

# The Alexander polynomial of a plane curve singularity and the ring of functions on it

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## Abstract

We give two formulae which express the Alexander polynomial  $\Delta^C$  of several variables of a plane curve singularity  $C$  in terms of the ring  $\mathcal{O}_C$  of germs of analytic functions on the curve. One of them expresses  $\Delta^C$  in terms of dimensions of some factorspaces corresponding to a (multi-indexed) filtration on the ring  $\mathcal{O}_C$ . The other one gives the coefficients of the Alexander polynomial  $\Delta^C$  as Euler characteristics of some explicitly described spaces (complements to arrangements of projective hyperplanes).

A version of this text has been published in Russian Mathematical Surveys, v.54 (1999), N 3 (327), p.157–158.

The ring  $\mathcal{O}_X$  of germs of holomorphic functions on a germ  $X$  of an analytic set determines  $X$  itself (up to analytic equivalence). Thus all invariants of  $X$ , in particular, topological ones, can “be read” from  $\mathcal{O}_X$ . There arises a general problem to find expressions for invariants of  $X$  in terms of the ring  $\mathcal{O}_X$ . The Alexander polynomial  $\Delta^C$  of several variables is a complete topological invariant of a plane curve singularity  $C \subset (\mathbb{C}^2, 0)$  ([Y]). A formula of D.Eisenbud and W.Neumann ([EN]) expresses the Alexander polynomial in terms of an embedded resolution of the curve  $C$ . In this note we give two formulae for the Alexander polynomial directly in terms of the ring of germs of

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analytic functions on the curve  $C$ . One of them expresses the Alexander polynomial  $\Delta^C$  in terms of dimensions of some factorspaces corresponding to a (multi-indexed) filtration on the ring  $\mathcal{O}_C$ . The other one gives the coefficients of the Alexander polynomial  $\Delta^C$  as Euler characteristics of some explicitly described spaces (complements to arrangements of projective hyperplanes). It seems to be the first result which describes the coefficients of the Alexander polynomial (and thus of the zeta-function of the monodromy) as Euler characteristics of some spaces. Another formula which expresses the Lefschetz numbers of iterates of the monodromy (and therefore the zeta-function of it) for a hypersurface singularity of any dimension in terms of Euler characteristics of some subspaces of the space of (truncated) arcs is given in a paper of J.Denef and F.Loeser (xxx-Preprint series, math.AG/0001105).

Let  $C$  be a germ of a reduced plane curve at the origin in  $\mathbb{C}^2$  and let  $C = \bigcup_{i=1}^r C_i$  be its representation as the union of irreducible components (with a fixed numbering). Let  $\mathcal{O}_{\mathbb{C}^2,0}$  be the ring of germs of holomorphic functions at the origin in  $\mathbb{C}^2$  and let  $\{f = 0\}$  ( $f \in \mathcal{O}_{\mathbb{C}^2,0}$ ) be an equation of the curve  $C$ . Let  $\mathcal{O}_C$  be the ring of germs of analytic functions on  $C$  ( $\cong \mathcal{O}_{\mathbb{C}^2,0}/(f)$ ), and let  $\Delta^C(t_1, \dots, t_r)$  be the Alexander polynomial of the link  $C \cap S_\varepsilon^3 \subset S_\varepsilon^3$  for  $\varepsilon > 0$  small enough (see, e.g., [EN]).

**Remarks.** 1. According to the definition, the Alexander polynomial  $\Delta^C(t_1, \dots, t_r)$  is well defined only up to multiplication by monomials  $\pm \underline{t}^{\underline{m}} = \pm t_1^{m_1} \dots t_r^{m_r}$  ( $\underline{t} = (t_1, \dots, t_r)$ ,  $\underline{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r$ ). We fix the Alexander polynomial assuming that it is really a polynomial (i.e., it does not contain variables with negative powers) and  $\Delta^C(0, \dots, 0) = 1$ .

2. There is some difference in definitions (or rather in descriptions) of the Alexander polynomial for a curve with one branch ( $r = 1$ ) or with many branches ( $r > 1$ ) (see, e.g., [EN]). In order to have all the results (Theorems 1 and 2 below) valid for  $r = 1$  as well, for an irreducible curve  $C$ ,  $\Delta^C(t)$  should be not the Alexander polynomial, but rather the zeta-function  $\zeta_C(t)$  of the monodromy, equal to the Alexander polynomial divided by  $(1 - t)$ . In this case  $\Delta^C(t)$  is not a polynomial, but an infinite power series. However for uniformity of the statements we shall use the name "Alexander polynomial" for this  $\Delta^C(t)$  as well.

Let  $\varphi_i : (\mathbb{C}_i, 0) \rightarrow (\mathbb{C}^2, 0)$  be parametrizations (uniformizations) of the components  $C_i$  of the curve  $C$ , i.e., germs of analytic maps such that  $\text{Im } \varphi_i = C_i$  and  $\varphi_i$  is an isomorphism between  $\mathbb{C}_i$  and  $C_i$  outside of the origin. For

a germ  $g \in \mathcal{O}_{\mathbb{C}^2,0}$ , let  $v_i = v_i(g)$  and  $a_i = a_i(g)$  be the power of the leading term and the coefficient at it in the power series decomposition of the germ  $g \circ \varphi_i : (\mathbb{C}_i, 0) \rightarrow \mathbb{C} : g \circ \varphi_i(t_i) = a_i \cdot t_i^{v_i} + \text{terms of higher degree}$  ( $a_i \neq 0$ ). If  $g \circ \varphi_i(t) \equiv 0$ ,  $v_i(g)$  is assumed to be equal to  $\infty$  and  $a_i(g)$  is not defined. The numbers  $v_i(g)$  and  $a_i(g)$  are defined for elements  $g$  of the ring  $\mathcal{O}_C$  of functions on the curve  $C$  as well.

The semigroup  $S = S_C$  of the plane curve singularity  $C$  is the subsemigroup of  $\mathbb{Z}_{\geq 0}^r$  which consists of elements of the form  $\underline{v}(g) = (v_1(g), \dots, v_r(g))$  for all germs  $g \in \mathcal{O}_C$  with  $v_i(g) < \infty$ ;  $i = 1, \dots, r$ . The extended semigroup  $\hat{S} = \hat{S}_C$  of the plane curve singularity  $C$  is the subsemigroup of  $\mathbb{Z}_{\geq 0}^r \times (\mathbb{C}^*)^r$  which consists of elements of the form  $(\underline{v}(g); \underline{a}(g)) = (v_1(g), \dots, v_r(g); a_1(g), \dots, a_r(g))$  for all germs  $g \in \mathcal{O}_C$  with  $v_i(g) < \infty$ ,  $i = 1, \dots, r$  ([CDG1]).

It is known that both the semigroup  $S_C$  and the Alexander polynomial  $\Delta^C(t_1, \dots, t_r)$  are complete topological invariants of a plane curve singularity, i.e., each of them determines the germ  $C$  up to topological equivalence ([W], [Y]). Therefore it is interesting to understand a connection between them. In fact from the formula for the Alexander polynomial in terms of a resolution of a plane curve singularity (see [EN]) it is not difficult to understand that the Alexander polynomial  $\Delta^C(t_1, \dots, t_r)$  may contain with non-zero coefficients only monomials  $t^{\underline{v}}$  for  $\underline{v}$  from the semigroup  $S_C$  of the curve  $C$ . For the case of an irreducible curve  $C$  ( $r = 1$ ) the corresponding connection has been described in [CDG2]. In this case

$$\zeta_C(t) = \sum_{i \in S_C} t^i$$

( $S_C \subset \mathbb{Z}_{\geq 0}$ ).

Let  $\pi : \hat{S}_C \rightarrow \mathbb{Z}^r$  be the natural projection:  $(\underline{v}, \underline{a}) \mapsto \underline{v}$ . For an element  $\underline{v} \in \mathbb{Z}^r$ , let  $F_{\underline{v}} = \pi^{-1}(\underline{v}) \subset \{\underline{v}\} \times (\mathbb{C}^*)^r \subset \{\underline{v}\} \times \mathbb{C}^r$  be the corresponding fibre of the extended semigroup ([CDG1]). The fibre  $F_{\underline{v}}$  is not empty if and only if  $\underline{v} \in S_C$ . For  $\underline{v} = (v_1, \dots, v_r) \in \mathbb{Z}^r$ , let  $J(\underline{v}) = \{g \in \mathcal{O}_C : v_i(g) \geq v_i; i = 1, \dots, r\}$  be an ideal in  $\mathcal{O}_C$ . One has a natural linear map  $j_{\underline{v}} : J(\underline{v}) \rightarrow \mathbb{C}^r$ , which sends  $g \in J(\underline{v})$  to  $(a_1, \dots, a_r)$ , where  $a_i$  is the coefficient in the power series expansion  $g \circ \varphi_i(t_i) = a_i t_i^{v_i} + \dots$  (the number  $a_i$  may be equal to zero). Let  $C(\underline{v}) \subset \mathbb{C}^r$  be the image of the map  $j_{\underline{v}}$ , let  $c(\underline{v}) = \dim C(\underline{v})$ . It is not difficult to see that  $C(\underline{v}) \cong J(\underline{v})/J(\underline{v} + \underline{1})$ , where  $\underline{1} = (1, \dots, 1)$ , and that  $F_{\underline{v}} = C(\underline{v}) \cap (\mathbb{C}^*)^r$  (under the natural identification of  $\{\underline{v}\} \times (\mathbb{C}^*)^r$  and  $(\mathbb{C}^*)^r$ ). Therefore the fibre  $F_{\underline{v}}$  ( $\underline{v} \in S_C$ ) is the complement to an arrangement of linear

hyperplanes in the vector space  $C(\underline{v})$ . The extended semigroup  $\hat{S}_C$  contains some analytic information about the plane curve singularity  $C$ , however the dimensions  $c(v)$  depend only on the topological type of  $C$  (see [CDG1]).

Let  $\mathcal{L} = \mathbb{Z}[[t_1, \dots, t_r, t_1^{-1}, \dots, t_r^{-1}]]$  be the set of formal Laurent series in  $t_1, \dots, t_r$ . Elements of  $\mathcal{L}$  are expressions of the form  $\sum_{\underline{v} \in \mathbb{Z}^r} k(\underline{v}) \cdot \underline{t}^{\underline{v}}$  with  $k(\underline{v}) \in \mathbb{Z}$ , generally speaking, infinite in all directions.  $\mathcal{L}$  is not a ring, but a  $\mathbb{Z}[t_1, \dots, t_r]$ - (or even  $\mathbb{Z}[t_1, \dots, t_r, t_1^{-1}, \dots, t_r^{-1}]$ -) module. The polynomial ring  $\mathbb{Z}[t_1, \dots, t_r]$  can be in a natural way considered as being embedded into  $\mathcal{L}$ .

Let  $L_C(t_1, \dots, t_r) = \sum_{\underline{v} \in \mathbb{Z}^r} c(\underline{v}) \cdot \underline{t}^{\underline{v}} \in \mathcal{L}$ ,  $P'_C(t_1, \dots, t_r) = (t_1 - 1) \cdot \dots \cdot (t_r - 1) \cdot L_C(t_1, \dots, t_r)$ . One can easily see that  $P'_C(t_1, \dots, t_r)$  is in fact a polynomial, i.e.,  $P'_C(t_1, \dots, t_r) \in \mathbb{Z}[t_1, \dots, t_r]$ . This follows from the fact that, if  $v'_i$  and  $v''_i$  are negative, then  $c(v_1, \dots, v'_i, \dots, v_r) = c(v_1, \dots, v''_i, \dots, v_r)$ . Let  $P_C(t_1, \dots, t_r) = P'_C(t_1, \dots, t_r) / (t_1 \cdot \dots \cdot t_r - 1) \in \mathbb{Z}[[t_1, \dots, t_r]]$ .

**Proposition.** For  $r > 1$ , the polynomial  $P'_C(t_1, \dots, t_r)$  is divisible by  $(t_1 \cdot \dots \cdot t_r - 1)$ , i.e.,  $P_C(t_1, \dots, t_r) \in \mathbb{Z}[t_1, \dots, t_r]$ .

For  $r = 1$ ,  $P_C(t) = L_C(t)$ .

**Theorem 1**  $P_C(t_1, \dots, t_r) = \Delta^C(t_1, \dots, t_r)$ .

The fibre  $F_{\underline{v}}$  of the extended semigroup is invariant with respect to multiplication by non-zero complex numbers. Let  $\mathbb{P}(F_{\underline{v}})$  be the projectivization of the fibre  $F_{\underline{v}}$ , i.e.,  $\mathbb{P}(F_{\underline{v}}) = F_{\underline{v}}/\mathbb{C}^*$ . The projectivization  $\mathbb{P}(F_{\underline{v}})$  of the fibre  $F_{\underline{v}}$  is the complement to an arrangement of projective hyperplanes in a projective space. If  $\underline{v} \geq \underline{\delta}$ , where  $\underline{\delta}$  is the conductor of the semigroup  $S_C$  of the curve  $C$ , then the fibre  $F_{\underline{v}}$  coincides with  $(\mathbb{C}^*)^r$  and the Euler characteristic  $\chi(\mathbb{P}(F_{\underline{v}}))$  of its projectivization is equal to 1 for  $r = 1$  and to 0 for  $r > 1$ . Let  $\chi(\mathbb{P}\hat{S}_C) := \sum_{\underline{v} \in \mathbb{Z}_{\geq 0}^r} \chi(\mathbb{P}(F_{\underline{v}})) \cdot \underline{t}^{\underline{v}}$ .

**Theorem 2**

$$\Delta^C(t_1, \dots, t_r) = \chi(\mathbb{P}\hat{S}_C). \quad (*)$$

Let  $\zeta_C(t) (= \Delta^C(t, t, \dots, t))$  be the zeta-function of the monodromy of the germ  $f$  (the equation of the curve  $C$ ). Let  $|\underline{v}| := v_1 + \dots + v_r$ .

**Corollary.**  $\zeta_C(t) = \sum_{i=0}^{\infty} \chi\left(\bigcup_{\underline{v}: |\underline{v}|=i} \mathbb{P}(F_{\underline{v}})\right) \cdot t^i$ .

**Remark.** For an irreducible plane curve singularity all coefficients of the zeta–function of the monodromy are equal to 0 or 1. In terms of the equation (\*),  $0 = \chi(\emptyset)$ ,  $1 = \chi(\textit{point})$ .

The proof consists of calculation of the polynomial  $\chi(\mathbb{P}\hat{S}_C)$  in terms of a suitable (not minimal one) embedded resolution of the curve  $C \subset (\mathbb{C}^2, 0)$  and comparing it with the formula for the Alexander polynomial from [EN]. These calculations involve a detailed knowledge about the structure of the semigroup and its relation with the resolution of a singularity. In fact the polynomials  $P_C(t_1, \dots, t_r)$  and  $\chi(\mathbb{P}\hat{S}_C)$  coincide for any (not necessarily plane) curve. The proof will be published elsewhere.

A global version of the result from [CDG2] for a plane algebraic curve with one place at infinity was obtained in [CDG3].

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