}$$

Writing $(JH)^{\perp}$ in terms of $\Phi(\beta)$ and $\Phi(\bar{\beta})$,

$$2n(JH)^{\perp} = \sum_{\beta} \frac{4n}{\sin^{2}\theta} \Big(g(JH, \Phi(\beta)) \Phi(\bar{\beta}) + g(JH, \Phi(\bar{\beta})) \Phi(\beta) \Big)$$
$$= \sum_{\beta,\mu} \frac{8i\cos\theta}{\sin^{2}\theta} \Big(g(\nabla_{\bar{\mu}}dF(\mu), JdF(\beta)) \Phi(\bar{\beta}) - g(\nabla_{\bar{\mu}}dF(\mu), JdF(\bar{\beta})) \Phi(\beta) \Big),$$

and substituing these equations into (3.6), we prove Proposition 3.2.

3.1 The Weitzenböck formula for $F^*\omega$

For simplicity let us use the notation

$$g_X YZ = g(\nabla_X dF(Y), JdF(Z)).$$

We also observe that, from

$$\forall \mu \quad \frac{i}{2}\cos\theta = F^*\omega(\mu,\bar{\mu}),\tag{3.7}$$

valid on an open set, and from (2.4), we obtain $\forall \mu$

Then (3.3) is equivalent to $g(\nabla_X dF(\mu), JdF(\bar{\mu})) = g(\nabla_X dF(\bar{\mu}), JdF(\mu))$, $\forall \mu$ (or some μ). From $J_{\omega}Z_{\alpha} = iZ_{\alpha}$, $J_{\omega}Z_{\bar{\alpha}} = -iZ_{\bar{\alpha}}$ and the fact that J_{ω} is g_M -orthogonal, we get, on Ω^0_{2n} , $\forall \alpha, \beta$, and $\forall v \in TM$

$$\langle \nabla_v J_{\omega}(\alpha), \beta \rangle = 2i \langle \nabla_v \alpha, \beta \rangle, \qquad \langle \nabla_v J_{\omega}(\alpha), \bar{\beta} \rangle = 0.$$
 (3.9)

Recall that, if ξ is a r+1-form on M, $r \geq 0$, with values on a vector bundle E over M with a connection ∇^E , then $\delta \xi$, the divergence of ξ , is the r-form on M with values on E given by

$$\delta \xi(u_1, \dots, u_r) = -\sum_s \nabla_{e_s}^E \xi(e_s, u_1, \dots, u_r),$$

where e_1, \ldots, e_m is an orthonormal basis of T_pM , $u_i \in T_pM$, and $\nabla^E \xi$ is the covariant derivative of ξ on $\bigwedge^{r+1} T^*M \otimes E$. Thus, δ is the formal adjoint of d on forms (cf. [E-L]). Note that $\delta F^*\omega(X) = \langle \delta F^*\omega, X \rangle$, $\forall X \in T_pM$, considering on the left-hand side $F^*\omega$ a (closed) 2-form on M and on the right-hand side an endomorphism of TM.

Proposition 3.3 Let F be an immersion with equal Kähler angles and $\nabla \cos \theta$ denote the gradient with respect to g_M . On Ω^0_{2n} , and considering $F^*\omega$ an endomorphism of TM.

$$\delta F^* \omega = (n-2) J_{\omega}(\nabla \cos \theta), \qquad \cos \theta (\delta J_{\omega}) = (n-1) J_{\omega}(\nabla \cos \theta).$$

Thus,

- (i) For n = 1, $\delta J_{\omega} = 0$ (obviously!), and $\delta F^* \omega = 0$ iff θ is constant.
- (ii) For n=2, $\delta F^*\omega=0$. Moreover, $\delta J_{\omega}=0$ iff θ is constant.
- (iii) For $n \neq 1, 2$, $\delta F^* \omega = 0$ iff $\delta J_{\omega} = 0$ iff θ is constant.

Proof. Considering $F^*\omega$ a 2-form on M, using the symmetry of ∇dF and (2.4), if $X \in T_pM$,

$$\delta(F^*\omega)(X) = \sum_{\mu} -2\nabla_{\mu}F^*\omega(\bar{\mu}, X) - 2\nabla_{\bar{\mu}}F^*\omega(\mu, X) = \sum_{\mu} 2g_{\mu}\bar{\mu}X - 2g_{\mu}X\bar{\mu} + 2g_{\bar{\mu}}\mu X - 2g_{\bar{\mu}}X\mu$$
$$= 2\sum_{\mu} (-g_X\mu\bar{\mu} + g_X\bar{\mu}\mu) - 4\sum_{\mu} (g_{\bar{\mu}}X\mu - g_{\bar{\mu}}\mu X).$$

From (3.8), $\frac{ni}{2}d\cos\theta(X) = \sum_{\mu} -g_X\mu\bar{\mu} + g_X\bar{\mu}\mu$. Therefore,

$$\delta(F^*\omega)(X) = nid\cos\theta(X) - 4\sum_{\mu} \nabla_{\bar{\mu}} F^*\omega(\mu, X). \tag{3.10}$$

Since $F^*\omega$ is of type (1,1) with respect to J_{ω} , and $\forall Z \in T_p^c M$, $\forall \mu, \beta$, $\langle \nabla_Z \beta, \mu \rangle = -\langle \beta, \nabla_Z \mu \rangle$, we get using (3.9)

$$\nabla_{Z}F^{*}\omega(\mu,\beta) = d(F^{*}\omega(\mu,\beta))(Z) - F^{*}\omega(\nabla_{Z}\mu,\beta) - F^{*}\omega(\mu,\nabla_{Z}\beta)$$

$$= 2i\cos\theta\langle\nabla_{Z}\mu,\beta\rangle = \cos\theta\langle\nabla_{Z}J_{\omega}(\mu),\beta\rangle. \tag{3.11}$$

Note that, since $J_{\omega}^2 = -Id$, $\nabla_X J_{\omega}(J_{\omega}Y) = -J_{\omega}(\nabla_X J_{\omega}(Y))$, $\forall X, Y \in T_pM$. So

$$4\sum_{\mu} \nabla_{\bar{\mu}} J_{\omega}(\mu) = \sum_{\mu} \nabla_{X_{\mu}} J_{\omega}(X_{\mu}) + \nabla_{Y_{\mu}} J_{\omega}(Y_{\mu}) + i \nabla_{Y_{\mu}} J_{\omega}(X_{\mu}) - i \nabla_{X_{\mu}} J_{\omega}(Y_{\mu})$$
$$= -\delta J_{\omega} + i \sum_{\mu} (-\nabla_{X_{\mu}} J_{\omega}(J_{\omega} X_{\mu}) - \nabla_{Y_{\mu}} J_{\omega}(J_{\omega} Y_{\mu})) = -(\delta J_{\omega} + i J_{\omega}(\delta J_{\omega})).$$

Hence, from (3.11), and since J_{ω} is g_M -orthogonal, $\forall \beta$

$$\sum_{\mu} \nabla_{\bar{\mu}} F^* \omega(\mu, \beta) = -\frac{\cos \theta}{4} \langle \delta J_{\omega} + i J_{\omega}(\delta J_{\omega}), \beta \rangle = -\frac{\cos \theta}{2} \langle \delta J_{\omega}, \beta \rangle.$$

Moreover, $id\cos\theta(\beta) = d\cos\theta(J_{\omega}\beta) = \langle \nabla\cos\theta, J_{\omega}\beta \rangle = -\langle J_{\omega}(\nabla\cos\theta), \beta \rangle$. From (3.10), $\delta F^*\omega(\beta) = \langle -nJ_{\omega}(\nabla\cos\theta) + 2\cos\theta\delta J_{\omega}, \beta \rangle$. Thus, if we consider $F^*\omega$ an endomorphism of TM, and since \langle , \rangle , J_{ω} , and $F^*\omega$ are real operators,

$$\delta F^* \omega = -n J_{\omega}(\nabla \cos \theta) + 2 \cos \theta \, \delta J_{\omega}. \tag{3.12}$$

On the other hand, $F^*\omega = \cos\theta J_{\omega}$. Then, an elementary computation gives

$$\delta F^* \omega = -J_{\omega}(\nabla \cos \theta) + \cos \theta \, \delta J_{\omega}. \tag{3.13}$$

Comparing (3.12) with (3.13) we get the Proposition.

Remark 2. One may check the equation in Proposition 3.2 by using the equalities given

in the above Proposition and its proof.

If we apply the Weitzenböck formula to the 2-form $F^*\omega$, for an immersion $F:M\to N$ we get (see e.g [E-L] (1.32))

$$\frac{1}{2}\Delta \|F^*\omega\|^2 = -\langle \Delta F^*\omega, F^*\omega \rangle + \|\nabla F^*\omega\|^2 + \langle SF^*\omega, F^*\omega \rangle, \tag{3.14}$$

where \langle , \rangle denotes the Hilbert-Schmidt inner product for 2-forms, and S is the Ricci operator of $\bigwedge^2 T^*M$. We note that we use the sign convention $\triangle \phi = +Trace_{q_M}Hess \phi$, for ϕ a smooth real map on M. This sign is opposite to the one of [E-L], but here we use the same sign as in [E-L] for the Laplacian of forms $\Delta = d\delta + \delta d$. If \overline{R} denotes the curvature tensor of $\bigwedge^2 T^*M$, and $X, Y, u, v \in T_pM$, $\xi \in \bigwedge^2 T_p^*M$, then

$$\overline{R}(X,Y)\xi\ (u,v) = -\xi(R^M(X,Y)u\,,\,v) - \xi(u\,,\,R^M(X,Y)v),$$
$$SF^*\omega(X,Y) = \sum_{1\leq i\leq 2n} -\overline{R}(e_i,X)F^*\omega\ (e_i,Y) + \overline{R}(e_i,Y)F^*\omega\ (e_i,X),$$

Where \mathbb{R}^{M} denotes the curvature tensor of M. In general, we use the following sign convention for curvature tensors: $R^M(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z$. Then, $R^{M}(X,Y,Z,W)=g_{M}(R^{M}(X,Y)Z,W)$. It is straightforward to prove

Lemma 3.2 If $\{X_{\alpha}, Y_{\alpha}\}$ is a diagonalizing orthonormal basis of $F^*\omega$ at p,

$$\begin{split} \langle SF^*\omega, F^*\omega \rangle \; &=\; \sum_{\mu,\rho} 4\cos^2\theta_\mu Ricci^M(\mu,\bar{\mu}) + \sum_{\mu,\rho} 8\cos\theta_\mu \cos\theta_\rho R^M(\rho,\bar{\rho},\mu,\bar{\mu}) \\ &=\; \sum_{\mu,\rho} 4(\cos\theta_\mu + \cos\theta_\rho)^2 R^M(\rho,\mu,\bar{\rho},\bar{\mu}) + 4(\cos\theta_\mu - \cos\theta_\rho)^2 R^M(\bar{\rho},\mu,\rho,\bar{\mu}). \end{split}$$

In particular, if F has equal Kähler angles at p, then, at p,

$$\langle SF^*\omega, F^*\omega \rangle = 16\cos^2\theta \sum R^M(\rho, \mu, \bar{\rho}, \bar{\mu}).$$

 $\langle SF^*\omega, F^*\omega\rangle = 16\cos^2\theta \sum_{\rho,\mu} R^M(\rho,\mu,\bar{\rho},\bar{\mu}).$ Moreover, if (M,J_ω,g_M) is Kähler in a neighbourhood of p, then $\langle SF^*\omega, F^*\omega\rangle = 0$.

For example, if M has constant sectional curvature K, $\langle SF^*\omega, F^*\omega \rangle = 4(n-1)K\|F^*\omega\|^2$. If (M, J_{ω}, g_M) is a Kähler manifold of constant holomorphic sectional curvature K then $\langle SF^*\omega, F^*\omega \rangle = 4K \Big(n \sum_{\mu} \cos^2 \theta_{\mu} - \big(\sum_{\mu} \cos \theta_{\mu} \big)^2 \Big)$ has constant sign, with equality to zero iff K=0 or F has equal Kähler angles. If $F^*\omega$ is parallel, from (3.14), we obtain that $\langle SF^*\omega, F^*\omega \rangle = 0$. In the latter case, if $n \geq 2$ and M has constant sectional curvature, then, either F is Lagrangian, or K=0.

We recall the concept of non-negative isotropic sectional curvature, for M with dimension ≥ 4 , defined by Micallef and Moore in [Mi-Mo]. Let

$$K_{isot}(\sigma) = \frac{R^M(z, w, \bar{z}, \bar{w})}{||z \wedge w||^2},$$

where $\sigma = span_{\mathbb{C}}\{z,w\}$ is a totally isotropic complex two-plane in T^cM , that is, $u \in \sigma \Rightarrow g_M(u,u) = 0$, and where $R^M(x,y,u,v)$ is extend to the complexified tangent space by \mathbb{C} -multilinearity. The curvature of M is said to be non-negative (resp. positive) on totally isotropic two-planes at p, if $K(\sigma) \geq 0$ (resp. > 0) whenever $\sigma \subset T_p^cM$ is a totally isotropic two-plane over p. If M is compact, simply connected and has positive isotropic sectional curvature everywhere, then M is homeomorphic to a sphere ([Mi-Mo]). If $n \geq 1$, T^{2n} is the flat torus, and S^2 is the euclidean sphere of \mathbb{R}^3 , then $S^2 \times T^{2n}$, $S^2 \times S^2$, $S^2 \times S^2 \times T^{2n}$ have isotropic sectional curvature ≥ 0 but not > 0. If $\{X_\alpha, Y_\alpha\}$ is any orthonormal basis of T_pM , and " μ " denotes Z_μ as in (1.3), the expression

$$S_{isot}(\{Z_{\alpha}\}_{1 \leq \alpha \leq n}) = \sum_{\rho \neq \mu} K_{isot}(span_{\mathbb{C}}\{\rho, \mu\}) = 4 \sum_{\rho, \mu} R^{M}(\rho, \mu, \bar{\rho}, \bar{\mu})$$
(3.15)

is a hermitian trace of the curvature of M restricted to the maximal totally isotropic subspace $span_{\mathbb{C}}\{Z_1,\ldots,Z_n\}$ of T^cM . To require it to be ≥ 0 , for all maximal totally isotropic subspaces - and we will say that M has non-negative isotropic scalar curvature - seems to be strictly weaker than to have non-negative isotropic sectional curvature. We also note that, any other metric conformaly equivalent to the flat metric g_0 on the 2n-torus with non-negative isotropic scalar curvature is homothetically equivalent to g_0 , hence flat. In fact, in general, if $\hat{g} = e^{\phi}g_M$ is a conformaly equivalent metric on M, then, for each g_M -orthonormal basis $\{X_\alpha, Y_\alpha\}$, $\hat{S}_{isot}(\{\hat{Z}_\alpha\}) = e^{-\phi}S_{isot}(\{Z_\alpha\}) - (n-1)e^{-2\phi}(2\triangle\phi + (n-1)\|\nabla\phi\|^2)$, where \hat{Z}_α are defined by the \hat{g} -orthonormal basis $\{e^{-\frac{\phi}{2}}X_\alpha, e^{-\frac{\phi}{2}}Y_\alpha\}$. To require $2\triangle\phi + (n-1)\|\nabla\phi\|^2 \le 0$, implies, in case of M compact, ϕ constant. We observe that, if $dim_R M \ge 6$, then $S_{isot} \equiv 0$ does not imply M to be flat, but $K_{isot} \equiv 0$ implies so. We also note that, if $dim_R (T_p M) = 4$, the set of curvature operators at p with zero isotropic sectional curvature, is a vector space of dimension 9.

Recall that, for an immersion with equal Kähler angles, $F^*\omega$ is parallel iff θ is constant and if $\cos \theta \neq 0$, (M, J_{ω}, g_M) is a Kähler manifold. We are going to see that an extra condition on the scalar isotropic curvature of M may imply itself that the Kähler angle is constant and/or $\nabla J_{\omega} = 0$. From Proposition 3.3, for any $n \geq 1$, on $\Omega_{2n}^0 \cup \Omega_0^0$

$$\|\delta F^*\omega\|^2 = (n-2)^2 \|\nabla \cos \theta\|^2.$$
 (3.16)

In particular, if $n \neq 2$, $\|\nabla \cos \theta\|^2$ can be extended as a smooth map on all M (recall that $\Omega_{2n}^0 \cup \Omega_0^0$ is dense on M), and from (3.1) we get that $\cos^2 \theta \|\nabla J_\omega\|^2$ is also smooth. Observe

that $\|\delta F^*\omega\|^2$ has the same value considering $\delta F^*\omega$ a vector or a 1-form, but considering $F^*\omega$ a 2-form (as in (3.14)) $\|\nabla F^*\omega\|^2$ is half of the square norm when considering $F^*\omega$ an operator of TM (as in (3.1)). For n=2, $F^*\omega$ is co-closed, and so it is a harmonic 2-form. In fact, since F has equal Kähler angles, $F^*\omega = \cos\theta(X_*^1 \wedge Y_*^1 + X_*^2 \wedge Y_*^2)$, and so $*F^*\omega = \pm F^*\omega$, where * is the Hodge star-operator of (M,g), and the \pm sign depends on the orientation of the diagonalizing basis. In particular, $F^*\omega$ is co-closed. For $n \geq 3$, $F^*\omega$ is harmonic iff θ is constant.

Integrating (3.14) on M, using (3.16) and (3.1), and the fact that $\int_M \langle \triangle F^* \omega, F^* \omega \rangle Vol_M = \int_M \|\delta F^* \omega\|^2 Vol_M$, we have

$$0 = \int_{M} \left((n - (n-2)^{2}) \|\nabla \cos \theta\|^{2} + \frac{1}{2} \cos^{2} \theta \|\nabla J_{\omega}\|^{2} \right) Vol_{M} + \int_{M} \langle SF^{*}\omega, F^{*}\omega \rangle Vol_{M}.$$
 (3.17)

The first integrand is smooth on M, for all n (for n=2 it gives half of (3.1)). The factor $n-(n-2)^2$ is respectively, >0, =0, <0, according n=2 or 3, n=4, and $n\geq 5$. If M has non-negative isotropic scalar curvature, $\langle SF^*\omega, F^*\omega \rangle \geq 0$, by Lemma 3.2. We conclude:

Proposition 3.4 Let F be a non-Lagrangian immersion with equal Kähler angles of a compact orientable M with non-negative isotropic scalar curvature into a Kähler manifold N. If n = 2 or 3, then θ is constant and (M, J_{ω}, g_M) is a Kähler manifold. If n = 4, $(\Omega_{2n}^0, J_{\omega}, g_M)$ is a Kähler manifold (but θ does not need to be constant). For any $n \ge 1$ and θ constant, $F^*\omega$ is parallel, i.e., (M, J_{ω}, g_M) is a Kähler manifold.

4 Minimal immersions with equal Kähler angles

Let us assume that $F: M \to N$ is minimal with equal Kähler angles. On a open set of $M \sim \mathcal{L}$ where a orthonormal frame $\{X_{\alpha}, Y_{\alpha} = J_{\omega}(X_{\alpha})\}$ is defined, we have from (3.11) and (2.4), for any $p, Z \in T_pM$ and μ, γ ,

$$2\cos\theta\langle\nabla_Z\mu,\gamma\rangle = -i\nabla_Z F^*\omega(\mu,\gamma) = ig_Z\mu\gamma - ig_Z\gamma\mu. \tag{4.1}$$

Note that $F^*\omega(\nabla_Z\mu,\bar{\gamma})=i\cos\theta\langle\nabla_Z\mu,\bar{\gamma}\rangle=-i\cos\theta\langle\mu,\nabla_Z\bar{\gamma}\rangle=-F^*\omega(\mu,\nabla_Z\bar{\gamma})$. Hence, if $\mu\neq\gamma,\,\nabla_Z F^*\omega(\mu,\bar{\gamma})=d(F^*\omega(\mu,\bar{\gamma}))(Z)=0$. Thus

$$g_Z \mu \bar{\gamma} = g_Z \bar{\gamma} \mu, \quad \forall \mu \neq \gamma$$
 (4.2)

From (3.8), for each μ ,

$$-\frac{i}{2}d\cos\theta(Z) = -\nabla_Z F^*\omega(\mu,\bar{\mu}) = g_Z \mu\bar{\mu} - g_Z\bar{\mu}\mu \quad \text{(no sumation over } \mu\text{)}$$
 (4.3)

From (1.6), on $M \sim (\mathcal{L} \cup \mathcal{C})$

$$\Delta \kappa = 4i \sum_{\beta} Ricci^{N}(JdF(\beta), dF(\bar{\beta}))$$

$$+ \frac{32}{\sin^{2}\theta} \sum_{\beta,\mu} Im(R^{N}(dF(\beta), dF(\mu), dF(\bar{\beta}), JdF(\bar{\mu}) + i\cos\theta dF(\bar{\mu}))) \qquad (4.4)$$

$$- \frac{128\cos\theta}{\sin^{4}\theta} \sum_{\beta,\mu,\rho} Re(g_{\beta}\mu\bar{\rho}\,g_{\bar{\beta}}\rho\bar{\mu}) \qquad (4.5)$$

$$+ \frac{64\cos\theta}{\sin^{2}\theta} \sum_{\beta,\mu,\rho} \left(|\langle\nabla_{\beta}\mu,\rho\rangle|^{2} + |\langle\nabla_{\bar{\beta}}\mu,\rho\rangle|^{2} \right), \qquad (4.6)$$

where now $\kappa = n \log \left(\frac{1+\cos\theta}{1-\cos\theta} \right)$. Since R(X,Y,Z,JW) is skew-symmetric on (X,Y) and symmetric on (Z,W), $\sum_{\mu,\beta} R^N(dF(\beta),dF(\mu),dF(\bar{\beta}),JdF(\bar{\mu})) = 0$. Then, from the Gauss equation and minimality of F,

$$(4.4) = \sum_{\beta,\mu} \frac{32}{\sin^2 \theta} Im \Big(i \cos \theta R^N (dF(\beta), dF(\mu), dF(\bar{\beta}), dF(\bar{\mu})) \Big)$$
$$= \frac{32 \cos \theta}{\sin^2 \theta} \sum_{\beta,\mu} R^M (\beta, \mu, \bar{\beta}, \bar{\mu}) + g(\nabla dF(\beta, \bar{\mu}), \nabla dF(\mu, \bar{\beta})).$$

Using the unitary basis $\{\frac{\sqrt{2}}{\sin\theta}\Phi(\rho), \frac{\sqrt{2}}{\sin\theta}\Phi(\bar{\rho})\}$ of the normal bundle,

$$\frac{32\cos\theta}{\sin^2\theta} \sum_{\beta,\mu} g(\nabla dF(\beta,\bar{\mu}), \nabla dF(\mu,\bar{\beta})) = \frac{64\cos\theta}{\sin^4\theta} \sum_{\beta,\mu,\rho} (|g_{\beta}\bar{\mu}\rho|^2 + |g_{\beta}\bar{\mu}\bar{\rho}|^2) =
= \frac{64\cos\theta}{\sin^4\theta} \sum_{\beta,\mu,\rho} (|g_{\beta}\bar{\rho}\mu|^2 + |g_{\bar{\mu}}\beta\bar{\rho}|^2) = \frac{128\cos\theta}{\sin^4\theta} \sum_{\beta,\mu,\rho} |g_{\beta}\bar{\rho}\mu|^2.$$
(4.7)

From (4.2) and (4.3),

$$\sum_{\beta,\mu,\rho} Re \Big(g_{\beta}\mu\bar{\rho} g_{\bar{\beta}}\rho\bar{\mu} \Big) = \sum_{\beta,\mu} \sum_{\rho\neq\mu} |g_{\beta}\bar{\rho}\mu|^2 + \sum_{\beta,\mu} Re \Big(g_{\beta}\mu\bar{\mu} g_{\bar{\beta}}\mu\bar{\mu} \Big)$$

$$= \sum_{\beta,\mu,\rho} |g_{\beta}\bar{\rho}\mu|^2 - \sum_{\beta,\mu} |g_{\beta}\bar{\mu}\mu|^2 + \sum_{\beta,\mu} Re \Big(g_{\beta}\mu\bar{\mu} g_{\bar{\beta}}\mu\bar{\mu} \Big)$$

$$= \sum_{\beta,\mu,\rho} |g_{\beta}\bar{\rho}\mu|^2 - \sum_{\beta,\mu} Re \Big(\frac{i}{2} d\cos\theta(\beta) g_{\bar{\beta}}\mu\bar{\mu} \Big),$$

SO

$$(4.7) + (4.5) = \frac{128\cos\theta}{\sin^4\theta} \sum_{\beta,\mu} Re\left(\frac{i}{2}d\cos\theta(\beta)g_{\bar{\beta}}\mu\bar{\mu}\right).$$

On the other hand, Proposition 3.2 and minimality of F gives,

$$-\sum_{\beta,\mu} \frac{4\cos\theta}{\sin^2\theta} Re\left(ig_\beta \bar{\mu}\mu \cdot \bar{\beta}\right) = \frac{1-n}{4} \nabla \log \sin^2\theta = \frac{(n-1)\cos\theta}{2\sin^2\theta} \nabla \cos\theta.$$

Consequentely,

$$\frac{128\cos\theta}{\sin^4\theta} \sum_{\beta,\mu} Re\left(\frac{i}{2}d\cos\theta(\beta)g_{\bar{\beta}}\mu\bar{\mu}\right) = \frac{128\cos\theta}{\sin^4\theta} \sum_{\beta,\mu} Re\left(-\frac{i}{2}d\cos\theta(\bar{\beta})g_{\beta}\bar{\mu}\mu\right) \\
= -\frac{64\cos\theta}{\sin^4\theta} d\cos\theta\left(Re\left(\sum_{\beta,\mu} ig_{\beta}\bar{\mu}\mu \cdot \bar{\beta}\right)\right) = \frac{8(n-1)\cos\theta}{\sin^4\theta} \|\nabla\cos\theta\|^2.$$

That is,

$$(4.7) + (4.5) = \frac{8(n-1)\cos\theta}{\sin^4\theta} \|\nabla\cos\theta\|^2. \tag{4.8}$$

Using (3.9),

$$\|\nabla J_{\omega}\|^{2} = \sum_{\beta} 4\langle \nabla_{\beta} J_{\omega}, \nabla_{\bar{\beta}} J_{\omega} \rangle = \sum_{\beta} \sum_{\mu,\rho} 16 \left(|\langle \nabla_{\beta} J_{\omega}(\mu), \rho \rangle|^{2} + |\langle \nabla_{\beta} J_{\omega}(\bar{\mu}), \bar{\rho} \rangle|^{2} \right)$$

$$= 64 \sum_{\beta,\mu,\rho} \left(|\langle \nabla_{\beta} \mu, \rho \rangle|^{2} + |\langle \nabla_{\bar{\beta}} \mu, \rho \rangle|^{2} \right). \tag{4.9}$$

Thus we see that $(4.6) = \frac{\cos \theta}{\sin^2 \theta} ||\nabla J_{\omega}||^2$. So we have obtained the following formula:

Proposition 4.1 If N is Kähler-Einstein with Ricci tensor $Ricci^N = Rg$, and F is a minimal immersion with equal Kähler angles, on an open set without complex and Lagrangian points,

$$\triangle \kappa = \cos \theta \Big(-2nR + \frac{32}{\sin^2 \theta} \sum_{\beta,\mu} R^M(\beta,\mu,\bar{\beta},\bar{\mu}) + \frac{1}{\sin^2 \theta} \|\nabla J_{\omega}\|^2 + \frac{8(n-1)}{\sin^4 \theta} \|\nabla \cos \theta\|^2 \Big).$$
(4.10)

Note that if n = 1 we get the expression of Wolfson [W], $\Delta \kappa = -2R \cos \theta$.

Proposition 4.2 If N is Kähler-Einstein with Ricci tensor $Ricci^N = Rg$, and F is a minimal imersion with equal Kähler angles, then:

(i) If
$$n = 2$$
,
$$\int_{M} nR \sin^{2}\theta \cos^{2}\theta Vol_{M} = 0. \tag{4.11}$$

(ii) If $n \ge 3$ and F has no complex points, $\int_{M} nR \sin^{2} \theta \cos^{2} \theta \, Vol_{M} = \int_{M} (n-2)(n-2+2\cot^{2} \theta) \|\nabla \cos \theta\|^{2} \, Vol_{M}. \tag{4.12}$

Proof. Multiplying (4.10) by $\sin^2\theta\cos\theta$, we get, on $M\sim\mathcal{C}\cup\mathcal{L}$, and using Lemma 3.2,

$$\sin^2 \theta \cos \theta \triangle \kappa = -2n \sin^2 \theta \cos^2 \theta R + 2\langle SF^*\omega, F^*\omega \rangle + \cos^2 \theta \|\nabla J_\omega\|^2 + \frac{8(n-1)\cos^2 \theta}{\sin^2 \theta} \|\nabla \cos \theta\|^2.$$

On the other hand, $\kappa = n \log \left(\frac{1 + \cos \theta}{1 - \cos \theta} \right)$, and so, $\Delta \kappa = \frac{2n}{\sin^2 \theta} \Delta \cos \theta + \frac{4n \cos \theta}{\sin^4 \theta} \|\nabla \cos \theta\|^2$. Hence,

$$2n\cos\theta\triangle\cos\theta + \frac{4n\cos^2\theta}{\sin^2\theta}\|\nabla\cos\theta\|^2 =$$

$$= -2n\sin^2\theta\cos^2\theta R + 2\langle SF^*\omega, F^*\omega\rangle + \cos^2\theta\|\nabla J_\omega\|^2 + \frac{8(n-1)\cos^2\theta}{\sin^2\theta}\|\nabla\cos\theta\|^2.$$
(4.13)

Recall that, from (3.1), and considering $F^*\omega$ a 2-form, $\|\nabla F^*\omega\|^2 = \frac{1}{2}\cos^2\theta\|\nabla J_\omega\|^2 + n\|\nabla\cos\theta\|^2$. Since $\triangle\cos^2\theta = 2\cos\theta\triangle\cos\theta + 2\|\nabla\cos\theta\|^2$, substituting this into (4.13), we have

$$n\triangle\cos^2\theta = -2n\sin^2\theta\cos^2\theta R + 2\langle SF^*\omega, F^*\omega\rangle + 2\|\nabla F^*\omega\|^2 + \frac{4(n-2)\cos^2\theta}{\sin^2\theta}\|\nabla\cos\theta\|^2$$
(4.14)

and, for n=2,

$$n\triangle\cos^2\theta = -2n\sin^2\theta\cos^2\theta R + 2\langle SF^*\omega, F^*\omega\rangle + 2\|\nabla F^*\omega\|^2. \tag{4.15}$$

Let us now suppose that $n \geq 3$. Then, under the condition of no complex points, (4.14) is valied on Ω_{2n}^0 and also on Ω_0^0 . From smoothness over all M of all maps into consideration (the first three terms of the right-hand side of (4.14) are smooth, and the last term is also smooth for $n \neq 2$), and the fact that the set $M \sim (\Omega_0^0 \cup \Omega_0^{2n})$ is a set of Lagrangian points with no interior, formula (4.14) is valid on all M. Integrating over M, and using (3.17), we have

$$\int_{M} 2nR \sin^{2}\theta \cos^{2}\theta \, Vol_{M} = \int_{M} \left(-2(n-(n-2)^{2}) + \frac{4(n-2)\cos^{2}\theta}{\sin^{2}\theta} + 2n \right) \|\nabla \cos\theta\|^{2} \, Vol_{M},$$

leading to (4.12). If n=2, we see that (4.15) is also valid at Lagrangian and complex points. In fact (see Lemma 3.2 and (3.1)), all terms of (4.15) vanish at interior points of the Lagrangian and complex sets. Since they are smooth on all M, they must vanish at boundary points of its complementary in M. Thus, the above equation is valid on all M, with or without complex or Lagrangian points, and all its terms are smooth. Then, (4.11) follows by integration on M of (4.15), and use of (3.17).

Proof of Theorem 1.2. and Theorem 1.3 If n=2 and $R\neq 0$, (4.11) implies $\sin^2\theta\cos^2\theta=0$. Hence F is either Lagrangian or a complex submanifold. If $n\geq 3$, and F has no complex points, the right-hand side of (4.12) is non-negative, while the left-hand side is non-positive for R<0. Then, $\sin^2\theta\cos^2\theta=0$ must hold on all M, that is, F is Lagrangian. If R=0, the right-hand side of (4.12) must vanish. Then, for $n\geq 3$, $\cos\theta$ must be constant, and we have proved Theorem 1.2. If $\cos\theta$ is constant, and if F is not a complex submanifold, the right-hand side of (4.12) vanishes. Hence, if $R\neq 0$, F is Lagrangian, and Theorem 1.3 is proved. \square

Proof of Theorem 1.4. If M is not Lagrangian, under the curvature condition on M, by Proposition 3.4, for n=2, or 3, (M, J_{ω}, g_M) is a Kähler manifold and $\cos \theta$ is constant. So, if M is not a complex submanifold, it has no complex directions, and by (4.11), or (4.12), R=0. In general, if $n \geq 1$ and θ is constant, Proposition 3.4 also applies. \square

Under the conditions of Theorem 1.4, if M is homeomorphic to a 4 or a 6 dimensional sphere, minimaly immersed into a Kähler-Einstein manifold, and with equal Kähler angles, then it must be Lagrangian, for it is well known that such manifolds cannot carry a Kähler structure. Obviously, any Riemannian manifold M with strictly positive isotropic scalar curvature cannot carry any Kähler structure. Moreover, such condition for n=2 would imply M to be homeomorphic to a 4-sphere. We also remark that we only need to require $S_{isot}(\{Z_{\alpha}\}) \geq 0$ on the maximal totally isotropic subspace $\{Z_{\alpha}\}$ defined by a diagonalizing orthonormal basis of $F^*\omega$, and outside Lagrangian points, to obtain the same conclusion given in Theorem 1.4.

As an observation, Theorem 1.4 should be compared with the following lemma:

Lemma 4.1 Let F be a minimal immersion, and $n \ge 2$. If $\cos \theta$ is constant $\ne 1, 0$, then

- (i) $(A, B, C) \rightarrow g_A BC$ is symmetric whenever A, B, and C are not all of the same type.
- (ii) $\langle \nabla_{\bar{\beta}} \mu, \gamma \rangle = 0, \quad \forall \beta, \mu, \gamma.$
- $(iii) \ \ F^*\omega \ is \ an \ harmonic \ 2\text{-}form.$

(iv)
$$32 \sum_{\beta,\mu} R^M(\beta,\mu,\bar{\beta},\bar{\mu}) = -64 \sum_{\beta,\mu,\rho} |\langle \nabla_{\!\beta}\mu,\rho\rangle|^2 = -\|\nabla J_{\omega}\|^2 \le 0.$$

Proof. Since $\cos \theta$ is constant, we obtain (4.3) = 0. This, together (4.2), and the symmetry of ∇dF , proves (i). But (i) and (4.1) implys (ii). (iii) comes from (3.16). Now we prove (vi). Since $F^*\omega$ is harmonic, from Weitzenböck formula (3.14) we conclude $\langle SF^*\omega, F^*\omega \rangle = -\|\nabla F^*\omega\|^2$. Lemma 3.2 and (3.1) (but considering $F^*\omega$ a 2-form) gives (iv).

Remark 3. If N is a Kähler manifold of constant holomorphic sectional curvature equal to K (and so $R = \frac{(2n+1)K}{2}$), and the isotropic scalar curvature of M satisfies $S_{isot} \geq c = constant$, we get from Gauss equation, with $\{X_{\alpha}, Y_{\alpha}\}$ a diagonalizing orthonormal basis of $F^*\omega$,

$$\sum_{\rho,\mu} R^{M}(\mu,\rho,\bar{\mu},\bar{\rho}) = \frac{n(n-1)}{16} \sin^{2}\theta K - \sum_{\rho,\mu} \|\nabla dF(\mu,\bar{\rho})\|^{2}, \tag{4.16}$$

that $c \leq \frac{n(n-1)K}{4}$. Thus, non-negative isotropic scalar curvature on M is a possible condition for $K \geq 0$. In the case K = 0, that is, N is the flat complex torus, then (4.16) (with K = 0) is valied for any orthonormal basis $\{X_{\alpha}, Y_{\alpha}\}$. This implies that, for $n \geq 2$, F must be totally geodesic, and so M is flat.

We also note that if $c = \frac{nR}{4}$, the right-hand side of (4.10) becomes > 0, outside Lagrangian points. An application of the maximum principle at a maximum point of κ would conclude that F must be Lagrangian. But such a lower bound c is not possible for the scalar isotropic curvature of M minimally immersed in N with constant holomorphic sectional curvature K > 0.

Remark 3. If $n \geq 2$ and F is a pluriminimal immersion with equal Kähler angles into a Kähler-Einstein manifold N, and F is not a complex submanifold, then N must be Ricciflat. Moreover, since F has constant equal Kähler angles, the scalar isotropic curvature of M with respect to the maximal isotropic subspace defined by a diagonalizing orthonormal basis of $F^*\omega$ will be ≤ 0 , with equality to zero iff (M, J_ω, g_M) is Kähler (see Lemma 4.1). We leave the following question: Is (M, J_ω, g_M) Kähler manifold a sufficient condition for a minimal immersion F, with constant equal Kähler angle, immersed into a Ricci-flat Kähler manifold N, to be pluriminimal? If N is the flat complex torus and $F: M \to N$ is minimal, under the conditions stated in the question, the Gauss equation implies that F is pluriminimal. A way to find pluriminimal submanifolds in hyper-Kähler manifolds is given in the next example, where the assumption of non-negative isotropic curvature does not imply necessarely F totally geodesic (and M flat), since hyper-Kähler manifolds do not need to be flat.

Example. Let (N, I, J, g) be an hyper-Kähler manifold of real dimension 8. Thus, I and J are two g-orthogonal complex structures on N, such that IJ = -JI and $\nabla I = \nabla J = 0$, where ∇ is the Levi-Civita connection relative to g. It is known that such manifolds are Ricci-flat ([B]). Set K = IJ. For each ν , ϕ , we take " $\nu\phi$ " = $(\cos \nu, \sin \nu \cos \phi, \sin \nu \sin \phi) \in S^2$, and define $J_{\nu\phi} = \cos \nu I + \sin \nu \cos \phi J + \sin \nu \sin \phi K$. These $J_{\nu\phi}$ are the complex structures on N compatible with its hyper-Kähler structure, that is, they are g-orthogonal and $\nabla J_{\nu\phi} = 0$.

Two of such complex structures, $J_{\nu\phi}$ and $J_{\mu\rho}$, anti-commute at a point p iff $J_{\nu\phi}(X)$ and $J_{\mu\rho}(X)$ are orthogonal for some non-zero $X \in T_pN$, iff $\nu\phi$ and $\mu\rho$ are orthogonal in \mathbb{R}^3 . Thus, they anti-commute at a point p iff they anti-commute everywhere. If that is the case $J_{\nu\phi} \circ J_{\mu\rho} = J_{\sigma\epsilon}$, where $\{\nu\phi, \mu\rho, \sigma\epsilon\}$ is a direct orthonormal basis of \mathbb{R}^3 . For each unit vector $X \in T_pN$, set $H_X = span\{X, IX, JX, KX\} = span\{X, J_{\nu\phi}(X), J_{\mu\rho}(X), J_{\sigma\epsilon}(X)\}$, for any orthonormal basis $\{\nu\phi, \mu\rho, \sigma\epsilon\}$. If $Y \in H_X^{\perp}$ is another unit vector, then $H_X \perp H_Y$. Let $\omega_{\nu\phi}$ be the Kähler form of $(N, J_{\nu\phi}, g)$. Let E be a 4-dimensional vector sub-space of T_pN . We first note that $E = H_X$ for some $X \in E$, iff $J_{\nu\phi}(E) \subset E$ for any ν, ϕ . If that is the case, then E is not a Lagrangian subspace with respect to any complex structure $J_{\mu\rho}$. In general, E contains a $J_{\nu\phi}$ -complex line for some $\nu\phi$ iff $dim(E \cap H_X) \geq 2$ for some

 $X \in E$. If that is the case, and if E is a Lagrangian subspace of T_pN with respect to $J_{\mu\rho}$, then $\nu\phi\perp\mu\rho$. Furthermore, if E is a $J_{\nu\phi}$ -complex subspace, then E is $J_{\mu\rho}$ -Lagrangian iff there exist an orthonormal basis $\{X, J_{\nu\phi}X, Y, J_{\nu\phi}Y\}$ of E with $H_X\perp H_Y$. To see this, let us suppose E is $J_{\nu\phi}$ -complex subspace and $J_{\mu\rho}$ -Lagrangian. We take $\{X, J_{\nu\phi}X, Y, J_{\nu\phi}Y\}$ an ortnormal basis of E. Then $Y \in span\{X, J_{\nu\phi}X, J_{\mu\rho}X\}^{\perp}$. So $Y = tJ_{\sigma\epsilon}X + \tilde{Y}$, for some $t \in \mathbb{R}$ and $\tilde{Y} \in H_X^{\perp}$, and where $\{\nu\phi, \mu\rho, \sigma\epsilon\}$ is an ortnormal basis of \mathbb{R}^3 . As $E \neq H_X$, $\tilde{Y} \neq 0$. From $0 = \langle J_{\mu\rho}Y, J_{\nu\phi}X \rangle$, we get t = 0. Thus, $Y \in H_X^{\perp}$. We observe that, in general, $J_{\mu\rho}$ -Lagrangian subspaces do not need to be $J_{\nu\phi}$ -complex, as for example $E = \{X, J_{\nu\phi}X, Y, J_{\sigma\epsilon}Y\}$, with $Y \in H_X^{\perp}$, that contains two orthogonal complex lines for different complex strutures.

Any $J_{\nu\phi}$ -complex submanifold $F: M \to N$ of real dimension 4, such that, for each point $p \in M$, there exist an orthonormal basis $\{X, J_{\nu\phi}X, Y, J_{\nu\phi}Y\}$ of T_pM with $H_X \perp H_Y$, is, for each $\mu\rho$, a minimal submanifold of $(N, J_{\mu\rho}, g)$ with constant equal Kähler angles, and $\pm J_{\nu\phi}$ is also the complex structure of M which comes from polar decomposition of $\omega_{\mu\rho}$ restricted to M. In fact, such an orthonormal basis of T_pM diagonalizes $\omega_{\mu\rho}$ restricted to M, and the Kähler angle θ is such that $\cos \theta = \pm \langle \nu\phi, \mu\rho \rangle$, where <, > is the inner product of \mathbb{R}^3 . Next proposition is an application of Theorem 1.4, for 4-dimensional submanifolds of N, where ω_I is the Kähler form of (N, I, g):

Proposition 4.3 Let $F: M \to N$ be a minimal immersion of a compact, oriented 4-dimensional submanifold with non-negative isotropic scalar curvature, and such that $\forall \nu \phi \in S^2$, F has equal Kähler angles with respect to $J_{\nu \phi}$. If $\exists p \in M$ and $\exists X \in T_p M$, unit vector, such that $\dim(T_p M \cap H_X) \geq 2$, then there exists $\nu \phi \in S^2$ such that M is a $J_{\nu \phi}$ -complex submanifold. Furthermore, if $J_{\nu \phi} = I$ then $F: M \to (N, I, g)$ is obviously pluriminimal. If $J_{\nu \phi} \neq I$ but $T_p M \cap H_X^{\perp} \neq \{0\}$, then $F^*\omega_I = \cos \nu J_{\nu \phi}$, and if F is not J_I -Lagrangian, $F: M \to (N, I, g)$ is still pluriminimal.

Note that, if $T_pM = H_X$, then $J_{\nu\phi}$ can be chosen equal to I. The first conclusion of this result is the 4-dimensional version of a result of Wolfson [W], for M a real surface and N a Ricci-flat K3 surface. In the latter case, there is only one Kähler angle, $\forall X \dim(T_pM \cap H_X) = 2$ is automatically satisfied, and the isotropic scalar curvature is always zero.

Proof. From the assumption, $dim(T_pM \cap H_X) \geq 2$, we may take a unit vector $Z \in T_pM \cap H_X$ such that $Z \perp X$. Then, $Z = J_{\nu\phi}(X)$ for some $\nu\phi$. Thus, $span\{X, J_{\nu\phi}(X)\} \subset T_pM$. This implies $F^*\omega_{\nu\phi}(X, J_{\nu\phi}(X)) = 1$. As the Kähler angles are equal, $\cos\theta_{\nu\phi} = 1$ at p. Applying Theorem 1.4 to $F: M \to (N, J_{\nu\phi}, g)$, $F^*\omega_{\nu\phi} = \cos\theta_{\nu\phi}J_{\omega_{\nu\phi}}$ with $\cos\theta_{\nu\phi}$ constant. Then $\cos\theta_{\nu\phi} = 1$ everywhere. That is, M is a $J_{\nu\phi}$ -complex submanifold. Moreover, from the second assumption, $T_pM \cap H_X^{\perp} \neq \{0\}$, we may take a unit vector

 $Y \in T_pM \cap H_X^{\perp}$. Then $\{X, J_{\nu\phi}X, Y, J_{\nu\phi}Y\}$ constitutes an orthonormal basis of T_pM , that diagonalizes $F^*\omega_I$, and $F^*\omega_I = \cos\nu J_{\nu\phi}$. This means that ν or $\nu + \pi$ is the constant Kähler angle of $F: M \to (N, I, g)$, and, since M is a $J_{\nu\phi}$ -complex submanifold, it is pluriharmonic with respect to $\pm J_{\nu\phi}$, and so, if $\cos\nu \neq 0$, it is pluriminimal as an immersion into (N, I, g).

5 Appendix: The computation of $\triangle \kappa$

We prove (1.6) for F minimal and outside complex and Lagrangian points. First, we compute some derivative formulas of a determinant, which we will need.

Lemma 5.1 Let $A: M \to \mathcal{M}_{m \times m}(\mathcal{C})$ be a smooth map of matrices $p \to A(p) = [A_1, \ldots, A_m]$, where $A_i(p)$ is a column vector of \mathcal{C}^m and M is a Riemannian manifold with its Levi-Civita connection ∇ . Assume that, at p_0 , $A(p_0)$ is a diagonal matrix $D = D(\lambda_1, \ldots, \lambda_m)$. Then, at p_0

$$d (det A)(Z) = \sum_{1 \le j \le m} \left(\prod_{k \ne j} \lambda_k \right) dA_j^j(Z),$$

$$Hess (det A)(Z, W) = \nabla d(det A)(Z, W) =$$

$$= \sum_{1 \le i, k \le m} \left(\prod_{s \ne i, k} \lambda_s \right) det \begin{bmatrix} dA_j^j(Z) & dA_j^k(Z) \\ dA_j^j(W) & dA_k^k(W) \end{bmatrix} + \sum_{1 \le i \le m} \left(\prod_{s \ne i} \lambda_s \right) Hess A_j^j(Z, W).$$

In particular, if e_1, \ldots, e_r is an orthonormal basis of $T_{p_0}M$, then, at p_0 ,

$$\triangle (\det A) = Trace \ Hess \ (\det A) = \\ = \sum_{1 \leq \alpha \leq r} \sum_{1 \leq j,k \leq m} \Big(\prod_{s \neq j,k} \lambda_s \Big) det \left[\begin{array}{cc} dA^j_j(e_\alpha) & dA^k_j(e_\alpha) \\ dA^j_k(e_\alpha) & dA^k_k(e_\alpha) \end{array} \right] + \sum_{1 \leq j \leq m} \Big(\prod_{s \neq j} \lambda_s \Big) \triangle A^j_j.$$

On each Ω_{2k}^0 , the complex structure J_{ω} and the sub-vector bundle $\mathcal{K}_{\omega}^{\perp}$ are smooth. Moreover, J_{ω} is g_M -orthogonal. Thus, for each $p_0 \in \Omega_{2k}^0$, there exists a locally g_M -orthonormal frame of $\mathcal{K}_{\omega}^{\perp}$ defined on a neighbourhood of p_0 , of the form $X_1, J_{\omega}X_1, \ldots, X_k, J_{\omega}X_k$. We enlarge this frame to a g_M -orthonormal local frame on M, on a neighbourhood of p_0 :

$$X_1, Y_1 = J_{\omega} X_1, \dots, X_k, Y_k = J_{\omega} X_k, X_{k+1}, Y_{k+1}, \dots, X_n, Y_n$$
 (5.1)

where $X_{k+1}, Y_{k+1}, \dots X_n, Y_n$ is any g_M -orthonormal frame of \mathcal{K}_{ω} , and which at p_0 is a diagonalizing basis of $F^*\omega$. Note that in general it is not possible to get smooth diagonalizing g_M -orthonormal frames in a whole neighbourhood of a point p_0 , unless, for instance, $F^*\omega$ has equal Kähler angles. We use the notations in section 3.1. We define a local complex structure on a neighbourhood of $p_0 \in \Omega^0_{2k}$ as $\tilde{J} = J_\omega \oplus J'$, where J_ω is defined only on \mathcal{K}^\perp_ω , and J' is the local complex structure on \mathcal{K}_ω , defined on a neighbourhood of p_0 by

$$J'Z_{\alpha} = iZ_{\alpha}, \quad J'Z_{\bar{\alpha}} = -iZ_{\bar{\alpha}}, \quad \forall \alpha \ge k+1.$$
 (5.2)

Thus, the vectors Z_{α} are of type (1,0) with respect to \tilde{J} , for $\forall \alpha$. Since \tilde{J} is g_M -orthogonal, then, $\forall \alpha, \beta$, on a neighbourhood of p_0 ,

$$\langle \nabla_{Z} \tilde{J}(\alpha), \beta \rangle = 2i \langle \nabla_{Z} \alpha, \beta \rangle = -\langle \alpha, \nabla_{Z} \tilde{J}(\beta) \rangle, \qquad \langle \nabla_{Z} \tilde{J}(\alpha), \bar{\beta} \rangle = 0, \qquad (5.3)$$

Note that $F^*\omega$ and \tilde{g} , where \tilde{g} is given in (1.1), as 2-tensors, are both of type (1,1) with respect to \tilde{J} , and have the same kernel \mathcal{K}_{ω} . They are related by $\tilde{g}(X,Y) = F^*\omega(X,J_{\omega}Y) = F^*\omega(X,\tilde{J}Y)$. Set $\tilde{g}_{AB} = \tilde{g}(A,B)$, and define $\overline{B} = B, \ \forall A,B \in \{1,\ldots,n,\bar{1},\ldots,\bar{n}\}$, and set $\epsilon_{\alpha} = +1, \ \epsilon_{\bar{\alpha}} = -1, \ \forall 1 \leq \alpha \leq n$. Let $1 \leq \alpha,\beta \leq n,\ A,B \in \{1,\ldots,n,\bar{1},\ldots,\bar{n}\}$, and $C \in \{1,\ldots,n\} \cup \{\overline{k+1},\ldots,\bar{n}\}$. Then

$$\begin{array}{ll}
(n) & \text{of } \overline{k+1}, \dots, \overline{n} \}. \text{ Then} \\
F^*\omega(\alpha, C) &= g(JdF(\alpha), dF(C)) = 0 & \forall p \text{ near } p_0 \\
F^*\omega(\alpha, \overline{\beta}) &= g(JdF(\alpha), dF(\overline{\beta})) = \frac{i}{2}\delta_{\alpha\beta}\cos\theta_{\alpha} & \text{at } p_0 \\
\tilde{g}_{AB} &= i\epsilon_B F^*\omega(A, B) = i\epsilon_B g(JdF(A), dF(B)) & \forall p \text{ near } p_0 \\
\tilde{g}_{\alpha C} &= \tilde{g}_{\overline{\alpha}\overline{C}} &= 0 & \forall p \text{ near } p_0 \\
\tilde{g}_{\alpha \overline{\beta}} &= \tilde{g}_{\overline{\alpha}\beta} &= \frac{1}{2}\delta_{\alpha\beta}\cos\theta_{\alpha} & \text{at } p_0
\end{array} \right\}.$$
(5.4)

At a point p_0 , with Kähler angles θ_{α} , $g_M \pm \tilde{g}$ is represented in the unitary basis $\{\sqrt{2}\alpha, \sqrt{2}\bar{\alpha}\}$, by the diagonal matrix $g_M \pm \tilde{g} = D(1 \pm \cos\theta_1, \dots, 1 \pm \cos\theta_n, 1 \pm \cos\theta_1, \dots, 1 \pm \cos\theta_n)$, and so

$$det(g_M \pm \tilde{g}) = \prod_{1 \le \alpha \le n} (1 \pm \cos \theta_\alpha)^2.$$
 (5.5)

If p_0 is a point without complex directions, $\cos \theta_{\alpha} \neq 1$, $\forall \alpha \in \{1, ..., n\}$, then $\tilde{g} < g_M$. Thus, on a neighbourwood of p_0 , we may consider the map κ .

$$\kappa = \frac{1}{2} \log \left(\frac{\det(g_M + \tilde{g})}{\det(g_M - \tilde{g})} \right) = \sum_{1 \le \alpha \le n} \log \left(\frac{1 + \cos \theta_\alpha}{1 - \cos \theta_\alpha} \right). \tag{5.6}$$

This map is continuous outside the complex points, and smooth on each Ω_{2k}^0 . We wish to compute $\Delta \kappa$ on Ω_{2k}^0 .

Lemma 5.2 At $p_0 \in \Omega^0_{2k}$, without complex directions and for $Z, W \in T_{p_0}M$,

$$d(\det(g_M \pm \tilde{g}))(Z) = \pm 4 \sum_{1 \le \mu \le n} \frac{\prod_{1 \le \alpha \le n} (1 \pm \cos \theta_{\alpha})^2}{(1 \pm \cos \theta_{\mu})} d\tilde{g}_{\mu\bar{\mu}}(Z),$$

$$Hess(\det(g_M \pm \tilde{g}))(Z, W) =$$

$$= 16 \Big(\prod_{1 \le \alpha \le n} (1 \pm \cos \theta_{\alpha})^2 \Big) \sum_{\mu, \rho} \frac{1}{(1 \pm \cos \theta_{\mu})(1 \pm \cos \theta_{\rho})} d\tilde{g}_{\mu\bar{\mu}}(Z) d\tilde{g}_{\rho\bar{\rho}}(W)$$

$$-8 \Big(\prod_{1 \le \alpha \le n} (1 \pm \cos \theta_{\alpha})^2 \Big) \sum_{\mu, \rho} \frac{1}{(1 \pm \cos \theta_{\mu})(1 \pm \cos \theta_{\rho})} d\tilde{g}_{\mu\bar{\rho}}(W) d\tilde{g}_{\rho\bar{\mu}}(Z)$$

$$\pm 4 \Big(\prod_{1 \le \alpha \le n} (1 \pm \cos \theta_{\alpha})^2 \Big) \sum_{\mu} \frac{1}{(1 \pm \cos \theta_{\mu})} Hess\tilde{g}_{\mu\bar{\mu}}(Z, W).$$

Proof. Using the unitary basis $\{\sqrt{2}\alpha, \sqrt{2}\bar{\alpha}\}\$ of T_p^cM , for p near p_0 , $g_M + \tilde{g}$ is represented by the matrix

$$g_M \pm \tilde{g} = \begin{bmatrix} g_M \pm \tilde{g}(\sqrt{2}\alpha, \sqrt{2}\bar{\gamma}) & g_M \pm \tilde{g}(\sqrt{2}\alpha, \sqrt{2}\gamma) \\ g_M \pm \tilde{g}(\sqrt{2}\bar{\alpha}, \sqrt{2}\bar{\gamma}) & g_M \pm \tilde{g}(\sqrt{2}\bar{\alpha}, \sqrt{2}\gamma) \end{bmatrix} = \begin{bmatrix} \delta_{\alpha\gamma} \pm 2\tilde{g}_{\alpha\bar{\gamma}} & 0 \\ 0 & \delta_{\alpha\gamma} \pm 2\tilde{g}_{\bar{\alpha}\gamma} \end{bmatrix}$$

that at p_0 is the diagonal matrix $D(1\pm\cos\theta_1,\ldots,1\pm\cos\theta_n,1\pm\cos\theta_1,\ldots,1\pm\cos\theta_n)$. The lemma follows as a simple application of lemma 5.1, and noting that $\tilde{g}_{\mu\bar{\rho}}=\tilde{g}_{\bar{\rho}\mu}$.

On Ω_{2k}^0 ,

$$2\triangle\kappa = \triangle \log(\det(g_M + \tilde{g})) - \triangle \log(\det(g_M - \tilde{g}))$$

$$= \frac{\triangle(\det(g_M + \tilde{g}))}{\det(g_M + \tilde{g})} - \frac{\|d(\det(g_M + \tilde{g}))\|^2}{(\det(g_M + \tilde{g}))^2} - \frac{\triangle(\det(g_M - \tilde{g}))}{\det(g_M - \tilde{g})} + \frac{\|d(\det(g_M - \tilde{g}))\|^2}{(\det(g_M - \tilde{g}))^2}.$$

From the above lemma and

$$||d(\det(g_M \pm \tilde{g}))||^2 = 4 \sum_{\beta} d(\det(g_M \pm \tilde{g}))(\beta) d(\det(g_M \pm \tilde{g}))(\bar{\beta})$$

$$\triangle \det(g_M \pm \tilde{g}) = 4 \sum_{\beta} Hess(\det(g_M \pm \tilde{g}))(\beta, \bar{\beta})$$

we have at p_0 ,

$$2\triangle\kappa = \sum_{\beta,\mu,\rho} \frac{64(\cos\theta_{\mu} + \cos\theta_{\rho})}{\sin^{2}\theta_{\mu}\sin^{2}\theta_{\rho}} d\tilde{g}_{\mu\bar{\rho}}(\bar{\beta}) d\tilde{g}_{\rho\bar{\mu}}(\beta) + \sum_{\beta,\mu} \frac{32}{\sin^{2}\theta_{\mu}} Hess\tilde{g}_{\mu\bar{\mu}}(\beta,\bar{\beta}). \tag{5.7}$$

Recalling (2.4), and $d(F^*\omega(X,Y))(Z) = \nabla_Z F^*\omega(X,Y) + F^*\omega(\nabla_Z X,Y) + F^*\omega(X,\nabla_Z Y)$, using (5.4), we obtain

Lemma 5.3 $\forall p \ near \ p_0 \in \Omega^0_{2k}, \ Z \in T^c_pM, \ and \ \mu, \gamma \in \{1, \dots, n\}$

$$d\tilde{g}_{\mu\bar{\gamma}}(Z) = ig_Z \mu\bar{\gamma} - ig_Z\bar{\gamma}\mu + 2\sum_{\rho} \left(\langle \nabla_{\!Z}\mu, \bar{\rho} \rangle \tilde{g}_{\rho\bar{\gamma}} + \langle \nabla_{\!Z}\bar{\gamma}, \rho \rangle \tilde{g}_{\mu\bar{\rho}} \right)$$

$$0 = d\tilde{g}_{\mu\gamma}(Z) = -ig_Z \mu\gamma + ig_Z \gamma\mu + 2\sum_{\rho} \left(\langle \nabla_{\!Z}\mu, \rho \rangle \tilde{g}_{\bar{\rho}\gamma} - \langle \nabla_{\!Z}\gamma, \rho \rangle \tilde{g}_{\mu\bar{\rho}} \right).$$

In particular, at p_0

$$d\tilde{g}_{\mu\bar{\gamma}}(Z) = ig_Z\mu\bar{\gamma} - ig_Z\bar{\gamma}\mu - (\cos\theta_\mu - \cos\theta_\gamma)\langle\nabla_Z\mu,\bar{\gamma}\rangle$$

$$0 = d\tilde{g}_{\mu\gamma}(Z) = -ig_Z\mu\gamma + ig_Z\gamma\mu + (\cos\theta_\mu + \cos\theta_\gamma)\langle\nabla_Z\mu,\gamma\rangle.$$

Lemma 5.4 If F is minimal and $p_0 \in \Omega_{2k}^0$ is a point without complex directions, then for each $\mu \in \{1, \ldots, n\}$

$$\begin{split} \sum_{1 \leq \beta \leq n} Hess \tilde{g}_{\mu\bar{\mu}}(\beta,\bar{\beta}) &= \sum_{1 \leq \beta \leq n} d\Big(d\tilde{g}_{\mu\bar{\mu}}(\beta)\Big)(\bar{\beta}) - d\tilde{g}_{\mu\bar{\mu}}(\nabla_{\bar{\beta}}\beta) = \\ &= \sum_{1 \leq \beta \leq n} iR^N(dF(\beta),dF(\bar{\beta}),dF(\mu),JdF(\bar{\mu}) + i\cos\theta_\mu dF(\bar{\mu})) \\ &\quad + 2Im\Big(R^N(dF(\beta),dF(\mu),dF(\bar{\beta}),JdF(\bar{\mu}) + i\cos\theta_\mu dF(\bar{\mu}))\Big) \end{split}$$

$$+2\sum_{1\leq\rho\leq n}\frac{\left(\cos\theta_{\rho}-\cos\theta_{\mu}\right)}{\sin^{2}\theta_{\rho}}\left(|g_{\beta}\mu\rho|^{2}+|g_{\beta}\bar{\mu}\bar{\rho}|^{2}\right)$$

$$-2\sum_{1\leq\rho\leq n}\frac{\left(\cos\theta_{\rho}+\cos\theta_{\mu}\right)}{\sin^{2}\theta_{\rho}}\left(|g_{\beta}\mu\bar{\rho}|^{2}+|g_{\beta}\bar{\mu}\bar{\rho}|^{2}\right)$$

$$+\sum_{1\leq\rho\leq n}\frac{-2i\langle\nabla_{\mu}\beta,\bar{\rho}\rangle g_{\bar{\beta}}\rho\bar{\mu}-2i\langle\nabla_{\mu}\beta,\rho\rangle g_{\bar{\beta}}\bar{\rho}\bar{\mu}-2i\langle\nabla_{\mu}\bar{\beta},\bar{\rho}\rangle g_{\rho}\beta\bar{\mu}}{+\sum_{1\leq\rho\leq n}\frac{2i\langle\nabla_{\bar{\beta}}\mu,\bar{\rho}\rangle g_{\beta}\rho\bar{\mu}-2i\langle\nabla_{\mu}\bar{\beta},\rho\rangle g_{\bar{\rho}}\beta\bar{\mu}+2i\langle\nabla_{\bar{\beta}}\mu,\rho\rangle g_{\bar{\rho}}\beta\bar{\mu}}{+\sum_{1\leq\rho\leq n}\frac{2i\langle\nabla_{\bar{\mu}}\beta,\bar{\rho}\rangle g_{\bar{\beta}}\rho\mu+2i\langle\nabla_{\bar{\mu}}\beta,\rho\rangle g_{\bar{\rho}}\beta\mu+2i\langle\nabla_{\bar{\mu}}\bar{\beta},\bar{\rho}\rangle g_{\rho}\beta\mu}{+\sum_{1\leq\rho\leq n}\frac{2i\langle\nabla_{\bar{\beta}}\bar{\mu},\bar{\rho}\rangle g_{\rho}\beta\mu+2i\langle\nabla_{\bar{\mu}}\bar{\beta},\rho\rangle g_{\bar{\rho}}\beta\mu-2i\langle\nabla_{\bar{\beta}}\bar{\mu},\rho\rangle g_{\bar{\rho}}\beta\mu}{+\sum_{1\leq\rho\leq n}\frac{2i\langle\nabla_{\bar{\beta}}\bar{\mu},\bar{\rho}\rangle g_{\beta}\mu\rho+2i\langle\nabla_{\bar{\beta}}\bar{\mu},\rho\rangle g_{\bar{\beta}}\mu\bar{\rho}-2i\langle\nabla_{\bar{\beta}}\mu,\rho\rangle g_{\bar{\beta}}\bar{\mu}\rho}{+\sum_{1\leq\rho\leq n}\frac{2i\langle\nabla_{\bar{\beta}}\mu,\bar{\rho}\rangle g_{\bar{\beta}}\bar{\mu}\rho+2i\langle\nabla_{\beta}\mu,\bar{\rho}\rangle g_{\bar{\beta}}\bar{\rho}\mu}{+\sum_{1\leq\rho\leq n}\frac{2i\langle\nabla_{\bar{\beta}}\mu,\bar{\rho}\rangle g_{\bar{\beta}}\mu\bar{\rho}-2i\langle\nabla_{\beta}\mu,\bar{\rho}\rangle g_{\bar{\beta}}\bar{\mu}\rho}{+\sum_{1\leq\rho\leq n}\frac{2i\langle\nabla_{\bar{\beta}}\mu,\bar{\rho}\rangle g_{\bar{\beta}}\mu\bar{\rho}-2i\langle\nabla_{\beta}\mu,\bar{\rho}\rangle g_{\bar{\beta}}\bar{\mu}\rho}{+\sum_{1\leq\rho\leq n}\frac{2i\langle\nabla_{\bar{\beta}}\mu,\bar{\rho}\rangle g_{\bar{\beta}}\mu\bar{\rho}-2i\langle\nabla_{\beta}\mu,\bar{\rho}\rangle g_{\bar{\beta}}\bar{\mu}\rho}{+\sum_{1\leq\rho\leq n}\frac{2i\langle\nabla_{\beta}\mu,\bar{\rho}\rangle g_{\bar{\beta}}\bar{\mu}}{+\sum_{1\leq\rho\leq n}\frac{2i\langle\nabla_{\beta}\mu,\bar{\rho}\rangle g_{\bar{\beta}}\bar{\mu}}{+\sum_{1\leq\rho\leq n}\frac{2i\langle\nabla_{\beta}\mu,\bar{\rho}\rangle g_{\bar{\beta}}\bar{\mu}}{+\sum_{1\leq\rho\leq n}\frac{2i\langle\nabla_{\beta}\mu,\bar{\rho}\rangle g_{\bar{\beta}}\bar{\mu}}{+\sum_{1\leq\rho\leq n}\frac{2i\langle\nabla_{\beta}\mu,\bar{\rho}\rangle g_{\bar{\beta}}\bar{\mu}}{+\sum_{1\leq\rho\leq n}\frac{2i$$

Proof. We denote by $\nabla_X \nabla_Y dF$ the covariant derivative of $\nabla_Y dF$ in $T^*M \otimes F^{-1}TN$, and by $\overline{R}(X,Y)\xi$, the curvature tensor of this connection, namely $(\overline{R}(X,Y)\xi)(Z) = R^N(dF(X),dF(Y))\xi(Z) - \xi(R^M(X,Y)Z)$. From Lemma 5.3, for p on a neighbourhood of p_0 ,

 $d\tilde{g}_{\mu\bar{\mu}}(\beta) = ig(\nabla_{\!\beta}dF(\mu), JdF(\bar{\mu})) - ig(\nabla_{\!\beta}dF(\bar{\mu}), JdF(\mu)) + 2\sum_{\rho} \left(\langle \nabla_{\!\beta}\mu, \bar{\rho} \rangle \tilde{g}_{\rho\bar{\mu}} + \langle \nabla_{\!\beta}\bar{\mu}, \rho \rangle \tilde{g}_{\mu\bar{\rho}} \right).$ Then at p_0 ,

$$d(d\tilde{g}_{\mu\bar{\mu}}(\beta))(\bar{\beta}) =$$

$$= ig\left(\nabla_{\bar{\beta}}(\nabla_{\beta}dF(\mu)), JdF(\bar{\mu})\right) + ig\left(\nabla_{\beta}dF(\mu), \nabla_{\bar{\beta}}(JdF(\bar{\mu}))\right) - ig\left(\nabla_{\bar{\beta}}(\nabla_{\beta}dF(\bar{\mu})), JdF(\mu)\right) - ig\left(\nabla_{\beta}dF(\bar{\mu}), \nabla_{\bar{\beta}}(JdF(\mu))\right) + 2\sum_{\rho}\left(\nabla_{\bar{\beta}}(\langle\nabla_{\beta}\mu, \bar{\rho}\rangle)\tilde{g}_{\rho\bar{\mu}} + \nabla_{\bar{\beta}}(\langle\nabla_{\beta}\bar{\mu}, \rho\rangle)\tilde{g}_{\mu\bar{\rho}}\right) + \sum_{\rho}2\langle\nabla_{\beta}\mu, \bar{\rho}\rangle d\tilde{g}_{\rho\bar{\mu}}(\bar{\beta}) + 2\langle\nabla_{\beta}\bar{\mu}, \rho\rangle d\tilde{g}_{\mu\bar{\rho}}(\bar{\beta})$$

$$= ig\left(\nabla_{\bar{\beta}}(\nabla_{\beta}dF(\mu)), JdF(\bar{\mu})\right) + ig\left(\nabla_{\beta}dF(\mu), J\nabla_{\bar{\beta}}dF(\bar{\mu})\right) + ig\left(\nabla_{\beta}dF(\mu), JdF(\bar{\mu})\right) - ig\left(\nabla_{\beta}dF(\bar{\mu}), JdF(\mu)\right) - ig\left(\nabla_{\beta}dF(\bar{\mu}), JdF(\bar{\nu})\right) + \cos\theta_{\mu}\left(\nabla_{\bar{\beta}}(\langle\nabla_{\beta}\mu, \bar{\mu}\rangle) + \nabla_{\bar{\beta}}(\langle\mu, \nabla_{\beta}\bar{\mu}\rangle)\right) + (5.8)$$

$$= ig\left(\nabla_{\bar{\beta}}(\nabla_{\beta}dF(\mu)), JdF(\bar{\mu})\right) + \sum_{\rho} 2i\langle\nabla_{\bar{\beta}}\bar{\mu}, \rho\rangle g_{\beta}\mu\bar{\rho} + 2i\langle\nabla_{\bar{\beta}}\bar{\mu}, \bar{\rho}\rangle g_{\beta}\mu\rho$$

$$(5.9)$$

$$-ig\left(\nabla_{\bar{\beta}}\left(\nabla_{\beta}dF(\bar{\mu})\right), JdF(\mu)\right)$$

$$-ig\left(\nabla_{\beta}dF(\bar{\mu}), J\nabla_{\bar{\beta}}dF(\mu)\right) + \sum_{\bar{\beta}} -2i\langle\nabla_{\bar{\beta}}\mu, \rho\rangle g_{\beta}\bar{\mu}\bar{\rho} - 2i\langle\nabla_{\bar{\beta}}\mu, \bar{\rho}\rangle g_{\beta}\bar{\mu}\rho$$

$$+\cos\theta_{\mu}\left(\nabla_{\bar{\beta}}\left(\langle\nabla_{\beta}\mu, \bar{\mu}\rangle\right) + \nabla_{\bar{\beta}}\left(\langle\hat{\mu}, \nabla_{\beta}\bar{\mu}\rangle\right)\right)$$

$$+(5.8).$$

$$(5.10)$$

The term (5.11) vanish because $\langle \nabla_{\beta} \mu, \bar{\mu} \rangle = -\langle \mu, \nabla_{\beta} \bar{\mu} \rangle$ on a neighbourhood of p_0 . Minimality of F implies

$$\begin{split} &\sum_{\beta} \nabla_{\!\bar{\beta}} \Big(\nabla_{\!\beta} dF(\mu) \Big) = \\ &= \sum_{\beta} \nabla_{\!\bar{\beta}} \Big(\nabla_{\!\mu} dF(\beta) \Big) = \sum_{\beta} \nabla_{\!\bar{\beta}} \nabla_{\!\mu} dF(\beta) + \nabla_{\!\mu} dF(\nabla_{\!\bar{\beta}} \beta) \\ &= \sum_{\beta} \nabla_{\!\mu} \nabla_{\!\bar{\beta}} dF(\beta) - \nabla_{\![\mu,\bar{\beta}]} dF(\beta) + (\overline{R}(\mu,\bar{\beta}) dF)(\beta) + \nabla_{\!\mu} dF(\nabla_{\!\bar{\beta}} \beta) \\ &= \sum_{\beta} \nabla_{\!\mu} \Big(\nabla_{\!\bar{\beta}} dF(\beta) \Big) - \nabla_{\!\bar{\beta}} dF(\nabla_{\!\mu} \beta) - \nabla_{\![\mu,\bar{\beta}]} dF(\beta) \\ &= \sum_{\beta} \nabla_{\!\mu} \Big(dF(\mu) + (\partial_{\bar{\beta}} (\bar{\beta})) dF(\beta) - (\partial_{\bar{\beta}} (\bar{\beta}) (\bar{\beta})) + \nabla_{\!\mu} dF(\nabla_{\!\bar{\beta}} \beta) \Big) \\ &= \sum_{\beta} \sum_{\beta} -2 \langle \nabla_{\!\mu} \beta, \bar{\rho} \rangle \nabla_{\!\bar{\beta}} dF(\rho) + \sum_{\beta} -2 \langle \nabla_{\!\mu} \beta, \rho \rangle \nabla_{\!\bar{\beta}} dF(\bar{\rho}) \\ &- \sum_{\beta} (2 \langle \nabla_{\!\mu} \bar{\beta}, \bar{\rho} \rangle - 2 \langle \nabla_{\!\bar{\beta}} \mu, \bar{\rho} \rangle) \nabla_{\!\rho} dF(Z_{\beta}) \\ &- \sum_{\beta} (2 \langle \nabla_{\!\mu} \bar{\beta}, \bar{\rho} \rangle - 2 \langle \nabla_{\!\bar{\beta}} \mu, \bar{\rho} \rangle) \nabla_{\!\bar{\rho}} dF(Z_{\beta}) \\ &+ R^{N} (dF(\mu) + dF(\bar{\beta})) dF(\beta) - dF(R^{M}(\mu, \bar{\beta}) \beta) \\ &+ \sum_{\beta} (2 \langle \nabla_{\!\bar{\beta}} \beta, \bar{\rho} \rangle \nabla_{\!\mu} dF(\rho) + \sum_{\beta} 2 \langle \nabla_{\!\bar{\beta}} \beta, \bar{\rho} \rangle \nabla_{\!\mu} dF(\bar{\rho}). \end{split}$$

Hence

$$(5.9) = \sum_{\beta} iR^{N}(dF(\mu), dF(\bar{\beta}), dF(\beta), JdF(\bar{\mu})) - \cos\theta_{\mu}R^{M}(\mu, \bar{\beta}, \beta, \bar{\mu})$$

$$+ \sum_{\beta\rho} -2i\langle\nabla_{\mu}\beta, \bar{\rho}\rangle g_{\bar{\beta}}\rho\bar{\mu} - 2i\langle\nabla_{\mu}\beta, \rho\rangle g_{\bar{\beta}}\bar{\rho}\bar{\mu}$$

$$+ \sum_{\beta\rho} 2i(-\langle\nabla_{\mu}\bar{\beta}, \bar{\rho}\rangle + \langle\nabla_{\bar{\beta}}\mu, \bar{\rho}\rangle)g_{\rho}\beta\bar{\mu} + 2i(-\langle\nabla_{\mu}\bar{\beta}, \rho\rangle + \langle\nabla_{\bar{\beta}}\mu, \rho\rangle)g_{\bar{\rho}}\beta\bar{\mu}$$

$$+ \sum_{\beta\rho} 2i\langle\nabla_{\bar{\beta}}\beta, \bar{\rho}\rangle g_{\mu}\rho\bar{\mu} + 2i\langle\nabla_{\bar{\beta}}\beta, \rho\rangle g_{\mu}\bar{\rho}\bar{\mu}.$$

Similarly

$$-(5.10) = \sum_{j} iR^{N}(dF(\bar{\mu}), dF(\bar{\beta}), dF(\beta), JdF(\mu)) + \cos\theta_{\mu}R^{M}(\bar{\mu}, \bar{\beta}, \beta, \mu)$$

$$+ \sum_{\beta\rho}^{\beta} -2i\langle\nabla_{\bar{\mu}}\beta, \bar{\rho}\rangle g_{\bar{\beta}}\rho\mu - 2i\langle\nabla_{\bar{\mu}}\beta, \rho\rangle g_{\bar{\beta}}\bar{\rho}\mu$$

$$+ \sum_{\beta\rho}^{\beta\rho} 2i(-\langle\nabla_{\bar{\mu}}\bar{\beta}, \bar{\rho}\rangle + \langle\nabla_{\bar{\beta}}\bar{\mu}, \bar{\rho}\rangle)g_{\rho}\beta\mu + 2i(-\langle\nabla_{\bar{\mu}}\bar{\beta}, \rho\rangle + \langle\nabla_{\bar{\beta}}\bar{\mu}, \rho\rangle)g_{\bar{\rho}}\beta\mu$$

$$+ \sum_{\beta\rho}^{\beta\rho} 2i\langle\nabla_{\bar{\beta}}\beta, \bar{\rho}\rangle g_{\bar{\mu}}\rho\mu + 2i\langle\nabla_{\bar{\beta}}\beta, \rho\rangle g_{\bar{\mu}}\bar{\rho}\mu.$$

Using Bianchi identity,

$$\begin{split} iR^N(dF(\mu),dF(\bar{\beta}),dF(\beta),JdF(\bar{\mu})) - iR^N(dF(\bar{\mu}),dF(\bar{\beta}),dF(\beta),JdF(\mu)) &= \\ &= -iR^N(dF(\beta),dF(\mu),dF(\bar{\beta}),JdF(\bar{\mu})) - iR^N(dF(\bar{\beta}),dF(\beta),dF(\mu),JdF(\bar{\mu})) \\ &- iR^N(dF(\bar{\mu}),dF(\bar{\beta}),dF(\beta),JdF(\mu)) \\ &= iR^N(dF(\beta),dF(\bar{\beta}),dF(\mu),JdF(\bar{\mu})) + 2Im\Big(R^N(dF(\beta),dF(\mu),dF(\bar{\beta}),JdF(\bar{\mu}))\Big), \end{split}$$

and by Gauss equation, and minimality of F,

$$\begin{split} &\sum_{\beta} -R^{M}(\mu, \bar{\beta}, \beta, \bar{\mu}) - R^{M}(\bar{\mu}, \bar{\beta}, \beta, \mu) = \\ &= \sum_{\beta} R^{M}(\beta, \mu, \bar{\beta}, \bar{\mu}) + R^{M}(\bar{\beta}, \beta, \mu, \bar{\mu}) - R^{M}(\bar{\mu}, \bar{\beta}, \beta, \mu) \\ &= \sum_{\beta} -R^{M}(\beta, \bar{\beta}, \mu, \bar{\mu}) + 2R^{M}(\beta, \mu, \bar{\beta}, \bar{\mu}) \\ &= \sum_{\beta} -R^{N}(dF(\beta), dF(\bar{\beta}), dF(\mu), dF(\bar{\mu})) \\ &= \sum_{\beta} -R^{N}(dF(\beta), dF(\bar{\mu})) + g(\nabla_{\beta}dF(\bar{\mu}), \nabla_{\bar{\beta}}dF(\mu)) \\ &+ 2R^{N}(dF(\beta), dF(\mu), dF(\bar{\beta}), dF(\bar{\mu})) \\ &+ 2g(\nabla_{\beta}dF(\bar{\beta}), \nabla_{\mu}dF(\bar{\mu})) - 2g(\nabla_{\beta}dF(\bar{\mu}), \nabla_{\mu}dF(\bar{\beta})) \\ &= \sum_{\beta} -R^{N}(dF(\beta), dF(\bar{\beta}), dF(\mu), dF(\bar{\mu})) + 2R^{N}(dF(\beta), dF(\mu), dF(\bar{\mu})) \\ &- g(\nabla_{\beta}dF(\mu), \nabla_{\bar{\beta}}dF(\bar{\mu})) - g(\nabla_{\beta}dF(\bar{\mu}), \nabla_{\mu}dF(\bar{\beta})). \end{split}$$

Note that $R^N(dF(\beta), dF(\mu), dF(\bar{\beta}), dF(\bar{\mu})) = Im(iR^N(dF(\beta), dF(\mu), dF(\bar{\beta}), dF(\bar{\mu})))$, since it is real. Therefore,

$$\sum_{\beta} d(d\tilde{g}_{\mu\bar{\mu}}(\beta))(\bar{\beta}) = \\
= \sum_{\beta} iR^{N}(dF(\beta), dF(\bar{\beta}), dF(\mu), JdF(\bar{\mu}) + i\cos\theta_{\mu}dF(\bar{\mu})) \\
+ 2Im(R^{N}(dF(\beta), dF(\mu), dF(\bar{\beta}), JdF(\bar{\mu}) + i\cos\theta_{\mu}dF(\bar{\mu}))) \\
- \cos\theta_{\mu} g(\nabla_{\beta}dF(\mu), \nabla_{\bar{\beta}}dF(\bar{\mu})) - \cos\theta_{\mu} g(\nabla_{\beta}dF(\bar{\mu}), \nabla_{\mu}dF(\bar{\beta})) \\
+ \sum_{\rho} -2i\langle\nabla_{\mu}\beta, \bar{\rho}\rangle g_{\bar{\beta}}\rho\bar{\mu} - 2i\langle\nabla_{\mu}\beta, \rho\rangle g_{\bar{\beta}}\bar{\rho}\bar{\mu} \\
+ \sum_{\rho} 2i(-\langle\nabla_{\mu}\bar{\beta}, \bar{\rho}\rangle + \langle\nabla_{\bar{\beta}}\mu, \bar{\rho}\rangle)g_{\rho}\beta\bar{\mu} + 2i(-\langle\nabla_{\mu}\bar{\beta}, \rho\rangle + \langle\nabla_{\bar{\beta}}\mu, \rho\rangle)g_{\bar{\rho}}\beta\bar{\mu} \\
+ \sum_{\rho} 2i\langle\nabla_{\bar{\beta}}\beta, \bar{\rho}\rangle g_{\mu}\rho\bar{\mu} + 2i\langle\nabla_{\bar{\beta}}\beta, \rho\rangle g_{\mu}\bar{\rho}\bar{\mu} \\
+ \sum_{\rho} 2i\langle\nabla_{\bar{\mu}}\beta, \bar{\rho}\rangle g_{\bar{\beta}}\rho\mu + 2i\langle\nabla_{\bar{\mu}}\beta, \rho\rangle g_{\bar{\beta}}\bar{\rho}\mu \\
+ \sum_{\rho} 2i(\langle\nabla_{\bar{\mu}}\beta, \bar{\rho}\rangle - \langle\nabla_{\bar{\beta}}\bar{\mu}, \bar{\rho}\rangle)g_{\rho}\beta\mu + 2(\langle\nabla_{\bar{\mu}}\bar{\beta}, \rho\rangle - \langle\nabla_{\bar{\beta}}\bar{\mu}, \rho\rangle)g_{\bar{\rho}}\beta\mu \\
+ \sum_{\rho} 2i(\langle\nabla_{\bar{\beta}}\beta, \bar{\rho}\rangle g_{\bar{\mu}}\rho\mu - 2i\langle\nabla_{\bar{\beta}}\beta, \rho\rangle g_{\bar{\mu}}\bar{\rho}\mu \\
+ \sum_{\rho} -2i\langle\nabla_{\bar{\beta}}\beta, \bar{\rho}\rangle g_{\bar{\mu}}\rho\mu - 2i\langle\nabla_{\bar{\beta}}\beta, \rho\rangle g_{\bar{\mu}}\bar{\rho}\mu \\
+ ig(\nabla_{\beta}dF(\mu), J\nabla_{\bar{\beta}}dF(\bar{\mu})) - ig(\nabla_{\beta}dF(\bar{\mu}), J\nabla_{\bar{\beta}}dF(\mu)) \tag{5.15}$$

$$+ \sum_{\rho} 2i \langle \nabla_{\bar{\beta}} \bar{\mu}, \rho \rangle g_{\beta} \mu \bar{\rho} + 2i \langle \nabla_{\bar{\beta}} \bar{\mu}, \bar{\rho} \rangle g_{\beta} \mu \rho$$
$$\sum_{\rho} -2i \langle \nabla_{\bar{\beta}} \mu, \rho \rangle g_{\beta} \bar{\mu} \bar{\rho} - 2i \langle \nabla_{\bar{\beta}} \mu, \bar{\rho} \rangle g_{\beta} \bar{\mu} \rho + (5.8).$$

Using the unitary basis $\{\frac{\sqrt{2}}{\sin\theta_{\rho}}\Phi(\rho), \frac{\sqrt{2}}{\sin\theta_{\rho}}\Phi(\bar{\rho})\}$ of the normal bundle, and (2.1)

$$(5.12) + (5.15) =$$

$$= -\sum_{\beta,\rho} \frac{2\cos\theta_{\mu}}{\sin^{2}\theta_{\rho}} (|g_{\beta}\mu\rho|^{2} + |g_{\beta}\mu\bar{\rho}|^{2}) - \sum_{\beta,\rho} \frac{2\cos\theta_{\mu}}{\sin^{2}\theta_{\rho}} (|g_{\beta}\bar{\mu}\rho|^{2} + |g_{\beta}\bar{\mu}\bar{\rho}|^{2})$$

$$-\sum_{\beta,\rho} \frac{2\cos\theta_{\rho}}{\sin^{2}\theta_{\rho}} (|g_{\beta}\mu\bar{\rho}|^{2} - |g_{\beta}\mu\rho|^{2}) + \sum_{\beta,\rho} \frac{2\cos\theta_{\rho}}{\sin^{2}\theta_{\rho}} (|g_{\beta}\bar{\mu}\rho|^{2} - |g_{\beta}\bar{\mu}\rho|^{2})$$

$$= 2\sum_{\beta,\rho} \frac{(\cos\theta_{\rho} - \cos\theta_{\mu})}{\sin^{2}\theta_{\rho}} |g_{\beta}\mu\rho|^{2} - 2\sum_{\beta,\rho} \frac{(\cos\theta_{\rho} + \cos\theta_{\mu})}{\sin^{2}\theta_{\rho}} |g_{\beta}\mu\bar{\rho}|^{2}$$

$$-2\sum_{\beta,\rho} \frac{(\cos\theta_{\rho} + \cos\theta_{\mu})}{\sin^{2}\theta_{\rho}} |g_{\beta}\bar{\mu}\rho|^{2} + 2\sum_{\beta,\rho} \frac{(\cos\theta_{\rho} - \cos\theta_{\mu})}{\sin^{2}\theta_{\rho}} |g_{\beta}\bar{\mu}\bar{\rho}|^{2}.$$

Applying lemma 5.3 and since $\langle \nabla_Z \mu, \bar{\mu} \rangle + \langle \nabla_Z \bar{\mu}, \mu \rangle = 0$, we have

$$d\tilde{g}_{\mu\bar{\mu}}(\nabla_{\bar{\beta}}\beta) = \sum_{\rho} 2\langle \nabla_{\bar{\beta}}\beta, \bar{\rho}\rangle d\tilde{g}_{\mu\bar{\mu}}(\rho) + \sum_{\rho} 2\langle \nabla_{\bar{\beta}}\beta, \rho\rangle d\tilde{g}_{\mu\bar{\mu}}(\bar{\rho})$$

$$= 2i \sum_{\rho} \left(\langle \nabla_{\bar{\beta}}\beta, \bar{\rho}\rangle g_{\rho}\mu\bar{\mu} - \langle \nabla_{\bar{\beta}}\beta, \bar{\rho}\rangle g_{\rho}\bar{\mu}\mu + \langle \nabla_{\bar{\beta}}\beta, \rho\rangle g_{\bar{\rho}}\mu\bar{\mu} - \langle \nabla_{\bar{\beta}}\beta, \rho\rangle g_{\bar{\rho}}\bar{\mu}\mu \right)$$

$$= (5.13) + (5.14).$$

Finally

$$(5.8) = \sum_{\rho} 2\langle \nabla_{\beta}\mu, \bar{\rho}\rangle (ig_{\bar{\beta}}\rho\bar{\mu} - ig_{\bar{\beta}}\bar{\mu}\rho) - 2\langle \nabla_{\beta}\mu, \bar{\rho}\rangle (\cos\theta_{\rho} - \cos\theta_{\mu})\langle \nabla_{\bar{\beta}}\rho, \bar{\mu}\rangle + \sum_{\rho} 2\langle \nabla_{\beta}\bar{\mu}, \rho\rangle (ig_{\bar{\beta}}\mu\bar{\rho} - ig_{\bar{\beta}}\bar{\rho}\mu) - 2\langle \nabla_{\beta}\bar{\mu}, \rho\rangle (\cos\theta_{\mu} - \cos\theta_{\rho})\langle \nabla_{\bar{\beta}}\mu, \bar{\rho}\rangle = \sum_{\rho} 2i\langle \nabla_{\beta}\mu, \bar{\rho}\rangle g_{\bar{\beta}}\rho\bar{\mu} - 2i\langle \nabla_{\beta}\mu, \bar{\rho}\rangle g_{\bar{\beta}}\bar{\mu}\rho + 2i\langle \nabla_{\beta}\bar{\mu}, \rho\rangle g_{\bar{\beta}}\mu\bar{\rho} - 2i\langle \nabla_{\beta}\bar{\mu}, \rho\rangle g_{\bar{\beta}}\bar{\rho}\mu -2\sum_{\rho} (\cos\theta_{\mu} - \cos\theta_{\rho}) (|\langle \nabla_{\beta}\mu, \bar{\rho}\rangle|^{2} + |\langle \nabla_{\bar{\beta}}\mu, \bar{\rho}\rangle|^{2}).$$

These expressions lead to the expression of the lemma. \Box

Finally, we have

Proposition 5.1 If F is minimal without complex directions, then for each $0 \le k \le 2n$ at each $p_0 \in \Omega^0_{2k}$,

$$\triangle \kappa = 4i \sum_{\beta,\mu} Ricci^{N}(JdF(\beta), dF(\bar{\beta}))$$

$$+ \sum_{\beta,\mu}^{\beta} \frac{32}{\sin^{2}\theta_{\mu}} Im\left(R^{N}(dF(\beta), dF(\mu), dF(\bar{\beta}), JdF(\bar{\mu}) + i\cos\theta_{\mu}dF(\bar{\mu}))\right)$$

$$- \sum_{\beta,\mu,\rho} \frac{64(\cos\theta_{\mu} + \cos\theta_{\rho})}{\sin^{2}\theta_{\mu}\sin^{2}\theta_{\rho}} Re\left(g_{\beta}\mu\bar{\rho}g_{\bar{\beta}}\rho\bar{\mu}\right)$$

$$+ \sum_{\beta,\mu,\rho} \frac{32(\cos\theta_{\rho} - \cos\theta_{\mu})}{\sin^{2}\theta_{\mu}\sin^{2}\theta_{\rho}} \left(|g_{\beta}\mu\rho|^{2} + |g_{\bar{\beta}}\mu\rho|^{2} \right)$$

$$+ \sum_{\beta,\mu,\rho} \frac{32(\cos\theta_{\mu} + \cos\theta_{\rho})}{\sin^{2}\theta_{\mu}} \left(|\langle\nabla_{\beta}\mu,\rho\rangle|^{2} + |\langle\nabla_{\bar{\beta}}\mu,\rho\rangle|^{2} \right).$$

Proof. From (5.7) and Lemma 5.4 we get

$$\begin{split} & = + \sum_{\beta,\mu,\rho} \frac{64(\cos\theta_{\mu} + \cos\theta_{\rho})}{\sin^{2}\theta_{\mu}\sin^{2}\theta_{\rho}} d\bar{g}_{\mu\bar{\rho}}(\bar{\beta}) d\bar{g}_{\rho\bar{\mu}}(\beta) \\ & + \sum_{\beta,\mu} \frac{32i}{\sin^{2}\theta_{\mu}} R^{N} (dF(\beta), dF(\bar{\beta}), dF(\mu), JdF(\bar{\mu}) + i\cos\theta_{\mu}dF(\bar{\mu})) \\ & + \sum_{\beta,\mu,\rho} \frac{64}{\sin^{2}\theta_{\mu}} Im \Big(R^{N} (dF(\beta), dF(\mu), dF(\bar{\beta}), JdF(\bar{\mu}) + i\cos\theta_{\mu}dF(\bar{\mu})) \Big) \\ & + \sum_{\beta,\mu,\rho} \frac{64(\cos\theta_{\rho} - \cos\theta_{\mu})}{\sin^{2}\theta_{\mu}\sin^{2}\theta_{\rho}} (|g_{\beta}\mu\rho|^{2} + |g_{\beta}\bar{\mu}\bar{\rho}|^{2}) \\ & - \sum_{\beta,\mu,\rho} \frac{64(\cos\theta_{\rho} + \cos\theta_{\mu})}{\sin^{2}\theta_{\mu}\sin^{2}\theta_{\rho}} (|g_{\beta}\mu\bar{\rho}|^{2} + |g_{\beta}\bar{\mu}\rho|^{2}) \\ & + \sum_{\beta,\mu,\rho} -\frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\mu}\beta,\bar{\rho}\rangle g_{\beta}\rho\bar{\mu} - \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\mu}\beta,\rho\rangle g_{\bar{\beta}}\bar{\rho}\bar{\mu} - \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\mu}\bar{\beta},\bar{\rho}\rangle g_{\rho}\bar{\beta}\bar{\mu} \\ & + \sum_{\beta,\mu,\rho} \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\bar{\beta}}\mu,\bar{\rho}\rangle g_{\beta}\rho\bar{\mu} - \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\mu}\bar{\beta},\rho\rangle g_{\bar{\beta}}\bar{\rho}\mu + \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\bar{\beta}}\mu,\rho\rangle g_{\bar{\rho}}\bar{\beta}\bar{\mu} \\ & + \sum_{\beta,\mu,\rho} \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\bar{\mu}}\beta,\bar{\rho}\rangle g_{\beta}\rho\mu + \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\bar{\mu}}\beta,\rho\rangle g_{\bar{\beta}}\bar{\mu} + \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\bar{\mu}}\bar{\beta},\rho\rangle g_{\bar{\rho}}\bar{\beta}\mu - \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\bar{\beta}}\bar{\mu},\rho\rangle g_{\bar{\rho}}\bar{\beta}\mu \\ & + \sum_{\beta,\mu,\rho} \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\bar{\beta}}\bar{\mu},\bar{\rho}\rangle g_{\beta}\mu\rho + \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\bar{\mu}}\bar{\mu},\rho\rangle g_{\beta}\bar{\mu}\rho - \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\bar{\beta}}\mu,\rho\rangle g_{\bar{\beta}}\bar{\mu}\rho \\ & + \sum_{\beta,\mu,\rho} \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\bar{\beta}}\mu,\bar{\rho}\rangle g_{\beta}\bar{\mu}\rho + \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\bar{\beta}}\mu,\rho\rangle g_{\beta}\bar{\mu}\rho - \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\bar{\beta}}\mu,\rho\rangle g_{\bar{\beta}}\bar{\mu}\rho \\ & + \sum_{\beta,\mu,\rho} \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\bar{\beta}}\mu,\bar{\rho}\rangle g_{\bar{\beta}}\bar{\mu}\rho + \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\bar{\beta}}\mu,\rho\rangle g_{\bar{\beta}}\bar{\rho}\mu - \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\bar{\beta}}\mu,\rho\rangle g_{\bar{\beta}}\bar{\mu}\rho \\ & + \sum_{\beta,\mu,\rho} \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\bar{\beta}}\mu,\bar{\rho}\rangle g_{\bar{\beta}}\bar{\mu}\rho + \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\bar{\beta}}\mu,\bar{\rho}\rangle g_{\bar{\beta}}\bar{\rho}\mu - \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\bar{\beta}}\mu,\bar{\rho}\rangle g_{\bar{\beta}}\bar{\mu}\rho \\ & + \sum_{\beta,\mu,\rho} \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\bar{\beta}}\mu,\bar{\rho}\rangle g_{\bar{\beta}}\bar{\mu}\rho + \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\bar{\beta}}\mu,\bar{\rho}\rangle g_{\bar{\beta}}\bar{\mu}\rho \\ & + \sum_{\beta,\mu,\rho} \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\bar{\beta}}\mu,\bar{\rho}\rangle g_{\bar{\beta}}\bar{\mu}\rho + \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\bar{\beta}}\mu,\bar{\rho}\rangle g_{\bar{\beta}}\bar{\mu}\rho \\ & + \sum_{\beta,\mu,\rho} \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\bar{\beta}}\mu,\bar{\rho}\rangle g_{\bar{\beta}}\bar{\mu}\rho - \frac{64i}{\sin^{2}\theta_{\mu}} \langle \nabla_{\bar{\beta}$$

Interchanging ρ with β in the first term of (5.16) (that we named by (5.16)(1), and similarly to other equations), we see that (5.16)(1) + (5.17)(2) = 0. Interchanging ρ with β in (5.18)(1), we get (5.18)(1) + (5.19)(2) = 0. In (5.16)(2), $\langle \nabla_{\mu} \beta, \rho \rangle$ is skew-symmetric on ρ and β , and $g_{\bar{\beta}}\bar{\rho}\bar{\mu}$ is symmetric on ρ and β . Hence (5.16)(2) = 0. Similarly (5.16)(3) = (5.18)(2) = (5.18)(3) = 0. If we interchange ρ with μ in (5.17)(1),

$$(5.17)(1) + (5.20)(2) = -\sum_{\beta,\mu,\rho} \frac{64i(\sin^2\theta_{\mu} - \sin^2\theta_{\rho})}{\sin^2\theta_{\mu}\sin^2\theta_{\rho}} \langle \nabla_{\bar{\beta}}\bar{\mu}, \rho \rangle g_{\beta}\mu\bar{\rho}.$$

Interchanging ρ with μ in (5.17)(3), we get

$$(5.17)(3) + (5.20)(3) = -\sum_{\beta,\mu,\rho} \frac{64i(\sin^2\theta_\mu + \sin^2\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} \langle \nabla_{\bar{\beta}}\mu,\rho \rangle g_{\beta}\bar{\mu}\bar{\rho}.$$

Interchanging ρ with μ in (5.19)(1), we get

$$(5.19)(1) + (5.20)(1) = \sum_{\beta,\mu,\rho} \frac{64i(\sin^2\theta_{\mu} + \sin^2\theta_{\rho})}{\sin^2\theta_{\mu}\sin^2\theta_{\rho}} \langle \nabla_{\bar{\beta}}\bar{\mu}, \bar{\rho} \rangle g_{\beta}\mu\rho.$$

Interchanging ρ with μ in (5.19)(3), we get

$$(5.19)(3) + (5.21)(1) = \sum_{\beta,\mu,\rho} \frac{64i(\sin^2\theta_{\mu} - \sin^2\theta_{\rho})}{\sin^2\theta_{\mu}\sin^2\theta_{\rho}} \langle \nabla_{\bar{\beta}}\mu, \bar{\rho} \rangle g_{\beta}\bar{\mu}\rho.$$

Interchanging ρ with μ in (5.21)(2)

$$(5.21)(2) + (5.22)(1) = \sum_{\beta,\mu,\rho} \frac{64i(-\sin^2\theta_{\mu} + \sin^2\theta_{\rho})}{\sin^2\theta_{\mu}\sin^2\theta_{\rho}} \langle \nabla_{\!\beta}\bar{\mu}, \rho \rangle g_{\bar{\beta}}\mu\bar{\rho}.$$

Interchanging ρ with μ in (5.22)(2), we obtain

$$(5.22)(2) + (5.21)(3) = \sum_{\beta,\mu,\rho} \frac{64i(\sin^2\theta_\mu - \sin^2\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} \langle \nabla_\beta \mu, \bar{\rho} \rangle g_{\bar{\beta}}\bar{\mu}\rho.$$

Therefore

$$2\triangle\kappa = \sum_{\beta,\mu,\rho} \frac{64(\cos\theta_{\mu} + \cos\theta_{\rho})}{\sin^{2}\theta_{\mu}\sin^{2}\theta_{\rho}} d\tilde{g}_{\mu\bar{\rho}}(\bar{\beta}) d\tilde{g}_{\rho\bar{\mu}}(\beta)$$

$$(5.23)$$

$$+\sum_{\beta,\mu} \frac{32i}{\sin^2 \theta_{\mu}} R^N(dF(\beta), dF(\bar{\beta}), dF(\mu), JdF(\bar{\mu}) + i\cos \theta_{\mu} dF(\bar{\mu}))$$

$$(5.24)$$

$$+\sum_{\beta,\mu}^{\infty} \frac{64}{\sin^2 \theta_{\mu}} Im \Big(R^N(dF(\beta), dF(\mu), dF(\bar{\beta}), JdF(\bar{\mu}) + i \cos \theta_{\mu} dF(\bar{\mu})) \Big)$$

$$+\sum_{\beta,\mu,\rho} \frac{64(\cos\theta_{\rho} - \cos\theta_{\mu})}{\sin^{2}\theta_{\mu}\sin^{2}\theta_{\rho}} |g_{\beta}\mu_{\rho}|^{2}$$

$$(5.25)$$

$$-\sum_{\beta,\mu,\rho} \frac{64(\cos\theta_{\rho} + \cos\theta_{\mu})}{\sin^2\theta_{\mu}\sin^2\theta_{\rho}} |g_{\beta}\mu\bar{\rho}|^2$$
(5.26)

$$-\sum_{\beta,\mu,\rho} \frac{64(\cos\theta_{\rho} + \cos\theta_{\mu})}{\sin^2\theta_{\mu}\sin^2\theta_{\rho}} |g_{\beta}\bar{\mu}_{\rho}|^2$$
(5.27)

$$+\sum_{\beta,\mu,\rho} \frac{64(\cos\theta_{\rho} - \cos\theta_{\mu})}{\sin^{2}\theta_{\mu}\sin^{2}\theta_{\rho}} |g_{\beta}\bar{\mu}\bar{\rho}|^{2}$$

$$(5.28)$$

$$-\sum_{\beta,\mu,\rho} \frac{64i(\sin^2\theta_{\mu} - \sin^2\theta_{\rho})}{\sin^2\theta_{\mu}\sin^2\theta_{\rho}} \langle \nabla_{\bar{\beta}}\bar{\mu}, \rho \rangle g_{\beta}\mu\bar{\rho}$$
(5.29)

$$-\sum_{\beta,\mu,\rho} \frac{\sin^2 \theta_{\mu} \sin^2 \theta_{\rho}}{\sin^2 \theta_{\mu} \sin^2 \theta_{\rho}} \langle \nabla_{\bar{\beta}} \mu, \rho \rangle g_{\beta} \bar{\mu} \bar{\rho}$$

$$(5.30)$$

$$+\sum_{\beta,\mu,\rho} \frac{64i(\sin^2\theta_{\mu} + \sin^2\theta_{\rho})}{\sin^2\theta_{\mu}\sin^2\theta_{\rho}} \langle \nabla_{\bar{\beta}}\bar{\mu}, \bar{\rho}\rangle g_{\beta}\mu\rho$$
(5.31)

$$+\sum_{\beta,\mu,\rho} \frac{64i(\sin^2\theta_{\mu} - \sin^2\theta_{\rho})}{\sin^2\theta_{\mu}\sin^2\theta_{\rho}} \langle \nabla_{\bar{\beta}}\mu, \bar{\rho} \rangle g_{\beta}\bar{\mu}\rho \tag{5.32}$$

$$+\sum_{\beta,\mu,\rho} \frac{64i(-\sin^2\theta_{\mu} + \sin^2\theta_{\rho})}{\sin^2\theta_{\mu}\sin^2\theta_{\rho}} \langle \nabla_{\!\beta}\bar{\mu}, \rho \rangle g_{\bar{\beta}}\mu\bar{\rho}$$
(5.33)

$$+\sum_{\beta,\mu,\rho} \frac{64i(\sin^2\theta_{\mu} - \sin^2\theta_{\rho})}{\sin^2\theta_{\mu}\sin^2\theta_{\rho}} \langle \nabla_{\!\beta}\mu, \bar{\rho} \rangle g_{\bar{\beta}}\bar{\mu}\rho \tag{5.34}$$

$$-\sum_{\beta,\mu,\rho} \frac{64(\cos\theta_{\mu} - \cos\theta_{\rho})}{\sin^2\theta_{\mu}} (|\langle \nabla_{\!\beta}\mu,\bar{\rho}\rangle|^2 + |\langle \nabla_{\!\bar{\beta}}\mu,\bar{\rho}\rangle|^2). \tag{5.35}$$

By Lemma 5.3,

$$(5.23) = \sum_{\beta,\mu,\rho} \frac{64(\cos\theta_{\mu} + \cos\theta_{\rho})}{\sin^{2}\theta_{\mu}\sin^{2}\theta_{\rho}} \cdot \left(ig_{\bar{\beta}}\mu\bar{\rho} - ig_{\bar{\beta}}\bar{\rho}\mu - (\cos\theta_{\mu} - \cos\theta_{\rho})\langle\nabla_{\bar{\beta}}\mu,\bar{\rho}\rangle\right) \cdot \left(ig_{\beta}\rho\bar{\mu} - ig_{\beta}\bar{\mu}\rho - (\cos\theta_{\rho} - \cos\theta_{\mu})\langle\nabla_{\beta}\rho,\bar{\mu}\rangle\right)$$

$$= -\sum_{\beta,\mu,\rho} \frac{64(\cos\theta_{\mu} + \cos\theta_{\rho})}{\sin^{2}\theta_{\mu}\sin^{2}\theta_{\rho}} g_{\bar{\beta}}\mu\bar{\rho}g_{\beta}\rho\bar{\mu}$$

$$+ \sum_{\beta,\mu,\rho} \frac{64(\cos\theta_{\mu} + \cos\theta_{\rho})}{\sin^{2}\theta_{\mu}\sin^{2}\theta_{\rho}} |g_{\beta}\bar{\mu}\rho|^{2}$$

$$(5.36)$$

$$+\sum_{\beta,\mu,\rho} \frac{64i(\cos^2\theta_{\mu} - \cos^2\theta_{\rho})}{\sin^2\theta_{\mu}\sin^2\theta_{\rho}} g_{\bar{\beta}}\mu\bar{\rho}\langle\nabla_{\!\beta}\rho,\bar{\mu}\rangle \tag{5.37}$$

$$+\sum_{\beta,\mu,\rho} \frac{64(\cos\theta_{\mu} + \cos\theta_{\rho})}{\sin^2\theta_{\mu}\sin^2\theta_{\rho}} |g_{\beta}\rho\bar{\mu}|^2 \tag{5.38}$$

$$-\sum_{\beta,\mu,\rho} \frac{64(\cos\theta_{\mu} + \cos\theta_{\rho})}{\sin^{2}\theta_{\mu}\sin^{2}\theta_{\rho}} g_{\beta}\bar{\mu}\rho g_{\bar{\beta}}\bar{\rho}\mu$$

$$-\sum_{\beta,\mu,\rho} \frac{64i(\cos^2\theta_\mu - \cos^2\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} \langle \nabla_{\!\beta}\rho, \bar{\mu} \rangle g_{\bar{\beta}}\bar{\rho}\mu \tag{5.39}$$

$$-\sum_{\beta,\mu,\rho} \frac{64i(\cos^2\theta_\mu - \cos^2\theta_\rho)}{\sin^2\theta_\mu \sin^2\theta_\rho} \langle \nabla_{\bar{\beta}}\mu, \bar{\rho} \rangle g_\beta \rho \bar{\mu}$$
 (5.40)

$$+\sum_{\beta,\mu,\rho} \frac{64i(\cos^2\theta_{\mu} - \cos^2\theta_{\rho})}{\sin^2\theta_{\mu}\sin^2\theta_{\rho}} \langle \nabla_{\bar{\beta}}\mu, \bar{\rho} \rangle g_{\beta}\bar{\mu}\rho$$
(5.41)

$$+\sum_{\beta,\mu,\rho} \frac{64(\cos^2\theta_{\mu} - \cos^2\theta_{\rho})}{\sin^2\theta_{\mu}\sin^2\theta_{\rho}} (\cos\theta_{\rho} - \cos\theta_{\mu}) \langle \nabla_{\bar{\beta}}\mu, \bar{\rho} \rangle \langle \nabla_{\beta}\rho, \bar{\mu} \rangle. \tag{5.42}$$

Immediately we have, (5.27) + (5.36) = (5.32) + (5.41) = (5.33) + (5.37) = 0, and interchanging μ with ρ in (5.26), (5.34) and in (5.40), we get, (5.26) + (5.38) = (5.29) + (5.40) = (5.34) + (5.39) = 0. Note that

$$\sum_{\mu,\rho} \frac{(\cos\theta_{\mu} - \cos\theta_{\rho})}{\sin^2\theta_{\mu}} |\langle \nabla_{\!\beta}\mu, \bar{\rho} \rangle|^2 = \sum_{\mu,\rho} \frac{(\cos\theta_{\rho} - \cos\theta_{\mu})}{\sin^2\theta_{\rho}} |\langle \nabla_{\!\bar{\beta}}\mu, \bar{\rho} \rangle|^2.$$

Hence (5.35) + (5.42) = 0. Then,

$$2\triangle\kappa = \sum_{\beta,\mu} \frac{32i}{\sin^2 \theta_{\mu}} R^N(dF(\beta), dF(\bar{\beta}), dF(\mu), JdF(\bar{\mu}) + i\cos\theta_{\mu}dF(\bar{\mu}))$$

$$+ \sum_{\beta,\mu} \frac{64}{\sin^2 \theta_{\mu}} Im \Big(R^N(dF(\beta), dF(\mu), dF(\bar{\beta}), JdF(\bar{\mu}) + i\cos\theta_{\mu}dF(\bar{\mu})) \Big)$$
(5.43)

$$+\sum_{\beta,\mu,\rho} -\frac{64(\cos\theta_{\mu} + \cos\theta_{\rho})}{\sin^2\theta_{\mu}\sin^2\theta_{\rho}} (g_{\bar{\beta}}\mu\bar{\rho}g_{\beta}\rho\bar{\mu} + g_{\beta}\bar{\mu}\rho g_{\bar{\beta}}\bar{\rho}\mu)$$

$$(5.44)$$

$$+\sum_{\beta,\mu,\rho} \frac{64(\cos\theta_{\rho} - \cos\theta_{\mu})}{\sin^2\theta_{\mu}\sin^2\theta_{\rho}} \left(|g_{\beta}\mu\rho|^2 + |g_{\bar{\beta}}\mu\rho|^2 \right)$$

$$-\sum_{\beta,\mu,\rho} \frac{64i(\sin^2\theta_{\mu} + \sin^2\theta_{\rho})}{\sin^2\theta_{\mu}\sin^2\theta_{\rho}} \langle \nabla_{\bar{\beta}}\mu, \rho \rangle g_{\beta}\bar{\mu}\bar{\rho}$$
(5.45)

$$+\sum_{\beta,\mu,\rho} \frac{64i(\sin^2\theta_{\mu} + \sin^2\theta_{\rho})}{\sin^2\theta_{\mu}\sin^2\theta_{\rho}} \langle \nabla_{\bar{\beta}}\bar{\mu}, \bar{\rho} \rangle g_{\beta}\mu\rho. \tag{5.46}$$

Using Lemma 5.3, and interchanging ρ by μ when necessary,

$$(5.45) + (5.46) =$$

$$= \sum_{\beta,\mu,\rho} -\frac{64i}{\sin^2 \theta_{\rho}} \langle \nabla_{\bar{\beta}} \mu, \rho \rangle g_{\beta} \bar{\mu} \bar{\rho} - \frac{64i}{\sin^2 \theta_{\mu}} \langle \nabla_{\bar{\beta}} \mu, \rho \rangle g_{\beta} \bar{\mu} \bar{\rho} + \frac{64i}{\sin^2 \theta_{\mu}} \langle \nabla_{\bar{\beta}} \bar{\mu}, \bar{\rho} \rangle g_{\beta} \mu \rho + \frac{64i}{\sin^2 \theta_{\rho}} \langle \nabla_{\bar{\beta}} \bar{\mu}, \bar{\rho} \rangle g_{\beta} \mu \rho$$

$$= \sum_{\beta,\mu,\rho} \frac{-64i}{\sin^2 \theta_{\mu}} \langle \nabla_{\bar{\beta}} \mu, \rho \rangle \Big(g_{\beta} \bar{\mu} \bar{\rho} - g_{\beta} \bar{\rho} \bar{\mu} \Big) + \sum_{\beta,\mu,\rho} \frac{64i}{\sin^2 \theta_{\mu}} \langle \nabla_{\bar{\beta}} \bar{\mu}, \bar{\rho} \rangle \Big(g_{\beta} \mu \rho - g_{\beta} \rho \mu \Big)$$

$$= \sum_{\beta,\mu,\rho} \frac{64}{\sin^2 \theta_{\mu}} \langle \nabla_{\bar{\beta}} \mu, \rho \rangle (\cos \theta_{\mu} + \cos \theta_{\rho}) \langle \nabla_{\beta} \bar{\mu}, \bar{\rho} \rangle + \frac{64}{\sin^2 \theta_{\mu}} \langle \nabla_{\bar{\beta}} \bar{\mu}, \bar{\rho} \rangle (\cos \theta_{\mu} + \cos \theta_{\rho}) \langle \nabla_{\beta} \mu, \rho \rangle$$

$$= \sum_{\beta,\mu,\rho} \frac{64(\cos \theta_{\mu} + \cos \theta_{\rho})}{\sin^2 \theta_{\mu}} \Big(|\langle \nabla_{\beta} \mu, \rho \rangle|^2 + |\langle \nabla_{\bar{\beta}} \mu, \rho \rangle|^2 \Big).$$

Obviously

$$(5.44) = \sum_{\beta,\mu,\rho} \frac{-128(\cos\theta_{\mu} + \cos\theta_{\rho})}{\sin^2\theta_{\mu}\sin^2\theta_{\rho}} Re(g_{\beta}\mu\bar{\rho}g_{\bar{\beta}}\rho\bar{\mu}).$$

From (1.4), (2.1), and the *J*-invariance of Ricci, (5.43) = $8i \sum_{\beta} Ricci^{N}(JdF(\beta), dF(\bar{\beta}))$, and the expression of the Proposition follows.

Acknowledgments

We would like to thank very much Professor James Eells for helpful discussions and encouragement.

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